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STABILITY ANALYSIS AND MODEL ORDER REDUCTION FOR COUPLED SYSTEMS

TIMO REIS* AND TATJANA STYKEL†

Abstract. In this paper we discuss the stability and model order reduction of coupled linear time-invariant systems. Sufficient conditions for a closed-loop system to be asymptotically stable are given. We present a model reduction approach for coupled systems based on reducing the order of the subsystems and coupling the reduced-order subsystems by the same interconnection matrices as for the original model. Such an approach allows to obtain error bounds for the reduced-order closed-loop system in terms of the errors in the reduced-order subsystems. Model reduction of coupled systems with unstable subsystems is also considered. Numerical examples are given.

Key words. stability, model reduction, coupled systems, balanced truncation, coprime factorization

AMS subject classifications. 34A09, 93C, 93D20

1. Introduction.

Recent technological and industrial developments have caused a considerable interest in the study of dynamical processes modelled by coupled systems of ordinary differential equations, differential-algebraic equations and partial differential equations. Application areas of coupled systems include very large system interconnected (VLSI) chip design, micro-electro-mechanical systems (MEMS) and structural dynamics [9, 24, 25, 29]. Modeling of complex physical and technical processes leads to very large-scale coupled systems. Simulation, control and optimization of such systems is difficult even on modern computers because of computational complexity and memory requirements. This motivates model order reduction that consists in approximation of a large-scale system by a reduced-order model that preserves important properties of the original one. There exist various model reduction methods for dynamical systems, see [1, 3] and references therein. However, model order reduction for coupled systems as well as stability and accuracy issues related to the coupling of the reduced-order subsystems received only little attention [26, 31].

In this paper we consider a system of k coupled linear time-invariant generalized state space subsystems of the form

$$\begin{array}{rcl} E_j\dot{x}_j(t) &=& A_jx_j(t)+B_ju_j(t),\\ y_j(t) &=& C_jx_j(t), \end{array} \qquad (1.1)$$

where $E_j,\ A_j \in \mathbb{R}^{n_j,n_j}$, $B_j \in \mathbb{R}^{n_j,m_j}$, $C_j \in \mathbb{R}^{p_j,n_j}$, $x_j(t) \in \mathbb{R}^{n_j}$ is a state vector, $u_j(t) \in \mathbb{R}^{m_j}$ is an internal input and $y_j(t) \in \mathbb{R}^{p_j}$ is an internal output of the jth subsystem. The input $u_j(t)$ is a linear combination of the internal outputs of the subsystems and the external input $u(t) \in \mathbb{R}^m$, i.e.

$$u_j(t) = K_{j1}y_1(t) + \ldots + K_{jk}y_k(t) + H_ju(t), \quad j = 1,\ldots,k \qquad (1.2)$$

with some matrices $K_{j,l} \in \mathbb{R}^{m_j,p_l}$ and $H_j \in \mathbb{R}^{m_j,m}$. The external output $y(t) \in \mathbb{R}^p$ is

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a linear combination of the internal outputs of the subsystems and has the form

$$y(t) = R_1 y_1(t) + \ldots + R_k y_k(t) \tag{1.3}$$

with $R_j \in \mathbb{R}^{p,p_j}$. Coupled systems of the form (1.1)–(1.3) are known also as *interconnected* or *composite systems*. We will assume that the pencil $\lambda E_j - A_j$, $j = 1, \ldots k$, is *regular*, i.e., $\det(\lambda E_j - A_j) \not\equiv 0$. A *transfer function* of the subsystem (1.1) is given by $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$. It describes the input-output relation of (1.1) in the frequency domain. We will also denote the generalized state space representation (1.1) of $\boldsymbol{G}_j$ by $\boldsymbol{G}_j = [\, E_j,\, A_j,\, B_j,\, C_j \,]$. The transfer function $\boldsymbol{G}_j(s)$ is called *proper* if $\lim\limits_{s\to\infty} \boldsymbol{G}_j(s) < \infty$, and *improper*, otherwise.

Let $p_0 = p_1 + \ldots + p_k$ and $m_0 = m_1 + \ldots + m_k$. Introduce the interconnection matrices

$$R = [\, R_1,\, \ldots,\, R_k \,] \in \mathbb{R}^{p,p_0}, \qquad\qquad H = [\, H_1^T,\, \ldots,\, H_k^T \,]^T \in \mathbb{R}^{m_0,m},$$

and $K = [K_{j,l}]_{j,l=1}^k \in \mathbb{R}^{m_0,p_0}$ as well as the block diagonal matrices

$$\begin{aligned} E &= \mathrm{diag}(E_1, \ldots, E_k), \qquad & A &= \mathrm{diag}(A_1, \ldots, A_k),\\ B &= \mathrm{diag}(B_1, \ldots, B_k), \qquad & C &= \mathrm{diag}(C_1, \ldots, C_k). \end{aligned} \tag{1.4}$$

