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Balanced truncation model reduction of second-order systems

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In this paper we consider structure-preserving model reduction of second-order systems using a balanced truncation approach. Several sets of singular values are introduced for such systems, which lead to different concepts of balancing and different second-order balanced truncation methods. A comparison of these methods with other second-order balanced truncation techniques is presented. We also show that, in general, none of the existing structure-preserving balanced truncation methods for second-order systems preserves stability in the reduced models. Numerical examples are given that demonstrate the properties of the new methods.

Keywords: Second-order systems; Model reduction; Balanced truncation; Gramians; Singular values

AMS Subject Classification: 15A18; 15A24; 93C; 93D20

1. Introduction

Consider a linear time-invariant second-order system

$$\begin{aligned} \mathbf{M}\,\ddot{\boldsymbol{q}}(t) + \mathbf{D}\,\dot{\boldsymbol{q}}(t) + \mathbf{K}\,\boldsymbol{q}(t) &= \mathbf{B}_2\boldsymbol{u}(t), \\ \mathbf{C}_2\dot{\boldsymbol{q}}(t) + \mathbf{C}_1\boldsymbol{q}(t) &= \boldsymbol{y}(t), \end{aligned} \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{n,n}$ is nonsingular, $\mathbf{D} \in \mathbb{R}^{n,n}$, $\mathbf{K} \in \mathbb{R}^{n,n}$, $\mathbf{B}_2 \in \mathbb{R}^{n,m}$, $\mathbf{C}_1, \mathbf{C}_2 \in \mathbb{R}^{p,n}$, $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is a control input and $\boldsymbol{y}(t) \in \mathbb{R}^p$ is an output. Such systems arise in many practical applications including electrical circuits, mechanical systems, large structures and microsystem technology [1–4]. In mechanical engineering, the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are known as the mass, the damping and the stiffness matrices, respectively. Often, the number of equations and variables in system (1) exceeds tens of millions. Simulation, real-time controller design and optimization of such large-scale systems is unfeasible within a reasonable computation time. This motivates model reduction that consists in approximation of (1) by a reduced system

$$\begin{aligned} \tilde{\mathbf{M}}\,\ddot{\tilde{\boldsymbol{q}}}(t) + \tilde{\mathbf{D}}\,\dot{\tilde{\boldsymbol{q}}}(t) + \tilde{\mathbf{K}}\,\tilde{\boldsymbol{q}}(t) &= \tilde{\mathbf{B}}_2\boldsymbol{u}(t), \\ \tilde{\mathbf{C}}_2\dot{\tilde{\boldsymbol{q}}}(t) + \tilde{\mathbf{C}}_1\tilde{\boldsymbol{q}}(t) &= \tilde{\boldsymbol{y}}(t), \end{aligned} \tag{2}$$

where $\tilde{\mathbf{M}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{K}} \in \mathbb{R}^{\ell,\ell}$, $\tilde{\mathbf{B}}_2 \in \mathbb{R}^{\ell,m}$ and $\tilde{\mathbf{C}}_1$, $\tilde{\mathbf{C}}_2 \in \mathbb{R}^{p,\ell}$ with $\ell \ll n$. It is required that the approximate system (2) preserves essential properties of (1) like stability and passivity and that the approximation error is small.

A classical model reduction approach for second-order systems is first to rewrite (1) as a first-order generalized state space system

$$\begin{aligned} \boldsymbol{\mathcal{E}}\,\dot{\boldsymbol{x}}(t) &= \boldsymbol{\mathcal{A}}\,\boldsymbol{x}(t) + \boldsymbol{\mathcal{B}}\,\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{\mathcal{C}}\,\boldsymbol{x}(t), \end{aligned} \tag{3}$$

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where $\boldsymbol{x}(t) = [\,\boldsymbol{q}(t)^{\mathrm{T}},\ \dot{\boldsymbol{q}}(t)^{\mathrm{T}}\,]^{\mathrm{T}}$ and

$$\boldsymbol{\mathcal{E}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \qquad \boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}, \qquad \boldsymbol{\mathcal{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}, \qquad \boldsymbol{\mathcal{C}} = [\,\mathbf{C}_1,\ \mathbf{C}_2\,], \tag{4}$$

and then apply any model reduction method to (3). If a projection-based method like moment matching approximation [5, 6] or balanced truncation [7, 8] is used, we obtain a reduced model

$$\begin{aligned} \tilde{\boldsymbol{\mathcal{E}}}\,\dot{\tilde{\boldsymbol{x}}}(t) &= \tilde{\boldsymbol{\mathcal{A}}}\,\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{\mathcal{B}}}\,\boldsymbol{u}(t), \\ \tilde{\boldsymbol{y}}(t) &= \tilde{\boldsymbol{\mathcal{C}}}\,\tilde{\boldsymbol{x}}(t), \end{aligned} \tag{5}$$

where $\tilde{\boldsymbol{\mathcal{E}}} = \boldsymbol{\mathcal{W}}^{\mathrm{T}}\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{T}}$, $\tilde{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{W}}^{\mathrm{T}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{T}}$, $\tilde{\boldsymbol{\mathcal{B}}} = \boldsymbol{\mathcal{W}}^{\mathrm{T}}\boldsymbol{\mathcal{B}}$, $\tilde{\boldsymbol{\mathcal{C}}} = \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{T}}$ and the projection matrices $\boldsymbol{\mathcal{W}},\ \boldsymbol{\mathcal{T}} \in \mathbb{R}^{2n,k}$ determine subspaces of interest. Note that instead of (4) one can also take other first-order systems that keep the structure in $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{A}}$ for structured $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$, see [3, 4, 9].

Unfortunately, the reduced system (5) cannot, in general, be turned into the second-order form (2), see [10, 11] for special cases when it can be done. Note that preservation of the second-order structure in the reduced model allows a meaningful physical interpretation and usually provides more accurate approximations. In addition, software tools specially developed for second-order systems can also be used for the reduced models.

Recently, structure-preserving model reduction of second-order systems received a lot of attention [3, 10–16, 20]. Moment matching approximation based on Krylov subspace methods is one of the most used model reduction techniques for large-scale systems, see [5, 6] for surveys on these methods. Two different modifications of this approach have been proposed for second-order systems [3, 12–14, 16, 17].

Balanced truncation is another model reduction approach that has been proved to be an efficient for first-order systems [7, 8, 18, 19]. Important properties of this approach are that stability is preserved in the reduced model (5) and an a priori error bound exists. Balancing-related model reduction of second-order systems has been previously considered in [10, 14, 15, 20]. The goal of this paper is to present a general framework for this type of second-order model reduction. Using the position and velocity Gramians introduced in [14, 20], we define in Section 2 several concepts of singular values and balanced realizations for (1). The singular values play a crucial role in identifying which states are important and which states can be truncated without changing the system properties significantly. In Section 3, we present different variants of the second-order balanced truncation method and compare them with existing approaches from [10, 15]. We also discuss the symmetric case and stability issues. Numerical examples are given in Section 4.

Throughout the paper we denote by $\mathbb{R}^{n,m}$ the space of $n \times m$ real matrices. The matrix $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A} \in \mathbb{R}^{n,m}$ and $\mathbf{A}^{-\mathrm{T}} = (\mathbf{A}^{-1})^{\mathrm{T}}$. An identity matrix of order n is denoted by $\mathbf{I}_n$ or simply by $\mathbf{I}$. We denote by rank($\mathbf{A}$) the rank of the matrix $\mathbf{A}$, $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, eigenvalues and singular values of a matrix or an operator. We use $\mathbb{L}_2(\mathbb{I}, \mathbb{R}^m)$ to denote the Hilbert space of vector-valued functions of dimension m whose elements are quadratically integrable on $\mathbb{I} \subseteq \mathbb{R}$.