Let $\boldsymbol{G}(s) = C(sE - A)^{-1}B = \mathrm{diag}\big(\boldsymbol{G}_1(s), \ldots, \boldsymbol{G}_k(s)\big)$. If $I - \boldsymbol{G}(s)K$ is invertible, then the input-output relation of the coupled system (1.1)–(1.3) can be written as $\boldsymbol{y}(s) = \boldsymbol{G}_{cl}(s)\boldsymbol{u}(s)$, where $\boldsymbol{y}(s)$ and $\boldsymbol{u}(s)$ are the Laplace transforms of the external output $y(t)$ and the external input $u(t)$, respectively, and the closed-loop transfer function $\boldsymbol{G}_{cl}(s)$ has the form

$$\boldsymbol{G}_{cl}(s) = R\big(I - \boldsymbol{G}(s)K\big)^{-1}\boldsymbol{G}(s)H = R\,\boldsymbol{G}(s)\big(I - K\boldsymbol{G}(s)\big)^{-1}H. \tag{1.5}$$

A generalized state space realization of $\boldsymbol{G}_{cl}(s)$ is given by

$$\begin{aligned} \mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t),\\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{1.6}$$

where

$$\mathcal{E} = E, \qquad \mathcal{A} = A + BKC, \qquad \mathcal{B} = BH, \qquad \mathcal{C} = RC. \tag{1.7}$$

Note that $I - \boldsymbol{G}(s)K$ is invertible if and only if the pencil $\lambda E - A - BKC$ is regular. Moreover, if $\boldsymbol{G}(s)$ and $(I - \boldsymbol{G}(s)K)^{-1}$ are proper, then the coupled descriptor system (1.1)–(1.3) is well-posed in the sense that the closed-loop transfer function $\boldsymbol{G}_{cl}(s)$ exists and is proper.

The first question that will be addressed in this paper is the stability theory for coupled systems. Based on the work [17], we will propose sufficient conditions for the closed-loop system (1.6), (1.7) to be asymptotically stable.

The second problem that we will consider is model order reduction of coupled systems. Instead of reducing the order of the entire system (1.6), it seems to be more efficient to reduce the order of the subsystems (1.1) using the interconnection structure of the coupled system. The first approach is a projection method proposed in [31]. It consists in projecting the subsystems (1.1) onto the appropriate subspaces computed either by a structure preserving balanced truncation method or by a Krylov subspace method applied to (1.6). The disadvantages of this approach are that we need to work with very large matrices as in (1.7) and no stability and accuracy results are known.

In this paper we consider another approach that consists in reducing the order of the subsystems (1.1) by some (possibly different) model reduction methods and coupling the reduced-order subsystems through the same interconnection matrices R, H and K. This approach is attractive for parallelization, since all k subsystems may be reduced simultaneously using k processors. We present a priori error bounds for the closed-loop reduced system in terms of the absolute errors in the reduced subsystems. If these subsystems are obtained using a balanced truncation method [12, 21, 27], then a priori error bounds can be computed using Hankel singular values of the subsystems. Such bounds allow us to estimate how well the subsystems should be approximated to attain a prescribed accuracy in the reduced-order closed-loop system.

Throughout the paper we will denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{F}^{n,m}$ the space of $n \times m$ real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) matrices. The imaginary axis is denoted by $i\mathbb{R}$. The matrix A^T stands for the transpose of A. An identity matrix of order m is denoted by I_m or simply by I. The matrix $A = [a_{jl}]_{j,l=1}^{n} \in \mathbb{R}^{n,n}$ is called *nonnegative* if $a_{jl} \geq 0$ for $j, l = 1, \ldots, n$. For two nonnegative matrices A and B, we will write $A \preceq B$ if the matrix $B - A$ is nonnegative. The spectral radius of $A \in \mathbb{F}^{n,n}$ is denoted by $\rho(A)$. The induced matrix norm of $A \in \mathbb{F}^{n,m}$ is defined by $\|A\|_{\alpha,\beta} = \max_{x\neq 0} \|Ax\|_\beta / \|x\|_\alpha$, where $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are the vector norms on $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. We will denote by $\|A\|_2 = \|A\|_{2,2}$ the spectral norm of $A \in \mathbb{F}^{n,m}$. Let $\mathbb{H}_\infty$ be a space of all proper rational transfer functions that are analytic and bounded in the open right half-plane. The $\mathbb{H}_\infty$-*norm* of $\boldsymbol{G} \in \mathbb{H}_\infty$ is defined by

$$\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = \sup_{\Re e(s)>0} \|\boldsymbol{G}(s)\|_2 = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega)\|_2.$$

The paper is organized as follows. In Section 2 we analyze the stability of coupled systems and give sufficient criteria for that. Section 3 deals with model reduction of the coupled system (1.1)–(1.3), where all k subsystems are asymptotically stable. Bounds for the $\mathbb{H}_\infty$-norm of the difference between the coupled system and its reduced-order model are presented. In Section 4 we briefly review model reduction for descriptor systems by balanced truncation and apply this technique to coupled systems. In Section 5 we extend the results of Section 3 to the case where several subsystems are unstable. Finally, in Section 6 some numerical examples are presented.

2. Stability of coupled systems.

In this section we generalize some results from [17] on stability of coupled standard state space systems to the systems in descriptor form. We obtain sufficient conditions for the closed-loop system (1.6) to be asymptotically stable that can be verified without computing the spectrum of the pencil $\lambda\mathcal{E} - \mathcal{A}$.

Firstly, we briefly review some known results from [7, 16, 30] on stability and structured stability radius for descriptor systems that we will need in the following. Consider a descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

where $E,\ A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,m}$, $C \in \mathbb{F}^{p,n}$ and the pencil $\lambda E - A$ is assumed to be regular. System (2.1) is *asymptotically stable* if the pencil $\lambda E - A$ is *stable*, i.e., all its finite eigenvalues have negative real part. A transfer function of (2.1) given by

$\boldsymbol{G}(s) = C(sE-A)^{-1}B$ is called *stable* if it has no poles in the closed right half-plane $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re e(s) \geq 0\}$. Clearly, the asymptotically stable system (2.1) has the stable transfer function $\boldsymbol{G}(s)$. But the stability of $\boldsymbol{G}(s)$ does not imply, in general, that $\lambda E - A$ is stable. However, for any stable transfer function $\boldsymbol{G}(s)$ one can find a generalized state space representation (2.1) such that $\boldsymbol{G}(s) = C(sE-A)^{-1}B$ and $\lambda E - A$ is stable, see [16].

Let $\Xi \subseteq \mathbb{F}^{m,p}$ be a vector space and let $\|\cdot\|$ be a norm on Ξ. A *structured stability radius* of the matrix quadruple $[\,E,\,A,\,B,\,C\,]$ with respect to the pair $(\Xi, \|\cdot\|)$ is defined via

$$\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big) = \inf\{\ \|\Delta\| \ \ : \ \ \Delta \in \Xi,\ \ \mathrm{Sp}_f(E, A + B\Delta C) \cap i\mathbb{R} \neq \emptyset\ \},$$

where $\mathrm{Sp}_f(E, A + B\Delta C)$ denotes the finite spectrum of the pencil $\lambda E - A - B\Delta C$. The structured stability radius $\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big)$ measures the smallest perturbation $\Delta \in \Xi$ with respect to the norm $\|\cdot\|$ such that the pencil $\lambda E - A - B\Delta C$ is unstable for given matrices E, A, B, C. In other words, $\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big)$ estimates the distance of a stable pencil to the set of all unstable pencils under structured perturbations. The structured stability radius for standard state space systems ($E = I$) has been first considered in [13, 14] and extended to the generalized problem in [30].

Furthermore, consider a *μ-function* of a matrix $M \in \mathbb{F}^{p,m}$ with respect to the pair $(\Xi, \|\cdot\|)$ given by

$$\mu_\Xi\big(M, \|\cdot\|\big) = \big(\inf\big\{\ \|\Delta\| \ \ : \ \ \Delta \in \Xi,\ \ \det(I - M\Delta) = 0\ \big\}\big)^{-1}.$$

The function $1/\mu_\Xi\big(M, \|\cdot\|\big)$ measures the smallest perturbation $\Delta \in \Xi$ with respect to the norm $\|\cdot\|$ such that the matrix $I - M\Delta$ is singular.

The structured stability radius is closely related to the μ-function of the transfer function $\boldsymbol{G}(s) = C(sE-A)^{-1}B$ of the descriptor system (2.1). It follows from

$$\begin{aligned}
\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big) &= \inf_{\omega\in\mathbb{R}} \big(\inf\big\{\ \|\Delta\| \ \ : \ \ \Delta \in \Xi,\ \det(i\omega E - A - B\Delta C) = 0\ \big\}\big) \\
&= \inf_{\omega\in\mathbb{R}} \big(\inf\big\{\ \|\Delta\| \ \ : \ \ \Delta \in \Xi,\ \det(I - \boldsymbol{G}(i\omega)\Delta) = 0\ \big\}\big)
\end{aligned}$$

that

$$\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big) = \frac{1}{\sup\limits_{\omega\in\mathbb{R}} \mu_\Xi\big(\boldsymbol{G}(i\omega), \|\cdot\|\big)}. \tag{2.2}$$

The following theorem gives equivalent conditions for the pencil $\lambda E - A - B\Delta C$ to be stable for all $\Delta \in \Xi$ with $\|\Delta\| \leq 1$.

THEOREM 2.1. *Let $\boldsymbol{G}(s) = C(sE-A)^{-1}B$ be a transfer function of system* (2.1)*. Consider a vector space $\Xi \subseteq \mathbb{F}^{m,p}$ with a norm $\|\cdot\|$. The following statements are equivalent:*

1. *the pencil $\lambda E - A - B\Delta C$ is stable for all $\Delta \in \Xi$ with $\|\Delta\| \leq 1$;*
2. $\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big) > 1$*;*
3. $\sup\limits_{\omega\in\mathbb{R}} \mu_\Xi\big(\boldsymbol{G}(i\omega), \|\cdot\|\big) < 1$.