2. Second-order systems

Consider the second-order system (1). The *transfer function* of (1) is given by

$$\mathbf{G}(s) = (s\mathbf{C}_2 + \mathbf{C}_1)(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}_2. \tag{6}$$

It describes the input-output behavior of (1) in the frequency domain. For simplicity, we will also denote system (1) by $\mathbf{G} = [\,\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2\,]$. Two systems $\mathbf{G} = [\,\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2\,]$ and $\hat{\mathbf{G}} = [\,\hat{\mathbf{M}}, \hat{\mathbf{D}}, \hat{\mathbf{K}}, \hat{\mathbf{B}}_2, \hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2\,]$ are called *restricted system equivalent* if there exist nonsingular matrices $\mathbf{T}_l, \mathbf{T}_r \in \mathbb{R}^{n,n}$ such that

$$\hat{\mathbf{M}} = \mathbf{T}_l\,\mathbf{M}\,\mathbf{T}_r, \quad \hat{\mathbf{D}} = \mathbf{T}_l\,\mathbf{D}\,\mathbf{T}_r, \quad \hat{\mathbf{K}} = \mathbf{T}_l\,\mathbf{K}\,\mathbf{T}_r, \quad \hat{\mathbf{B}}_2 = \mathbf{T}_l\mathbf{B}_2, \quad \hat{\mathbf{C}}_1 = \mathbf{C}_1\mathbf{T}_r, \quad \hat{\mathbf{C}}_2 = \mathbf{C}_2\mathbf{T}_r. \tag{7}$$

The pair $(\mathbf{T}_l, \mathbf{T}_r)$ is called *system equivalence transformation*. A characteristic quantity of (1) is *system invariant* if it is preserved under a system equivalence transformation. For example, the transfer function $\mathbf{G}(s)$ is system invariant, since

$$\begin{aligned}\mathbf{G}(s) &= (s\mathbf{C}_2 + \mathbf{C}_1)(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{B}_2 \\ &= (s\hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_1)\,\mathbf{T}_r^{-1}(s^2\,\mathbf{T}_l^{-1}\hat{\mathbf{M}}\,\mathbf{T}_r^{-1} + s\,\mathbf{T}_l^{-1}\hat{\mathbf{D}}\,\mathbf{T}_r^{-1} + \mathbf{T}_l^{-1}\hat{\mathbf{K}}\,\mathbf{T}_r^{-1})^{-1}\,\mathbf{T}_l^{-1}\hat{\mathbf{B}}_2 = \hat{\mathbf{G}}(s).\end{aligned}$$

Let $[\,\mathbf{M},\, \mathbf{D},\, \mathbf{K},\, \mathbf{B}_2,\, \mathbf{C}_1,\, \mathbf{C}_2\,]$ and $[\,\hat{\mathbf{M}},\, \hat{\mathbf{D}},\, \hat{\mathbf{K}},\, \hat{\mathbf{B}}_2,\, \hat{\mathbf{C}}_1,\, \hat{\mathbf{C}}_2\,]$ be restricted system equivalent. Then the associated first-order systems with the matrix coefficients $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$ as in (4) and

$$\hat{\boldsymbol{\mathcal{E}}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{M}} \end{bmatrix}, \qquad \hat{\mathcal{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\hat{\mathbf{K}} & -\hat{\mathbf{D}} \end{bmatrix}, \qquad \hat{\boldsymbol{\mathcal{B}}} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{B}}_2 \end{bmatrix}, \qquad \hat{\boldsymbol{\mathcal{C}}} = [\,\hat{\mathbf{C}}_1,\;\; \hat{\mathbf{C}}_2\,]$$

are also restricted system equivalent, i.e.,

$$\hat{\boldsymbol{\mathcal{E}}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{E}}\,\boldsymbol{\mathcal{T}}_r, \qquad \hat{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{A}}\,\boldsymbol{\mathcal{T}}_r, \qquad \hat{\boldsymbol{\mathcal{B}}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{B}}, \qquad \hat{\boldsymbol{\mathcal{C}}} = \boldsymbol{\mathcal{C}}\,\boldsymbol{\mathcal{T}}_r,$$

with the transformation matrices

$$\boldsymbol{\mathcal{T}}_l = \begin{bmatrix} \mathbf{T}_r^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_l \end{bmatrix}, \qquad \boldsymbol{\mathcal{T}}_r = \begin{bmatrix} \mathbf{T}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_r \end{bmatrix}.$$

The second-order system (1) is called *asymptotically stable* if the matrix polynomial $\mathbf{P}(\lambda) = \lambda^2\mathbf{M}+\lambda\mathbf{D}+\mathbf{K}$ is *stable*, i.e., all the zeros of $\mathbf{P}(\lambda)$ have negative real part. System (1) is *controllable* if

$$\operatorname{rank}[\,\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K},\; \mathbf{B}_2\,] = n \quad \text{for all } \lambda \in \mathbb{C},$$

and it is *observable* if

$$\operatorname{rank}[\,\lambda^2\mathbf{M}^{\mathrm{T}} + \lambda\mathbf{D}^{\mathrm{T}} + \mathbf{K}^{\mathrm{T}},\; \lambda\mathbf{C}_2^{\mathrm{T}} + \mathbf{C}_1^{\mathrm{T}}\,] = n \quad \text{for all } \lambda \in \mathbb{C}.$$

It has been shown in [21] that the second-order system (1) is controllable (observable) if and only if the first-order system (3) is controllable (observable), i.e., $\operatorname{rank}[\,\lambda\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{A}},\, \boldsymbol{\mathcal{B}}\,] = 2n$ ($\operatorname{rank}[\,\lambda\boldsymbol{\mathcal{E}}^{\mathrm{T}} - \boldsymbol{\mathcal{A}}^{\mathrm{T}},\, \boldsymbol{\mathcal{C}}^{\mathrm{T}}\,] = 2n$) for all $\lambda \in \mathbb{C}$.

2.1. *Position and velocity Gramians*

The Gramians play an important role in balanced truncation model reduction. For second-order systems, different types of Gramians have been proposed in the literature [10, 14, 15, 20]. In this subsection, we consider the position and velocity controllability and observability Gramians as introduced in [14, 20].

Assume that the matrix polynomial $\mathbf{P}(\lambda) = \lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K}$ is stable. Then all the eigenvalues of the pencil $\lambda\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{A}}$ with $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{A}}$ as in (4) have negative real part. In this case the generalized Lyapunov equations

$$\boldsymbol{\mathcal{E}}\,\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}}^{\mathrm{T}} + \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{P}}\,\boldsymbol{\mathcal{E}}^{\mathrm{T}} = -\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\mathrm{T}}, \qquad \boldsymbol{\mathcal{E}}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\,\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\,\mathcal{E} = -\boldsymbol{\mathcal{C}}^{\mathrm{T}}\boldsymbol{\mathcal{C}} \tag{8}$$

have unique symmetric, positive semidefinite solutions $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ which define, respectively, the *controllability Gramian* and the *observability Gramian* of the first-order system (3). These Gramians have the following integral representations

$$\boldsymbol{\mathcal{P}} = \int_0^\infty \boldsymbol{\mathcal{F}}(t)\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\mathrm{T}}\boldsymbol{\mathcal{F}}(t)^{\mathrm{T}}dt, \qquad \boldsymbol{\mathcal{Q}} = \int_0^\infty \boldsymbol{\mathcal{F}}(t)^{\mathrm{T}}\boldsymbol{\mathcal{C}}^{\mathrm{T}}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{F}}(t)\,dt,$$

where $\boldsymbol{\mathcal{F}}(t) = \mathrm{e}^{\boldsymbol{\mathcal{E}}^{-1}\boldsymbol{\mathcal{A}}t}\boldsymbol{\mathcal{E}}^{-1}$ is the fundamental solution matrix of (3). Equivalently, the Gramians can be represented as $\boldsymbol{\mathcal{P}} = \Psi_c\Psi_c^*$ and $\boldsymbol{\mathcal{Q}} = \Psi_o^*\Psi_o$ with the *controllability operator* Ψ_c and the *observability operator* Ψ_o given by

$$\begin{aligned} \Psi_c &: \mathbb{L}_2((-\infty,0],\mathbb{R}^m) \to \mathbb{R}^{2n}, & \Psi_o &: \mathbb{R}^{2n} \to \mathbb{L}_2([0,\infty),\mathbb{R}^p), \\ \boldsymbol{u} &\mapsto \int_{-\infty}^{0} \boldsymbol{\mathcal{F}}(-t)\boldsymbol{\mathcal{B}}\boldsymbol{u}(t)dt, & \boldsymbol{x}_0 &\mapsto \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{F}}(t)\boldsymbol{x}_0. \end{aligned}$$

The operators Ψ_c^* and Ψ_o^* denote the adjoint operators of Ψ_c and Ψ_o, respectively. Let the Gramians be partitioned as

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{\mathcal{P}}_p & \boldsymbol{\mathcal{P}}_{12} \\ \boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}} & \boldsymbol{\mathcal{P}}_v \end{bmatrix}, \qquad \boldsymbol{\mathcal{Q}} = \begin{bmatrix} \boldsymbol{\mathcal{Q}}_p & \boldsymbol{\mathcal{Q}}_{12} \\ \boldsymbol{\mathcal{Q}}_{12}^{\mathrm{T}} & \boldsymbol{\mathcal{Q}}_v \end{bmatrix},$$

where all the blocks are of size $n\times n$. Then $\boldsymbol{\mathcal{P}}_p$ and $\boldsymbol{\mathcal{P}}_v$ are the *position* and *velocity controllability Gramians* of the second-order system (1), whereas $\boldsymbol{\mathcal{Q}}_p$ and $\boldsymbol{\mathcal{Q}}_v$ are the *position* and *velocity observability Gramians* of (1). Defining the *position* and *velocity controllability operators* by

$$\Psi_{c,p} = [\,\mathbf{I},\ \mathbf{0}\,]^{\mathrm{T}}\Psi_c, \qquad \Psi_{c,v} = [\,\mathbf{0},\ \mathbf{I}\,]^{\mathrm{T}}\Psi_c$$

and the *position* and *velocity observability operators* by

$$\Psi_{o,p} = \Psi_o[\,\mathbf{I},\ \mathbf{0}\,]^{\mathrm{T}}, \qquad \Psi_{o,v} = \Psi_o[\,\mathbf{0},\ \mathbf{I}\,]^{\mathrm{T}},$$

the position and velocity Gramians can be represented as

$$\boldsymbol{\mathcal{P}}_p = \Psi_{c,p}\Psi_{c,p}^*, \qquad \boldsymbol{\mathcal{P}}_v = \Psi_{c,v}\Psi_{c,v}^*, \qquad \boldsymbol{\mathcal{Q}}_p = \Psi_{o,p}^*\Psi_{o,p}, \qquad \boldsymbol{\mathcal{Q}}_v = \Psi_{o,v}^*\Psi_{o,v}.$$

The energy interpretation for these Gramians can be found in [10, 15].

2.2. *Singular values*

Under a system equivalence transformation $(\mathbf{T}_l, \mathbf{T}_r)$ the position and velocity Gramians are transformed into

$$\hat{\boldsymbol{\mathcal{P}}}_p = \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_p\,\mathbf{T}_r^{-\mathrm{T}}, \qquad \hat{\boldsymbol{\mathcal{P}}}_v = \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_v\,\mathbf{T}_r^{-\mathrm{T}}, \qquad \hat{\boldsymbol{\mathcal{Q}}}_p = \mathbf{T}_r^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_p\,\mathbf{T}_r, \qquad \hat{\boldsymbol{\mathcal{Q}}}_v = \mathbf{T}_l^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\,\mathbf{T}_l^{-1}.$$

Then it follows from the equations

$$\begin{aligned} \hat{\boldsymbol{\mathcal{P}}}_p\hat{\boldsymbol{\mathcal{Q}}}_p &= \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p\mathbf{T}_r, & \hat{\boldsymbol{\mathcal{P}}}_v\hat{\boldsymbol{\mathcal{Q}}}_p &= \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_v\boldsymbol{\mathcal{Q}}_p\mathbf{T}_r, \\ \hat{\boldsymbol{\mathcal{P}}}_p\hat{\mathbf{M}}^{\mathrm{T}}\hat{\boldsymbol{\mathcal{Q}}}_v\hat{\mathbf{M}} &= \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_p\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}\mathbf{T}_r, & \hat{\boldsymbol{\mathcal{P}}}_v\hat{\mathbf{M}}^{\mathrm{T}}\hat{\boldsymbol{\mathcal{Q}}}_v\hat{\mathbf{M}} &= \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}_v\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}\mathbf{T}_r \end{aligned}$$

that the eigenvalues of the matrices $\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p$, $\boldsymbol{\mathcal{P}}_v\boldsymbol{\mathcal{Q}}_p$, $\boldsymbol{\mathcal{P}}_p\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}$ and $\boldsymbol{\mathcal{P}}_v\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}$ are system invariant. Furthermore, since these matrices are the products of two symmetric, positive semidefinite matrices, they are diagonalizable and have real non-negative eigenvalues [22, p.76]. Using these eigenvalues, we can define different sets of singular values for the second-order system (1).

Definition 2.1: Consider a second-order system (1) with the stable matrix polynomial $\lambda^2\mathbf{M}+\lambda\mathbf{D}+\mathbf{K}$.

1. The square roots of the eigenvalues of the matrix $\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p$, denoted by ξ_j^p, are called the *position singular values* of (1).
2. The square roots of the eigenvalues of the matrix $\boldsymbol{\mathcal{P}}_v\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}$, denoted by ξ_j^v, are called the *velocity singular values* of (1).

3. The square roots of the eigenvalues of the matrix $\boldsymbol{\mathcal{P}}_p\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}$, denoted by ξ_j^{pv}, are called the *position-velocity singular values* of (1).
4. The square roots of the eigenvalues of the matrix $\boldsymbol{\mathcal{P}}_v\boldsymbol{\mathcal{Q}}_p$, denoted by ξ_j^{vp}, are called the *velocity-position singular values* of (1).

We will assume that the position, velocity, position-velocity and velocity-position singular values of (1) are ordered decreasingly. Note that the position singular values coincide with the free velocity singular values defined in [10] and are the singular values of the operator $\mathbf{\Gamma}_p = \Psi_{o,p}\Psi_{c,p}$. This holds due to

$$\lambda_j(\mathbf{\Gamma}_p^*\mathbf{\Gamma}_p) = \lambda_j(\Psi_{c,p}^*\Psi_{o,p}^*\Psi_{o,p}\Psi_{c,p}) = \lambda_j(\Psi_{c,p}\Psi_{c,p}^*\Psi_{o,p}^*\Psi_{o,p}) = \lambda_j(\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p).$$

Analogously, we can show that $\xi_1^v,\ldots,\xi_n^v$ are the singular values of the operator $\mathbf{\Gamma}_v = \Psi_{o,v}\mathbf{M}\Psi_{c,v}$, the numbers $\xi_1^{pv},\ldots,\xi_n^{pv}$ are the singular values of $\mathbf{\Gamma}_{pv} = \Psi_{o,v}\mathbf{M}\Psi_{c,p}$, and $\xi_1^{vp},\ldots,\xi_n^{vp}$ are the singular values of $\mathbf{\Gamma}_{vp} = \Psi_{o,p}\Psi_{c,v}$.

If the second-order system (1) is controllable and observable, then the position and velocity Gramians of (1) are positive definite, since they are principal submatrices of $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$. In this case all the singular values of (1) are strictly positive. However, the positivity of ξ_j^p, ξ_j^v, ξ_j^{pv} and ξ_j^{vp} does not imply that system (1) is controllable and observable.

2.3. *Balancing*

Using different singular values we can define different balanced realizations for second-order systems.

Definition 2.2: Consider a second-order system (1) with the stable matrix polynomial $\lambda^2\mathbf{M}+\lambda\mathbf{D}+\mathbf{K}$.
1. System (1) is called *position balanced* if $\boldsymbol{\mathcal{P}}_p = \boldsymbol{\mathcal{Q}}_p = \mathrm{diag}(\xi_1^p,\ldots,\xi_n^p)$.
2. System (1) is called *velocity balanced* if $\boldsymbol{\mathcal{P}}_v = \boldsymbol{\mathcal{Q}}_v = \mathrm{diag}(\xi_1^v,\ldots,\xi_n^v)$.
3. System (1) is called *position-velocity balanced* if $\boldsymbol{\mathcal{P}}_p = \boldsymbol{\mathcal{Q}}_v = \mathrm{diag}(\xi_1^{pv},\ldots,\xi_n^{pv})$.
4. System (1) is called *velocity-position balanced* if $\boldsymbol{\mathcal{P}}_v = \boldsymbol{\mathcal{Q}}_p = \mathrm{diag}(\xi_1^{vp},\ldots,\xi_n^{vp})$.