Proof. The equivalence of the first and the second statements follows from the definition of the structured stability radius $\mathbf{r}_\Xi\big(E, A, B, C, \|\cdot\|\big)$. The equivalence of the second and the third statements immediately follows from equation (2.2). ∎

Consider now the coupled system (1.1)–(1.3). Let $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m_0}$ and $C \in \mathbb{R}^{p_0,n}$ be the block diagonal matrices as in (1.4) and let $\Phi = [\phi_{jl}]_{j,l=1}^k \in \mathbb{R}^{k,k}$ be a nonnegative matrix with a nonempty index set

$$\mathcal{I}_\Phi = \left\{ \, (j,l) \in \mathbb{N} \times \mathbb{N} \;\; : \;\; \phi_{jl} > 0 \, \right\}.$$

Furthermore, let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$ be k-tuples such that $\| \cdot \|_{\alpha_l}$ and $\| \cdot \|_{\beta_j}$ are vector norms on $\mathbb{C}^{p_l}$ and $\mathbb{C}^{m_j}$, respectively. On a vector space of block matrices given by

$$\Xi_\Phi = \left\{ \, \Delta = [\Delta_{jl}]_{j,l=1}^k \quad : \quad \Delta_{jl} \in \mathbb{C}^{m_j,p_l}, \quad \Delta_{jl} = 0 \;\text{ for }\; (j,l) \notin \mathcal{I}_\Phi \, \right\}, \tag{2.3}$$

we define a weighted maximum norm as

$$\|\Delta\|_\Phi^{\alpha,\beta} = \max_{(j,l)\in\mathcal{I}_\Phi} \frac{\|\Delta_{j,l}\|_{\alpha_l,\beta_j}}{\phi_{jl}}. \tag{2.4}$$

As a consequence of Theorem 2.1 we obtain the following sufficient condition for the closed-loop system (1.6), (1.7) to be asymptotically stable.

Corollary 2.2. *Consider a coupled descriptor system* (1.1)–(1.3)*. Let E, A, B and C be the block diagonal matrices as in* (1.4) *and let*

$$\boldsymbol{G}(s) = C(sE - A)^{-1}B = \operatorname{diag}(\boldsymbol{G}_1(s), \ldots, \boldsymbol{G}_k(s))$$

with $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$. Consider a nonnegative matrix $\Phi = [\phi_{jl}]_{j,l=1}^k$ with $\phi_{jl} = \|K_{jl}\|_{\alpha_l,\beta_j}$. If

$$\sup_{\omega\in\mathbb{R}} \rho\big(\Phi \;\operatorname{diag}(\|\boldsymbol{G}_1(i\omega)\|_{\beta_1,\alpha_1}, \ldots, \|\boldsymbol{G}_k(i\omega)\|_{\beta_k,\alpha_k})\big) < 1, \tag{2.5}$$

then $\lambda E_j - A_j$ is stable for $j = 1, \ldots, k$, the transfer function $I - \boldsymbol{G}(s)K$ is invertible and the closed-loop descriptor system (1.6), (1.7) *is asymptotically stable.*

Proof. It has been shown in [17, Theorem 4.2.1] that

$$\mu_{\Xi_\Phi}(\boldsymbol{G}(i\omega), \|\cdot\|_\Phi^{\alpha,\beta}) = \rho\big(\Phi \;\operatorname{diag}(\|\boldsymbol{G}_1(i\omega)\|_{\beta_1,\alpha_1}, \ldots, \|\boldsymbol{G}_k(i\omega)\|_{\beta_k,\alpha_k})\big), \tag{2.6}$$

where the pair $(\Xi_\Phi, \|\cdot\|_\Phi^{\alpha,\beta})$ is as in (2.3), (2.4). Since

$$\sup_{\omega\in\mathbb{R}} \mu_{\Xi_\Phi}(\boldsymbol{G}(i\omega), \|\cdot\|_\Phi^{\alpha,\beta}) < 1,$$

by Theorem 2.1 the pencil $\lambda E - A - B\Delta C$ is stable for all $\Delta \in \Xi_\Phi$ with $\|\Delta\|_\Phi^{\alpha,\beta} \le 1$. In this case for $\Delta = 0$, the pencil $\lambda E - A$ is stable, i.e., $\lambda E_j - A_j$ is stable for $j = 1, \ldots, k$. Moreover, we have $K \in \Xi_\Phi$ and $\|K\|_\Phi^{\alpha,\beta} = 1$. Hence, the pencil $\lambda E - A - BKC$ is also stable. Thus, $I - \boldsymbol{G}(s)K$ is invertible and the closed-loop system (1.6), (1.7) is asymptotically stable. $\square$

Note that the asymptotic stability of the closed-loop system (1.6), (1.7) does not imply, in general, that all the interconnected subsystems are asymptotically stable.