We will now show that if system (1) is controllable and observable, then there exist nonsingular matrices $\mathbf{T}_l$ and $\mathbf{T}_r$ that transform (1) into one of the balanced forms. For this purpose consider the Cholesky factorizations of the position and velocity Gramians

$$\boldsymbol{\mathcal{P}}_p = \mathbf{R}_p\mathbf{R}_p^{\mathrm{T}}, \qquad \boldsymbol{\mathcal{P}}_v = \mathbf{R}_v\mathbf{R}_v^{\mathrm{T}}, \qquad \boldsymbol{\mathcal{Q}}_p = \mathbf{L}_p\mathbf{L}_p^{\mathrm{T}}, \qquad \boldsymbol{\mathcal{Q}}_v = \mathbf{L}_v\mathbf{L}_v^{\mathrm{T}}, \tag{9}$$

where $\mathbf{R}_p$, $\mathbf{R}_v$, $\mathbf{L}_p$, $\mathbf{L}_v \in \mathbb{R}^{n,n}$ are nonsingular lower triangular Cholesky factors. Then the position singular values of (1) can be computed as the classical singular values of the matrix $\mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p$. Indeed, we have

$$(\xi_j^p)^2 = \lambda_j(\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p) = \lambda_j(\mathbf{R}_p\mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p\mathbf{L}_p^{\mathrm{T}}) = \lambda_j(\mathbf{L}_p^{\mathrm{T}}\mathbf{R}_p\mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p) = \sigma_j^2(\mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p).$$

Similarly, we can show that the velocity, position-velocity and velocity-position singular values of (1) are the classical singular values of the matrices $\mathbf{R}_v^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v$, $\mathbf{R}_p^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v$ and $\mathbf{R}_v^{\mathrm{T}}\mathbf{L}_p$, respectively. Let

$$\begin{aligned} \mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p &= \mathbf{U}_p\mathbf{\Sigma}_p\mathbf{V}_p^{\mathrm{T}}, & \mathbf{R}_v^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v &= \mathbf{U}_v\mathbf{\Sigma}_v\mathbf{V}_v^{\mathrm{T}}, \\ \mathbf{R}_p^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v &= \mathbf{U}_{pv}\mathbf{\Sigma}_{pv}\mathbf{V}_{pv}^{\mathrm{T}}, & \mathbf{R}_v^{\mathrm{T}}\mathbf{L}_p &= \mathbf{U}_{vp}\mathbf{\Sigma}_{vp}\mathbf{V}_{vp}^{\mathrm{T}} \end{aligned} \tag{10}$$

be the singular value decompositions, where $\mathbf{\Sigma}_\alpha = \mathrm{diag}(\xi_1^\alpha,\ldots,\xi_n^\alpha)$ with $\alpha \in \{p, v, pv, vp\}$. Using (10) we can determine the required balancing transformation matrices $\mathbf{T}_l$ and $\mathbf{T}_r$. These matrices are collected in Table 1.

Note that the velocity and position-velocity balanced systems have the normalized mass matrix $\hat{\mathbf{M}} = \mathbf{T}_l\mathbf{M}\mathbf{T}_r = \mathbf{I}$. Furthermore, for the position and velocity-position balancing, the left transformation matrix $\mathbf{T}_l$ can be chosen arbitrarily. We can use this freedom to impose additional conditions on the transformed system. For instance, computing the position balanced realization one could take

Table 1. The balancing transformations.

$\hat{\boldsymbol{\mathcal{P}}}_p = \hat{\boldsymbol{\mathcal{Q}}}_p = \boldsymbol{\Sigma}_p$	$\mathbf{T}_r = \mathbf{R}_p\mathbf{U}_p\boldsymbol{\Sigma}_p^{-1/2}$,	$\mathbf{T}_l$ arbitrary
$\hat{\boldsymbol{\mathcal{P}}}_v = \hat{\boldsymbol{\mathcal{Q}}}_v = \boldsymbol{\Sigma}_v$	$\mathbf{T}_r = \mathbf{R}_v\mathbf{U}_v\boldsymbol{\Sigma}_v^{-1/2}$,	$\mathbf{T}_l = \boldsymbol{\Sigma}_v^{-1/2}\mathbf{V}_v^{\mathrm{T}}\mathbf{L}_v^{\mathrm{T}}$
$\hat{\boldsymbol{\mathcal{P}}}_p = \hat{\boldsymbol{\mathcal{Q}}}_v = \boldsymbol{\Sigma}_{pv}$	$\mathbf{T}_r = \mathbf{R}_p\mathbf{U}_{pv}\boldsymbol{\Sigma}_{pv}^{-1/2}$,	$\mathbf{T}_l = \boldsymbol{\Sigma}_{pv}^{-1/2}\mathbf{V}_{pv}^{\mathrm{T}}\mathbf{L}_v^{\mathrm{T}}$
$\hat{\boldsymbol{\mathcal{P}}}_v = \hat{\boldsymbol{\mathcal{Q}}}_p = \boldsymbol{\Sigma}_{vp}$	$\mathbf{T}_r = \mathbf{R}_v\mathbf{U}_{vp}\boldsymbol{\Sigma}_{vp}^{-1/2}$,	$\mathbf{T}_l$ arbitrary

$\mathbf{T}_l = \boldsymbol{\Sigma}_p^{-1/2}\mathbf{V}_p^{\mathrm{T}}\mathbf{L}_p^{\mathrm{T}}\mathbf{M}^{-1}$ to normalize the mass matrix $\hat{\mathbf{M}}$ to the identity or $\mathbf{T}_l = \boldsymbol{\Sigma}^{-1/2}\mathbf{V}_v^{\mathrm{T}}\mathbf{L}_v^{\mathrm{T}}$ to balance the velocity observability Gramian $\hat{\boldsymbol{\mathcal{Q}}}_v = \mathbf{T}_l^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\,\mathbf{T}_l^{-1} = \boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma}$ being $\boldsymbol{\Sigma}_p$ or $\boldsymbol{\Sigma}_v$.

Remark 1: As noted in [15], it is impossible to simultaneously balance both pairs of the Gramians $(\boldsymbol{\mathcal{P}}_p, \boldsymbol{\mathcal{Q}}_p)$ and $(\boldsymbol{\mathcal{P}}_v, \boldsymbol{\mathcal{Q}}_v)$ using only a second-order system equivalence transformation. However, it can be done in the state space context working with the first-order system (3). If we allow for the diagonal blocks of the transformation matrices

$$\boldsymbol{\mathcal{T}}_l = \begin{bmatrix} \mathbf{T}_{l1} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{l2} \end{bmatrix}, \qquad \boldsymbol{\mathcal{T}}_r = \begin{bmatrix} \mathbf{T}_{r1} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{r2} \end{bmatrix}$$

to be different, then choosing

$$\mathbf{T}_{l1} = \boldsymbol{\Sigma}_1^{-1/2}\mathbf{V}_p^{\mathrm{T}}\mathbf{L}_p^{\mathrm{T}}, \qquad \mathbf{T}_{l2} = \boldsymbol{\Sigma}_2^{-1/2}\mathbf{V}_v^{\mathrm{T}}\mathbf{L}_v^{\mathrm{T}}, \qquad \mathbf{T}_{r1} = \mathbf{R}_p\mathbf{U}_p\boldsymbol{\Sigma}_1^{-1/2}, \qquad \mathbf{T}_{r2} = \mathbf{R}_v\mathbf{U}_v\boldsymbol{\Sigma}_2^{-1/2}$$

with some diagonal positive definite matrices $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$, we obtain

$$\begin{aligned} \hat{\boldsymbol{\mathcal{P}}}_p &= \mathbf{T}_{r1}^{-1}\,\boldsymbol{\mathcal{P}}_p\,\mathbf{T}_{r1}^{-\mathrm{T}} = \boldsymbol{\Sigma}_1 = \mathbf{T}_{l1}^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}_p\mathbf{T}_{l1}^{-1} = \hat{\boldsymbol{\mathcal{Q}}}_p, \\ \hat{\boldsymbol{\mathcal{P}}}_v &= \mathbf{T}_{r2}^{-1}\,\boldsymbol{\mathcal{P}}_v\,\mathbf{T}_{r2}^{-\mathrm{T}} = \boldsymbol{\Sigma}_2 = \mathbf{T}_{l2}^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{T}_{l2}^{-1} = \hat{\boldsymbol{\mathcal{Q}}}_v. \end{aligned}$$

For $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_p$ and $\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_v$, the balancing transformation is as in [15]. Taking $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$, we balance all four Gramians at the same time.

3. Second-order balanced truncation

Similarly to balanced truncation model reduction of first-order systems [7, 8], the approximate second-order model (2) can be computed by the transformation of system (1) into one of the balanced forms and truncation of the position and velocity components corresponding to the small singular values. Such components are less involved in the energy transfer from inputs to outputs, see [10, 15] for details. In practice, balancing and truncation can be combined by performing the projection onto the subspaces corresponding to the dominant singular values.

In the following we present only two algorithms related to the position and position-velocity balancing which are obvious generalizations of the square root balanced truncation method [23, 24] for the second-order system (1). Other algorithms can be stated in a similar way.

Algorithm 1: Second-order balanced truncation model reduction method with position balancing (SOBTp).