Example 2.3. Let $\mathbf{G}_1(s) = \dfrac{-3}{2s-1}$ *and* $\mathbf{G}_2(s) = \dfrac{1}{2s+2}$ *be coupled as shown in Figure* 2.1*. According to our notation, we have*

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

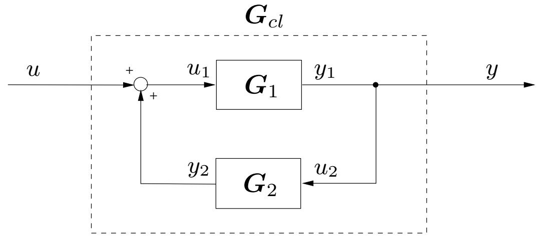


FIG. 2.1. *A coupled system.*

The transfer function of the closed-loop system given by

$$\mathbf{G}_{cl}(s) = \frac{\mathbf{G}_1(s)}{1-\mathbf{G}_1(s)\mathbf{G}_2(s)} = \frac{-6s-6}{4s^2+2s+1}$$

is proper and has poles at $(-1\pm i\sqrt{3})/4$. *Thus,* $\mathbf{G}_{cl}$ *can be realized by an asymptotically stable system, although* $\mathbf{G}_1$ *is unstable.*

Taking in (2.6) the spectral matrix norm, we obtain the following result.

COROLLARY 2.4. *Consider a coupled descriptor system* (1.1)–(1.3). *Let*

$$\Psi = \Phi_2 \operatorname{diag}(\|\boldsymbol{G}_1\|_{\mathbb{H}_\infty}, \ldots, \|\boldsymbol{G}_k\|_{\mathbb{H}_\infty}) \tag{2.7}$$

be a nonnegative matrix with $\Phi_2 = [\,\|K_{jl}\|_2\,]_{j,l=1}^{k}$. *If* $\rho(\Psi) < 1$, *then* $I - \boldsymbol{G}(s)K$ *is invertible and the closed-loop descriptor system* (1.6), (1.7) *is asymptotically stable.*

Proof. Taking into account that $\|\boldsymbol{G}_j(i\omega)\|_2 \le \|\boldsymbol{G}_j\|_{\mathbb{H}_\infty}$ for $j = 1,\ldots,k$ and for all $\omega \in \mathbb{R}$, we obtain the estimate $\Phi_2 \operatorname{diag}(\|\boldsymbol{G}_1(i\omega)\|_2, \ldots, \|\boldsymbol{G}_k(i\omega)\|_2) \preceq \Psi$. Then, using the monotonicity property for the spectral radius of nonnegative matrices [15], we get

$$\rho\big(\Phi_2 \operatorname{diag}(\|\boldsymbol{G}_1(i\omega)\|_2, \ldots, \|\boldsymbol{G}_k(i\omega)\|_2)\big) \le \rho(\Psi) < 1$$

for all $\omega \in \mathbb{R}$. Hence, by Corollary 2.2 with $\alpha = \beta = (2,\ldots,2)$ we have that $I - \boldsymbol{G}(s)K$ is invertible and the closed-loop system (1.6), (1.7) is asymptotically stable. ∎

It should be noted that for the asymptotically stable closed-loop system (1.6), the condition $\rho(\Psi) < 1$ is not necessarily fulfilled.

Example 2.5. Let $\mathbf{G}_1(s) = \dfrac{2}{s+1}$ *and* $\mathbf{G}_2(s) = \dfrac{-2}{s+1}$ *have the same interconnection structure as in Figure* 2.1. *The transfer function of the closed-loop system given by*

$$\mathbf{G}_{cl}(s) = \frac{\mathbf{G}_1(s)}{1-\mathbf{G}_1(s)\mathbf{G}_2(s)} = \frac{2s+2}{s^2+2s+5}.$$

is proper and has poles at $-1 \pm 2i$. *Thus,* $\mathbf{G}_{cl}(s)$ *can be realized by an asymptotically stable system, although* $\rho(\Psi) = 2$.

Note that the condition $\rho(\Psi) < 1$ in Corollary 2.4 can be replaced by the stronger condition $\|\Psi\| < 1$, where $\|\cdot\|$ is a induced matrix norm.

It is well known that the property of a matrix to have all eigenvalues inside the unite circle can be characterized in terms of discrete-time matrix Lyapunov equations, e.g., [10]. As a consequence we have the following result.

COROLLARY 2.6. *Consider a coupled descriptor system* (1.1)–(1.3). *Let* Ψ *be as in* (2.7). *If the discrete-time Lyapunov equation*

$$\Psi X \Psi^T - X = -I \tag{2.8}$$

has a symmetric, positive definite solution X, then $I - \boldsymbol{G}(s)K$ is invertible and the closed-loop descriptor system (1.6), (1.7) *is asymptotically stable.*

We see that the problem whether all the finite eigenvalues of the pencil $\lambda\mathcal{E} - \mathcal{A}$ as in (1.7) have negative real part, is reduced to the computation of the $\mathbb{H}_\infty$-norm of k transfer functions (that can be done in parallel) and examination whether all eigenvalues of the (usually much smaller) matrix $\Psi \in \mathbb{R}^{k,k}$ lie inside the unit circle. It should be noted that computing the $\mathbb{H}_\infty$-norm of the transfer function is a difficult problem by itself, particularly for large-scale systems, see [4, 6, 11] for recent results.