Given $\mathbf{G} = [\,\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{B}_2, \mathbf{C}_1, \mathbf{C}_2\,]$ such that $\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K}$ is stable, compute a reduced system $\tilde{\mathbf{G}} = [\,\tilde{\mathbf{M}}, \tilde{\mathbf{D}}, \tilde{\mathbf{K}}, \tilde{\mathbf{B}}_2, \tilde{\mathbf{C}}_1, \tilde{\mathbf{C}}_2\,]$.

1. Compute the Cholesky factors $\mathbf{R}_p$, $\mathbf{R}_v$, $\mathbf{L}_p$ and $\mathbf{L}_v$ of the position and velocity Gramians by solving the Lyapunov equations (8) with $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ as in (4).

2. Compute the singular value decompositions

$$\mathbf{R}_p^{\mathrm{T}}\mathbf{L}_p = [\,\mathbf{U}_{p1},\, \mathbf{U}_{p2}\,] \begin{bmatrix} \mathbf{\Sigma}_{p1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{p2} \end{bmatrix} [\,\mathbf{V}_{p1},\, \mathbf{V}_{p2}\,]^{\mathrm{T}},$$
$$\mathbf{R}_v^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v = [\,\mathbf{U}_{v1},\, \mathbf{U}_{v2}\,] \begin{bmatrix} \mathbf{\Sigma}_{v1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{v2} \end{bmatrix} [\,\mathbf{V}_{v1},\, \mathbf{V}_{v2}\,]^{\mathrm{T}},$$

where $[\,\mathbf{U}_{p1},\, \mathbf{U}_{p2}\,]$, $[\,\mathbf{V}_{p1},\, \mathbf{V}_{p2}\,]$, $[\,\mathbf{U}_{v1},\, \mathbf{U}_{v2}\,]$ and $[\,\mathbf{V}_{v1},\, \mathbf{V}_{v2}\,]$ are orthogonal and

$$\mathbf{\Sigma}_{p1} = \mathrm{diag}(\xi_1^p, \ldots, \xi_\ell^p), \qquad \mathbf{\Sigma}_{p2} = \mathrm{diag}(\xi_{\ell+1}^p, \ldots, \xi_n^p),$$
$$\mathbf{\Sigma}_{v1} = \mathrm{diag}(\xi_1^v, \ldots, \xi_\ell^v), \qquad \mathbf{\Sigma}_{v2} = \mathrm{diag}(\xi_{\ell+1}^v, \ldots, \xi_n^v).$$

3. Compute the reduced system

$$\tilde{\mathbf{M}} = \mathbf{W}^{\mathrm{T}}\mathbf{M}\mathbf{T}, \quad \tilde{\mathbf{D}} = \mathbf{W}^{\mathrm{T}}\mathbf{D}\mathbf{T}, \quad \tilde{\mathbf{K}} = \mathbf{W}^{\mathrm{T}}\mathbf{K}\mathbf{T}, \quad \tilde{\mathbf{B}}_2 = \mathbf{W}^{\mathrm{T}}\mathbf{B}_2, \quad \tilde{\mathbf{C}}_1 = \mathbf{C}_1\mathbf{T}, \quad \tilde{\mathbf{C}}_2 = \mathbf{C}_2\mathbf{T}$$

with the projection matrices $\mathbf{W} = \mathbf{L}_v\mathbf{V}_{v1}\mathbf{\Sigma}_{p1}^{-1/2}$ and $\mathbf{T} = \mathbf{R}_p\mathbf{U}_{p1}\mathbf{\Sigma}_{p1}^{-1/2}$.

Note that in this algorithm we choose the left projection matrix $\mathbf{W}$ such that the Gramians of the reduced model $\tilde{\mathbf{G}}$ satisfy

$$\tilde{\boldsymbol{\mathcal{P}}}_p = \tilde{\boldsymbol{\mathcal{Q}}}_p = \tilde{\boldsymbol{\mathcal{Q}}}_v = \mathrm{diag}(\xi_1^p, \ldots, \xi_\ell^p). \tag{11}$$

Thereby the velocity controllability Gramian takes the form

$$\tilde{\boldsymbol{\mathcal{P}}}_v = \mathbf{\Sigma}_{p1}^{-1/2}\mathbf{V}_{p1}^{\mathrm{T}}\mathbf{L}_p^{\mathrm{T}}\mathbf{R}_v\mathbf{R}_v^{\mathrm{T}}\mathbf{L}_p\mathbf{V}_{p1}\mathbf{\Sigma}_{p1}^{-1/2} = \mathbf{\Sigma}_{p1}^{-1/2}\mathbf{V}_{p1}^{\mathrm{T}}\mathbf{V}_{vp}\mathbf{\Sigma}_{vp}^{2}\mathbf{V}_{vp}^{\mathrm{T}}\mathbf{V}_{p1}\mathbf{\Sigma}_{p1}^{-1/2}.$$

The balanced truncation method with position-velocity balancing is summarized in the following algorithm.

Algorithm 2: Second-order balanced truncation model reduction method with position-velocity balancing (SOBTpv).

Given $\mathbf{G} = [\,\mathbf{M},\, \mathbf{D},\, \mathbf{K},\, \mathbf{B}_2,\, \mathbf{C}_1,\, \mathbf{C}_2\,]$ such that $\lambda^2\mathbf{M} + \lambda\mathbf{D} + \mathbf{K}$ is stable, compute a reduced system $\tilde{\mathbf{G}} = [\,\tilde{\mathbf{M}},\, \tilde{\mathbf{D}},\, \tilde{\mathbf{K}},\, \tilde{\mathbf{B}}_2,\, \tilde{\mathbf{C}}_1,\, \tilde{\mathbf{C}}_2\,]$.

1. Compute the Cholesky factors $\mathbf{R}_p$ and $\mathbf{L}_v$ of the Gramians $\boldsymbol{\mathcal{P}}_p$ and $\boldsymbol{\mathcal{Q}}_v$ by solving the Lyapunov equations (8) with $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$ as in (4).
2. Compute the singular value decomposition

$$\mathbf{R}_p^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{L}_v = [\,\mathbf{U}_{pv,1},\, \mathbf{U}_{pv,2}\,] \begin{bmatrix} \mathbf{\Sigma}_{pv,1} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{pv,2} \end{bmatrix} [\,\mathbf{V}_{pv,1},\, \mathbf{V}_{pv,2}\,]^{\mathrm{T}}, \tag{12}$$

where $[\,\mathbf{U}_{pv,1},\, \mathbf{U}_{pv,2}\,]$ and $[\,\mathbf{V}_{pv,1},\, \mathbf{V}_{pv,2}\,]$ are orthogonal and

$$\mathbf{\Sigma}_{pv,1} = \mathrm{diag}(\xi_1^{pv}, \ldots, \xi_\ell^{pv}), \qquad \mathbf{\Sigma}_{pv,2} = \mathrm{diag}(\xi_{\ell+1}^{pv}, \ldots, \xi_n^{pv}).$$

3. Compute the reduced system

$$\tilde{\mathbf{M}} = \mathbf{I}_\ell, \quad \tilde{\mathbf{D}} = \mathbf{W}^{\mathrm{T}}\mathbf{D}\mathbf{T}, \quad \tilde{\mathbf{K}} = \mathbf{W}^{\mathrm{T}}\mathbf{K}\mathbf{T}, \quad \tilde{\mathbf{B}}_2 = \mathbf{W}^{\mathrm{T}}\mathbf{B}_2, \quad \tilde{\mathbf{C}}_1 = \mathbf{C}_1\mathbf{T}, \quad \tilde{\mathbf{C}}_2 = \mathbf{C}_2\mathbf{T}$$

with the projection matrices $\mathbf{W} = \mathbf{L}_v\mathbf{V}_{pv,1}\mathbf{\Sigma}_{pv,1}^{-1/2}$ and $\mathbf{T} = \mathbf{R}_p\mathbf{U}_{pv,1}\mathbf{\Sigma}_{pv,1}^{-1/2}$.

Next we compare our second-order balanced truncation methods with the methods presented in [10,15]. First of all note that unlike [10,15] our methods are applied to the second-order system (1) with a general nonsingular mass matrix $\mathbf{M}$ and do not require its inverse. If $\mathbf{M}$ is ill-conditioned, then the inversion of $\mathbf{M}$ may lead to the loss of accuracy. An extension of the second-order balanced truncation methods to (1) with singular $\mathbf{M}$ remains an open problem.

Comparing the SOBTp method with the (free velocity) second-order balanced truncation method (SOBTfv) proposed in [10], we see that the right projection matrices $\mathbf{T}$ are the same in both methods, but the left projection matrices $\mathbf{W}$ are, in general, different. We take $\mathbf{W}$ such that the Gramians of the reduced model satisfy the balancing condition (11), whereas in [10] it is chosen to be $\mathbf{W} = \mathbf{T}$. The latter makes sense for symmetric systems since the symmetry is preserved in the reduced model. But for general systems, it usually results in less accurate approximations.

If we compare the SOBTp method with the second-order balanced truncation method (SOBT) presented in [15], we find that although the same products $\boldsymbol{\mathcal{P}}_p\boldsymbol{\mathcal{Q}}_p$ and $\boldsymbol{\mathcal{P}}_v\mathbf{M}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_v\mathbf{M}$ are used to compute the reduced systems in both methods, the reduction results differ from each other. In our method, the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are multiplied by the same right projection matrix that determines the right subspace corresponding to the dominant position singular values. The reduced system in [15] has the form

$$\tilde{\mathbf{M}} = \mathbf{I}, \quad \tilde{\mathbf{D}} = (\mathbf{S}\,\mathbf{Y}_2^{\mathrm{T}})\,\mathbf{D}\,(\mathbf{X}_2\mathbf{S}^{-1}), \quad \tilde{\mathbf{K}} = (\mathbf{S}\,\mathbf{Y}_2^{\mathrm{T}})\,\mathbf{K}\,\mathbf{X}_1,$$
$$\tilde{\mathbf{B}}_2 = (\mathbf{S}\,\mathbf{Y}_2^{\mathrm{T}})\mathbf{B}_2, \quad \tilde{\mathbf{C}}_1 = \mathbf{C}_1\mathbf{X}_1, \quad \tilde{\mathbf{C}}_2 = \mathbf{C}_2(\mathbf{X}_2\mathbf{S}^{-1}),$$

where $\mathbf{S} = \mathbf{Y}_1^{\mathrm{T}}\mathbf{X}_2$, the columns of $\mathbf{X}_1, \mathbf{Y}_1 \in \mathbb{R}^{n,\ell}$ span, respectively, the right and left subspaces corresponding to the dominant position singular values and the columns of $\mathbf{X}_2, \mathbf{Y}_2 \in \mathbb{R}^{n,\ell}$ form, respectively, the right and left subspaces corresponding to the dominant velocity singular values. The motivation of using the different projection matrices for $\mathbf{D}$ and $\mathbf{K}$ is to balance both pairs $(\boldsymbol{\mathcal{P}}_p, \boldsymbol{\mathcal{Q}}_p)$ and $(\boldsymbol{\mathcal{P}}_v, \boldsymbol{\mathcal{Q}}_v)$ at the same time. However, it is unclear, whether it makes sense from a physical point of view to handle the position and velocity vectors independently. A second drawback of this method is that the inversion of $\mathbf{S} = \mathbf{Y}_1^{\mathrm{T}}\mathbf{X}_2$ is required. For ill-conditioned $\mathbf{S}$, the accuracy may get lost due to numerical round-off errors.

3.1. *Symmetric case*

In this subsection we consider the symmetric second-order system (1) with $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$, $\mathbf{K} = \mathbf{K}^{\mathrm{T}}$, $\mathbf{D} = \mathbf{D}^{\mathrm{T}}$, $\mathbf{B}_2 = \mathbf{C}_1^{\mathrm{T}}$ and $\mathbf{C}_2 = \mathbf{0}$. These assumptions on the system matrices imply that $\mathbf{G}(s)$ is a symmetric matrix for all $s \in \mathbb{C}$ for which $s^2\mathbf{M} + s\mathbf{D} + \mathbf{K}$ is invertible. We show that the SOBTpv method described in Algorithm **2** preserves the symmetry in a reduced model.

The following theorem establishes some special structure of the Gramians of a symmetric system.

Theorem 3.1: *The position controllability Gramian $\boldsymbol{\mathcal{P}}_p$ and the velocity observability Gramian $\boldsymbol{\mathcal{Q}}_v$ of the symmetric system* (1) *satisfy $\boldsymbol{\mathcal{P}}_p = \boldsymbol{\mathcal{Q}}_v$.*

Proof: Consider the first-order system (3) with

$$\boldsymbol{\mathcal{E}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}, \quad \boldsymbol{\mathcal{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_2 \end{bmatrix}, \quad \boldsymbol{\mathcal{C}} = [\,\mathbf{B}_2^{\mathrm{T}},\ \mathbf{0}\,]. \tag{13}$$

Applying the transformations

$$\boldsymbol{\mathcal{T}}_l = \begin{bmatrix} \mathbf{D} & \mathbf{I} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\mathcal{T}}_r = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix},$$

we obtain the transposed system, i.e.

$$\boldsymbol{\mathcal{E}}^{\mathrm{T}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{E}}\,\boldsymbol{\mathcal{T}}_r, \qquad \boldsymbol{\mathcal{A}}^{\mathrm{T}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{A}}\,\boldsymbol{\mathcal{T}}_r, \qquad \boldsymbol{\mathcal{C}}^{\mathrm{T}} = \boldsymbol{\mathcal{T}}_l\,\boldsymbol{\mathcal{B}}, \qquad \boldsymbol{\mathcal{B}}^{\mathrm{T}} = \boldsymbol{\mathcal{C}}\,\boldsymbol{\mathcal{T}}_r.$$

Substituting these matrices in (8), we have

$$\boldsymbol{\mathcal{E}}^{\mathrm{T}}(\boldsymbol{\mathcal{T}}_r^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{T}}_r^{-\mathrm{T}})\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\mathrm{T}}(\boldsymbol{\mathcal{T}}_r^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{T}}_r^{-\mathrm{T}})\boldsymbol{\mathcal{E}} = -\boldsymbol{\mathcal{C}}^{\mathrm{T}}\boldsymbol{\mathcal{C}},$$
$$\boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{T}}_l^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{T}}_l^{-1})\boldsymbol{\mathcal{A}}^{\mathrm{T}} + \boldsymbol{\mathcal{E}}(\boldsymbol{\mathcal{T}}_l^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{T}}_l^{-1})\boldsymbol{\mathcal{A}}^{\mathrm{T}} = -\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^{\mathrm{T}}.$$

Since these Lyapunov equations have unique solutions, we get $\boldsymbol{\mathcal{T}}_r^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{T}}_r^{-\mathrm{T}} = \boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\mathcal{T}}_l^{-\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{T}}_l^{-1} = \boldsymbol{\mathcal{P}}$ with $\boldsymbol{\mathcal{T}}_r = \boldsymbol{\mathcal{T}}_l^{-\mathrm{T}}$. Hence,

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \boldsymbol{\mathcal{P}}_p & \boldsymbol{\mathcal{P}}_{12} \\ \boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}} & \boldsymbol{\mathcal{P}}_v \end{bmatrix} = \boldsymbol{\mathcal{T}}_r\,\boldsymbol{\mathcal{Q}}\,\boldsymbol{\mathcal{T}}_r^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\mathcal{Q}}_v & \hat{\boldsymbol{\mathcal{Q}}}_{12} \\ \hat{\boldsymbol{\mathcal{Q}}}_{12}^{\mathrm{T}} & \hat{\boldsymbol{\mathcal{Q}}}_{22} \end{bmatrix}$$

with $\hat{\boldsymbol{\mathcal{Q}}}_{12} = (\boldsymbol{\mathcal{Q}}_{12}^{\mathrm{T}} - \boldsymbol{\mathcal{Q}}_v\mathbf{D})\mathbf{M}^{-1}$, $\hat{\boldsymbol{\mathcal{Q}}}_{22} = \mathbf{M}^{-1}(\boldsymbol{\mathcal{Q}}_p - \boldsymbol{\mathcal{Q}}_{12}\mathbf{D} - \mathbf{D}\boldsymbol{\mathcal{Q}}_{12}^{\mathrm{T}} + \mathbf{D}\boldsymbol{\mathcal{Q}}_v\mathbf{D})\mathbf{M}^{-1}$. Especially, we have $\boldsymbol{\mathcal{P}}_p = \boldsymbol{\mathcal{Q}}_v$. □

As a consequence of this theorem we obtain the following result.