3. Model order reduction. Consider the descriptor systems (1.1) that are coupled by (1.2) and (1.3). Such systems arise in electrical circuit simulations or by spatial discretization of partial differential equation and have very large order, whereas the number of inputs and outputs (or interconnection variables) is usually small compared to the order of the system. Reducing the order of coupled systems leads to a significant decrease in storage requirements and simulation time. The aim of model reduction for the coupled system (1.1) – (1.3) is to find a reduced-order system that well approximates the external input-output relation. One approach is to replace the subsystems (1.1), or some of them, by reduced-order subsystems

$$\begin{aligned} \widetilde{E}_j\dot{\widetilde{x}}_j(t) &= \widetilde{A}_j\widetilde{x}_j(t) + \widetilde{B}_j\widetilde{u}_j(t), \\ \widetilde{y}_j(t) &= \widetilde{C}_j\widetilde{x}_j(t), \end{aligned} \tag{3.1}$$

with $\widetilde{E}_j,\, \widetilde{A}_j \in \mathbb{R}^{\ell_j,\ell_j}$, $\widetilde{B}_j \in \mathbb{R}^{\ell_j,m_j}$, $\widetilde{C}_j \in \mathbb{R}^{p_j,\ell_j}$ and $\ell_j \ll n_j$, and then to couple these subsystems through the same interconnection matrices R, H and K, i.e.,

$$\widetilde{u}_j(t) = K_{j1}\widetilde{y}_1(t) + \ldots + K_{jk}\widetilde{y}_k(t) + H_j u(t), \quad j = 1, \ldots, k, \tag{3.2}$$

$$\widetilde{y}(t) = R_1\widetilde{y}_1(t) + \ldots + R_k\widetilde{y}_k(t). \tag{3.3}$$

Note that since the internal outputs $y_j(t)$ are replaced by approximate outputs $\widetilde{y}_j(t)$, due to (3.2), the internal inputs $u_j(t)$ in (3.1) should also be replaced by approximate inputs $\widetilde{u}_j(t)$. Let

$$\begin{aligned} \widetilde{E} &= \mathrm{diag}(\widetilde{E}_1, \ldots, \widetilde{E}_k), & \widetilde{A} &= \mathrm{diag}(\widetilde{A}_1, \ldots, \widetilde{A}_k), \\ \widetilde{B} &= \mathrm{diag}(\widetilde{B}_1, \ldots, \widetilde{B}_k), & \widetilde{C} &= \mathrm{diag}(\widetilde{C}_1, \ldots, \widetilde{C}_k). \end{aligned}$$

If the reduced-order pencils $\lambda\widetilde{E} - \widetilde{A}$ and $\lambda\widetilde{E} - \widetilde{A} - \widetilde{B}K\widetilde{C}$ are regular, then the reduced-order closed-loop system is given by

$$\begin{aligned} \widetilde{\mathcal{E}}\,\dot{\widetilde{x}}(t) &= \widetilde{\mathcal{A}}\,\widetilde{x}(t) + \widetilde{\mathcal{B}}\,u(t), \\ y(t) &= \widetilde{\mathcal{C}}\,\widetilde{x}(t), \end{aligned} \tag{3.4}$$

where $\widetilde{\mathcal{E}} = \widetilde{E}$, $\widetilde{\mathcal{A}} = \widetilde{A} + \widetilde{B}K\widetilde{C}$, $\widetilde{\mathcal{B}} = \widetilde{B}H$ and $\widetilde{\mathcal{C}} = R\widetilde{C}$. The transfer function of (3.4) has the form

$$\widetilde{\boldsymbol{G}}_{cl}(s) = R\big(I - \widetilde{\boldsymbol{G}}(s)K\big)^{-1}\widetilde{\boldsymbol{G}}(s)H = R\,\widetilde{\boldsymbol{G}}(s)\big(I - K\widetilde{\boldsymbol{G}}(s)\big)^{-1}H,$$

where $\widetilde{\boldsymbol{G}}(s) = \mathrm{diag}(\widetilde{\boldsymbol{G}}_1(s), \ldots, \widetilde{\boldsymbol{G}}_k(s))$ with $\widetilde{\boldsymbol{G}}_j(s) = \widetilde{C}_j(s\widetilde{E}_j - \widetilde{A}_j)^{-1}\widetilde{B}_j$, $j = 1, \ldots, k$. This model reduction approach for coupled systems has some advantages compared to the approach when the model reduction method is applied to the closed-loop system (1.6). First of all note that there is no general model reduction technique, which can be considered as optimal, since the reliability, computation time and approximation quality of the reduced-order model strongly depend on the system features. On the other hand, the behavior of the coupled system is determined by different interconnected subsystems that are usually governed by entirely different physical laws and they often act in different scales. In the considered approach every subsystem can be reduced by a most suitable model reduction method that takes into consideration the structure and properties of the subsystems.

The main question that arises if we reduce the order of the subsystems is how the behavior of the reduced-order closed-loop system (3.4) changes. In other words, we have to investigate how well $\widetilde{\boldsymbol{G}}_{cl}$ approximates $\boldsymbol{G}_{cl}$. The following theorem gives two bounds on the $\mathbb{H}_\infty$-norm of the error $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$.