Corollary 3.2: *Consider the symmetric system* (1) *with positive definite* $\mathbf{M}$. *Then the reduced model* (2) *obtained by the SOBTpv method is symmetric and has the positive definite mass matrix* $\tilde{\mathbf{M}}$. *If, in addition,* $\mathbf{D}$ *and* $\mathbf{K}$ *are positive definite, then* $\tilde{\mathbf{D}}$ *and* $\tilde{\mathbf{K}}$ *are also positive definite.*

Proof: It follows from $\boldsymbol{\mathcal{P}}_p = \boldsymbol{\mathcal{Q}}_v$ that $\mathbf{R}_p = \mathbf{L}_v$, where $\mathbf{R}_p$ are $\mathbf{L}_v$ are the Cholesky factors of $\boldsymbol{\mathcal{P}}_p$ and $\boldsymbol{\mathcal{Q}}_v$ as in (9). Since $\mathbf{M}$ is symmetric and positive definite, we have $\mathbf{U}_{pv,1} = \mathbf{V}_{pv,1}$ in (12), and, hence, $\mathbf{W} = \mathbf{T} = \mathbf{R}_p\mathbf{U}_{pv,1}\boldsymbol{\Sigma}_{pv,1}^{-1/2}$. Then we obtain the reduced model (2) with

$$\tilde{\mathbf{M}} = \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}, \qquad \tilde{\mathbf{D}} = \mathbf{T}^{\mathrm{T}}\mathbf{D}\mathbf{T}, \qquad \tilde{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}, \qquad \tilde{\mathbf{B}}_2 = \mathbf{T}^{\mathrm{T}}\mathbf{B}_2, \qquad \tilde{\mathbf{C}}_1 = \mathbf{C}_1\mathbf{T}.$$

This completes the proof. □

As it was noted above, due to a special choice of the left projection matrix in the SOBTfv method of [10], this method also preserves symmetry in a reduced model. However, for other second-order balanced truncation methods (including the SOBT method of [15]), this property is not necessarily fulfilled even if we start with a symmetric first-order system

$$\begin{aligned} \boldsymbol{\mathcal{E}}_s\,\dot{\boldsymbol{x}}(t) &= \boldsymbol{\mathcal{A}}_s\,\boldsymbol{x}(t) + \boldsymbol{\mathcal{B}}_s\,\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{\mathcal{C}}_s\,\boldsymbol{x}(t), \end{aligned} \tag{14}$$

where $\boldsymbol{x}(t) = [\,\boldsymbol{q}(t)^{\mathrm{T}},\ \dot{\boldsymbol{q}}(t)^{\mathrm{T}}\,]^{\mathrm{T}}$ and

$$\boldsymbol{\mathcal{E}}_s = \begin{bmatrix} \mathbf{D} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\mathcal{A}}_s = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \boldsymbol{\mathcal{B}}_s = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\mathcal{C}}_s = [\,\mathbf{B}_2^{\mathrm{T}},\ \mathbf{0}\,]. \tag{15}$$

The controllability and observability Gramians $\boldsymbol{\mathcal{P}}_s$ and $\boldsymbol{\mathcal{Q}}_s$ of this system are equal and coincide with the controllability Gramian $\boldsymbol{\mathcal{P}}$ of (3), (13). The latter immediately follows from the fact that systems (3), (13) and (14), (15) are system restricted equivalent with the transformation matrices

$$\boldsymbol{\mathcal{T}}_l = \begin{bmatrix} \mathbf{D} & \mathbf{I} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\mathcal{T}}_r = \mathbf{I}_{2n}.$$

This implies that $\boldsymbol{\mathcal{P}}_s = \mathbf{T}_r^{-1}\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{T}}_r^{-\mathrm{T}} = \boldsymbol{\mathcal{P}}$. The position, velocity, position-velocity and velocity-position singular values of the symmetric system (1) are then defined from the corresponding combination of the diagonal blocks of the matrices

$$\boldsymbol{\mathcal{P}}_s = \begin{bmatrix} \boldsymbol{\mathcal{P}}_p & \boldsymbol{\mathcal{P}}_{12} \\ \boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}} & \boldsymbol{\mathcal{P}}_v \end{bmatrix}, \qquad \boldsymbol{\mathcal{E}}_s \boldsymbol{\mathcal{P}}_s \boldsymbol{\mathcal{E}}_s = \begin{bmatrix} \hat{\boldsymbol{\mathcal{P}}}_{11} & \hat{\boldsymbol{\mathcal{P}}}_{12} \\ \boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}} & \mathbf{M}\boldsymbol{\mathcal{P}}_p\mathbf{M} \end{bmatrix}$$

with $\hat{\boldsymbol{\mathcal{P}}}_{11} = \mathbf{D}\boldsymbol{\mathcal{P}}_p\mathbf{D} + \mathbf{M}\boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}}\mathbf{D} + \mathbf{D}\boldsymbol{\mathcal{P}}_{12}\mathbf{M} + \mathbf{M}\boldsymbol{\mathcal{P}}_v\mathbf{M}$ and $\hat{\boldsymbol{\mathcal{P}}}_{12} = \mathbf{D}\boldsymbol{\mathcal{P}}_p\mathbf{M} + \mathbf{M}\boldsymbol{\mathcal{P}}_{12}^{\mathrm{T}}\mathbf{M}$. The different balanced realizations and balanced truncation methods can be obtained in a similar way as above. Again, one can show that only position-velocity balancing will guarantee the preservation of symmetry in the reduced model.

3.2. *Stability issues*

It is well known that for first-order systems, the classical balanced truncation model reduction method guarantees stability in reduced models [7, 19]. This rises the question whether the second-order balanced truncation methods preserve stability as well.

Note that the symmetric second-order systems with positive definite matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are obviously asymptotically stable. In this case it follows from the previous subsection that the SOBTpv and the SOBTfv methods are stability-preserving. However, for general systems, neither the second-order balanced truncation methods presented in this paper nor the methods in [10, 15] guarantee the preservation of stability in reduced second-order systems. This can be demonstrated by the following simple counterexamples.

Example 3.3 Consider the second-order systems with

$$\begin{aligned}
&\text{(a)}\quad \mathbf{M} = \mathbf{I}_2,\ \mathbf{D} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix},\ \mathbf{B}_2 = \mathbf{C}_1^{\mathrm{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{C}_2 = 0;\\
&\text{(b)}\quad \mathbf{M} = \mathbf{I}_2,\ \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 3 & 4 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{C}_1 = [2,\ 1],\ \mathbf{C}_2 = 0;\\
&\text{(c)}\quad \mathbf{M} = \mathbf{I}_2,\ \mathbf{D} = \begin{bmatrix} 4 & 4 \\ 1 & 3 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix},\ \mathbf{C}_1 = [2,\ 1],\ \mathbf{C}_2 = 0.\\
&\text{(d)}\quad \mathbf{M} = \mathbf{I}_2,\ \mathbf{D} = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{C}_1 = [1,\ 1],\ \mathbf{C}_2 = 0.
\end{aligned}$$

These systems are asymptotically stable, controllable and observable. They have been approximated by the reduced second-order models of dimension $\ell = 1$ computed by the SOBT, the SOBTfv, the SOBTp and the SOBTpv methods as well as by the balanced truncation methods with velocity balancing (SOBTv) and velocity-position balancing (SOBTvp). Note that in the SOBTvp method we choose the left projection matrix $\mathbf{W}$ such that the Gramians of the reduced model satisfy $\tilde{\boldsymbol{\mathcal{P}}}_v = \tilde{\boldsymbol{\mathcal{Q}}}_p = \tilde{\boldsymbol{\mathcal{Q}}}_v = \mathrm{diag}(\xi_1^{vp}, \ldots, \xi_\ell^{vp})$. In Table 2 we present the singular values of all four systems. Table 3 shows whether the stability is preserved in the reduced models. The sign '+' indicates that the reduced system is asymptotically stable, whereas '−' means that the reduced system is unstable.

Table 2. Singular values

	ξ_j^p	ξ_j^v	ξ_j^{pv}	ξ_j^{vp}
(a)	0.969 0.228	0.252 0.127	0.319 0.075	1.004 0.296
(b)	5.477 4.024	1.618 0.370	5.816 0.233	6.734 1.448
(c)	0.702 0.194	0.274 0.134	0.206 0.053	1.766 0.260
(d)	2.201 0.099	2.200 0.032	1.242 0.014	3.901 0.226

Table 3. Stability properties of the reduced models

	SOBT	SOBTfv	SOBTp	SOBTv	SOBTpv	SOBTvp
(a)	−	+	−	−	+	−
(b)	+	−	+	+	+	−
(c)	+	+	−	+	−	+
(d)	−	−	−	−	−	−

Example 3.3(a) demonstrates that the SOBT, the SOBTp, the SOBTv and the SOBTvp methods do not preserve stability for symmetric systems.