THEOREM 3.1. *Consider the coupled system* (1.1)–(1.3) *and the reduced-order coupled system* (3.1)–(3.3). *Let* Π_l *and* Π_r *be the projectors such that*

$$\Pi_l\big(\widetilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s)\big) = \big(\widetilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s)\big)\Pi_r = \widetilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s).$$

1. *Let* $g_1 = \|\Pi_r K(I - \boldsymbol{G}K)^{-1}\|_{\mathbb{H}_\infty}$ *and* $g_2 = \|R(I - \boldsymbol{G}K)^{-1}\Pi_l\|_{\mathbb{H}_\infty}$. *If*

$$g_1 \max_{1\le j\le k} \|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty} < 1, \tag{3.5}$$

then the error $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_2\Big(\|H\|_2 + g_1\|\boldsymbol{G}H\|_{\mathbb{H}_\infty}\Big)\max\limits_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}}{1 - g_1\max\limits_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}}. \tag{3.6}$$

2. *Let* $g_3 = \|(I - K\boldsymbol{G})^{-1}K\Pi_l\|_{\mathbb{H}_\infty}$ *and* $g_4 = \|\Pi_r(I - K\boldsymbol{G})^{-1}H\|_{\mathbb{H}_\infty}$. *If*

$$g_3 \max_{1\le j\le k} \|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty} < 1, \tag{3.7}$$

then the error $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_4\Big(\|R\|_2 + g_3\|R\,\boldsymbol{G}\|_{\mathbb{H}_\infty}\Big)\max\limits_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}}{1 - g_3\max\limits_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}}. \tag{3.8}$$

Proof. 1. The error system has the form

$$\widetilde{\boldsymbol{G}}_{cl}(s) - \boldsymbol{G}_{cl}(s) = R\big(I - \widetilde{\boldsymbol{G}}(s)K\big)^{-1}\widetilde{\boldsymbol{G}}(s)H - R\big(I - \boldsymbol{G}(s)K\big)^{-1}\boldsymbol{G}(s)H.$$

Then for all $s \in \mathbb{C}^+$ we have

$$\begin{aligned}\|\widetilde{\boldsymbol{G}}_{cl}(s) - \boldsymbol{G}_{cl}(s)\|_2 \le{}& \|R(I - \widetilde{\boldsymbol{G}}(s)K)^{-1} - R(I - \boldsymbol{G}(s)K)^{-1}\|_2\|\widetilde{\boldsymbol{G}}(s)H\|_2\\ &+ \|R(I - \boldsymbol{G}(s)K)^{-1}\Pi_l\|_2\|(\widetilde{\boldsymbol{G}}(s) - \boldsymbol{G}(s))H\|_2\end{aligned}$$

and

$$\begin{aligned}\|\Pi_r K(I - \boldsymbol{G}(s)K)^{-1}\|_2 &\le \|\Pi_r K(I - \boldsymbol{G}K)^{-1}\|_{\mathbb{H}_\infty} = g_1,\\ \|R(I - \boldsymbol{G}(s)K)^{-1}\,\Pi_l\,\|_2 &\le \|R(I - \boldsymbol{G}K)^{-1}\,\Pi_l\,\|_{\mathbb{H}_\infty} = g_2.\end{aligned}$$

Taking into account that

$$(I-\widetilde{\boldsymbol{G}}(s)K)^{-1}=(I-\boldsymbol{G}(s)K)^{-1}\Big(I+\Pi_l(\widetilde{\boldsymbol{G}}(s)-\boldsymbol{G}(s))\Pi_r K(I-\widetilde{\boldsymbol{G}}(s)K)^{-1}\Big),$$

we can bound

$$\|R(I-\widetilde{\boldsymbol{G}}(s)K)^{-1}-R(I-\boldsymbol{G}(s)K)^{-1}\|_2\le\frac{g_1g_2\|\widetilde{\boldsymbol{G}}(s)-\boldsymbol{G}(s)\|_2}{1-g_1\|\widetilde{\boldsymbol{G}}(s)-\boldsymbol{G}(s)\|_2},$$

provided that $g_1\|\widetilde{\boldsymbol{G}}(s)-\boldsymbol{G}(s)\|_2<1$ for all $s\in\mathbb{C}^+$. Furthermore, using

$$\begin{aligned}&\|\widetilde{\boldsymbol{G}}(s)-\boldsymbol{G}(s)\|_2\le\|\widetilde{\boldsymbol{G}}-\boldsymbol{G}\|_{\mathbb{H}_\infty}=\max_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j-\boldsymbol{G}_j\|_{\mathbb{H}_\infty}<1/g_1,\\&\|\widetilde{\boldsymbol{G}}(s)H\|_2\le\|H\|_2\max_{1\le j\le k}\|\widetilde{\boldsymbol{G}}_j-\boldsymbol{G}_j\|_{\mathbb{H}_\infty}+\|\boldsymbol{G}H\|_{\mathbb{H}_\infty},\end{aligned}$$

we obtain the error bound (3.6).