4. Numerical examples

In this section we present numerical examples to compare different balanced truncation model reduction methods for second-order systems. We consider three models: the building model (B), the International Space Station model (ISS) and the clamped beam model (CB), see [25] for detailed description. Every model has been approximated by a reduced first-order system (5) of dimension 2ℓ computed using the balanced truncation (BT) method applied to (3), (4) and also by the reduced second-order systems of the form (2) of dimension ℓ computed using the SOBT, the SOBTfv, the SOBTp, the SOBTv and the SOBTpv methods. For comparison, we present the absolute errors $\|\tilde{\boldsymbol{\mathcal{G}}}(\mathrm{i}\omega)-\mathbf{G}(\mathrm{i}\omega)\|$ and $\|\tilde{\mathbf{G}}(\mathrm{i}\omega)-\mathbf{G}(\mathrm{i}\omega)\|$ for the frequency range $\omega\in[\,\omega_{\min},\ \omega_{\max}\,]$. Here,

$$\begin{aligned}\mathbf{G}(s) &= (s\mathbf{C}_2+\mathbf{C}_1)(s^2\mathbf{M}+s\mathbf{D}+\mathbf{K})^{-1}\mathbf{B}_2 = \boldsymbol{\mathcal{C}}(s\,\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{A}})^{-1}\boldsymbol{\mathcal{B}},\\ \tilde{\mathbf{G}}(s) &= (s\tilde{\mathbf{C}}_2+\tilde{\mathbf{C}}_1)(s^2\tilde{\mathbf{M}}+s\tilde{\mathbf{D}}+\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{B}}_2,\\ \tilde{\boldsymbol{\mathcal{G}}}(s) &= \tilde{\boldsymbol{\mathcal{C}}}(s\,\tilde{\boldsymbol{\mathcal{E}}}-\tilde{\boldsymbol{\mathcal{A}}})^{-1}\tilde{\boldsymbol{\mathcal{B}}},\end{aligned}$$

and $\|\cdot\|$ denotes the spectral matrix norm. Table 4 shows the relative errors

$$\|\tilde{\boldsymbol{\mathcal{G}}}-\mathbf{G}\|_{\mathbb{H}_\infty}/\|\mathbf{G}\|_{\mathbb{H}_\infty},\qquad \|\tilde{\mathbf{G}}-\mathbf{G}\|_{\mathbb{H}_\infty}/\|\mathbf{G}\|_{\mathbb{H}_\infty},$$

where the $\mathbb{H}_\infty$-norm is defined by $\|\mathbf{G}\|_{\mathbb{H}_\infty}=\sup_{\omega\in\mathbb{R}}\|\mathbf{G}(\mathrm{i}\omega)\|$.

Table 4. The relative errors for different balanced truncation methods.

	n	ℓ	BT	SOBT	SOBTfv	SOBTp	SOBTv	SOBTpv
B	24	4	$1.43e-01$	$3.54e-01$	$3.40e-01$	$3.48e-01$	$3.56e-01$	$2.96e-01$
ISS	135	13	$5.59e-03$	$5.61e-03$	$5.61e-03$	$5.61e-03$	$5.61e-03$	$1.07e-02$
CB	174	17	$1.75e-05$	$1.31e-04$	$6.65e-01$	$1.63e-04$	$1.53e-04$	$4.69e-04$

Example 4.1 Building model: $n=24$, $m=1$, $p=1$, $\ell=4$
Figure 1 shows that for low frequencies all three reduced second-order systems have the better approximation properties than the reduced first-order system, whereas for higher frequencies, all four approximation errors are about the same. If we compare the reduced second-order systems, we see that the SOBT and the SOBTp methods provide almost an equal result that is only slightly better than the approximation computed by the SOBTpv method.

Example 4.2 ISS model: $n=135$, $m=3$, $p=3$, $\ell=13$
Figure 2 demonstrates that the reduced first-order system and the reduced second-order systems computed by the SOBT and the SOBTp methods have almost the same errors that are smaller for high frequencies than the error for the system computed by the SOBTpv method. The latter system provides, however, a better approximation for low frequencies.

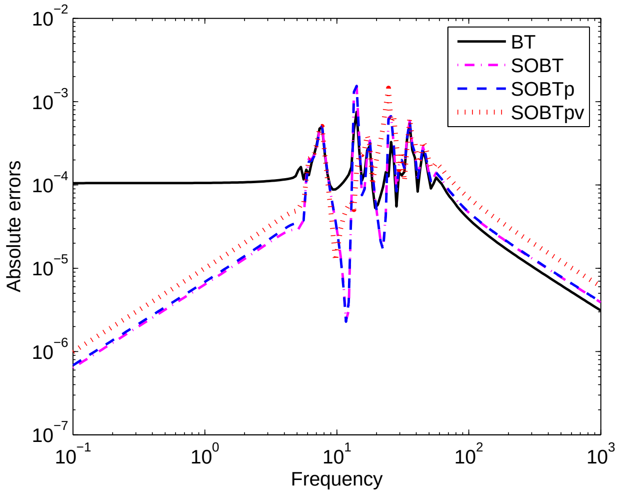

Figure 1. Building model: the absolute errors.

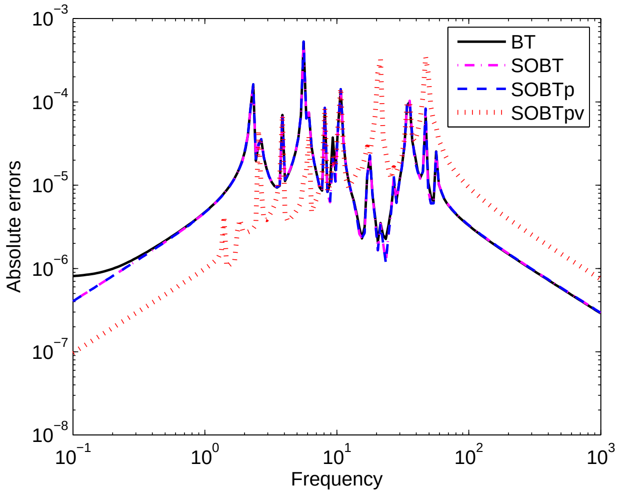

Figure 2. ISS model: the absolute errors.

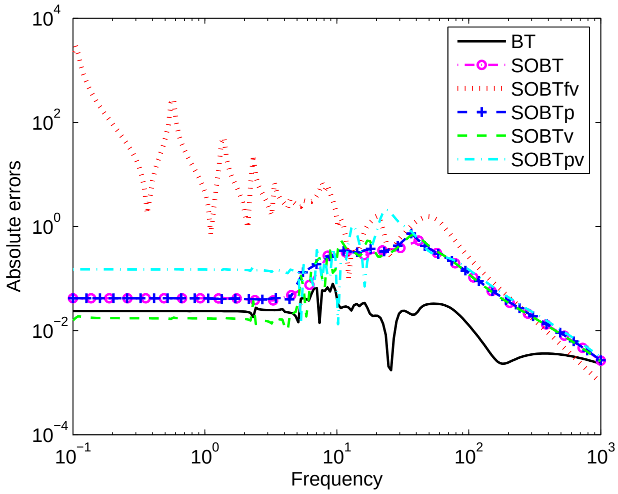

Figure 3. Clamped beam model: the absolute errors.

Example 4.3 Clamped beam model: $n = 174$, $m = 1$, $p = 1$, $\ell = 17$
Figure 3 shows that for low frequencies, the reduced second-order system computed by the SOBTfv method has much larger error compared with the systems obtained by other second-order balanced truncation methods. We also see that the SOBT and the SOBTp methods behave similarly.

5. Conclusions

In this paper we have considered structure-preserving model reduction of second-order systems based on balanced truncation. Using the pairs $(\boldsymbol{\mathcal{P}}_p, \boldsymbol{\mathcal{Q}}_p)$ and $(\boldsymbol{\mathcal{P}}_v, \boldsymbol{\mathcal{Q}}_v)$ of the position and velocity Gramians from [10, 15, 20], we have introduced the position, velocity, position-velocity and velocity-position singular values that can be used to characterize the importance of the position and velocity components. We have presented four new structure-preserving balanced truncation model reduction methods for second-order systems and compared these methods with the existing second-order balanced truncation techniques from [10, 15]. It has also been shown that the method based on position-velocity preserves stability for symmetric second-order systems with positive definite mass, damping and stiffness matrices. However, in general, none of the balanced truncation methods for second-order systems guarantees stability of the reduced models. Nevertheless, the numerical examples demonstrate that the structure-preserving second-order balanced truncation methods provide reduced models whose approximation error is comparable with that of the classical balanced truncation method.

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