2. Estimate (3.8) can be proved analogously, using the following representations $\widetilde{\boldsymbol{G}}_{cl}(s)=R\,\widetilde{\boldsymbol{G}}(s)\big(I-K\widetilde{\boldsymbol{G}}(s)\big)^{-1}H$ and $\boldsymbol{G}_{cl}(s)=R\,\boldsymbol{G}(s)\big(I-K\boldsymbol{G}(s)\big)^{-1}H$. □

The projectors Π_l and Π_r can be chosen, for example, as the block diagonal matrices $\Pi_l=\operatorname{diag}(\xi_1 I_{p_1},\ldots,\xi_k I_{p_k})$ and $\Pi_r=\operatorname{diag}(\xi_1 I_{m_1},\ldots,\xi_k I_{m_k})$, where $\xi_j=1$ if $\widetilde{\boldsymbol{G}}_j\ne\boldsymbol{G}_j$ and $\xi_j=0$, otherwise. Another possible choice is $\Pi_l=I_{p_0}$ and $\Pi_r=I_{m_0}$. However, in this case the error bounds (3.6) and (3.8) may be very conservative.

Bound (3.6) (or (3.8)) shows that if $\Pi_r K(I-\boldsymbol{G}K)^{-1}$, $R(I-\boldsymbol{G}K)^{-1}\Pi_l\in\mathbb{H}_\infty$ and $\boldsymbol{G}H\in\mathbb{H}_\infty$ (or $(I-K\boldsymbol{G})^{-1}K\Pi_l$, $\Pi_r(I-K\boldsymbol{G})^{-1}H$, $R\,\boldsymbol{G}\in\mathbb{H}_\infty$), then for sufficiently small $\|\widetilde{\boldsymbol{G}}_j-\boldsymbol{G}_j\|_{\mathbb{H}_\infty}$, the error $\|\widetilde{\boldsymbol{G}}_{cl}-\boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty}$ is also small. It should be noted that the condition $\boldsymbol{G}_{cl}\in\mathbb{H}_\infty$ implies that $\Pi_r K(I-\boldsymbol{G}K)^{-1}$, $R(I-\boldsymbol{G}K)^{-1}\Pi_l$, $(I-K\boldsymbol{G})^{-1}K\Pi_l$ and $\Pi_r(I-K\boldsymbol{G})^{-1}H$ are stable, but it does not guarantee that these transfer functions are proper. In fact, it may happen that g_1, g_2, g_3 or g_4 in Theorem 3.1 are not finite although both the original and the reduced-order closed-loop systems are asymptotically stable.

Remark 3.2. Note that computing the error bounds (3.6) and (3.8) for large-scale systems is time consuming, since we need to calculate the $\mathbb{H}_\infty$-norm of the transfer functions of the state space dimension $n_1+\ldots+n_k$. Reversing the role of $\boldsymbol{G}$ and $\widetilde{\boldsymbol{G}}$ in the proof of Theorem 3.1, we obtain a posteriori error bounds as in (3.6) and (3.8), where $\boldsymbol{G}$ is replaced by $\widetilde{\boldsymbol{G}}$. As numerical experiments show, for sufficiently small $\|\widetilde{\boldsymbol{G}}_j-\boldsymbol{G}_j\|_{\mathbb{H}_\infty}$ this does not change essentially the quality of the bounds but reduces the computation time significantly.

It should be noted that bounds (3.6) and (3.8) have been obtained independent of the model reduction method used to compute the reduced-order subsystems (3.1). For different methods with approximation bounds on $\|\widetilde{\boldsymbol{G}}_j-\boldsymbol{G}_j\|_{\mathbb{H}_\infty}$, we will obtain further error estimates for the reduced-order coupled system.

4. Balanced truncation. Balanced truncation is one of the well studied model reduction approaches proposed firstly for standard state space systems [8, 12, 21] and then generalized for descriptor systems in [23, 27]. An important property of this approach is that asymptotic stability is preserved in the reduced-order system. Moreover, the existence of a priori error bound [8, 12] allows an adaptive choice of the state space dimension ℓ of the approximate model. A disadvantage of balanced truncation is that matrix (generalized) Lyapunov equations have to be solved. However, recent results on low rank approximations to the solutions of Lyapunov equations

[18, 22, 28] make the balanced truncation model reduction approach attractive for large-scale systems.

For the continuous-time descriptor system (2.1) with the stable pencil $\lambda E - A$, the balanced truncation method is closely related to the *proper controllability* and *observability Gramians* $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ as well as the *improper controllability* and *observability Gramians* $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$. These Gramians are unique symmetric, positive semidefinite solutions of the projected generalized Lyapunov equations

$$E\,\mathcal{G}_{pc}A^T + A\,\mathcal{G}_{pc}E^T = -P_lBB^TP_l^T, \qquad \mathcal{G}_{pc} = P_r\mathcal{G}_{pc}P_r^T, \tag{4.1}$$

$$E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^TC^TCP_r, \qquad \mathcal{G}_{po} = P_l^T\mathcal{G}_{po}P_l, \tag{4.2}$$

$$A\,\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T = Q_lBB^TQ_l^T, \qquad \mathcal{G}_{ic} = Q_r\mathcal{G}_{ic}Q_r^T, \tag{4.3}$$

$$A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = Q_r^TC^TCQ_r, \qquad \mathcal{G}_{io} = Q_l^T\mathcal{G}_{io}Q_l, \tag{4.4}$$

where P_r and P_l are the spectral projectors onto the right and left deflating subspaces corresponding to the finite eigenvalues of the pencil $\lambda E - A$, $Q_r = I - P_r$ and $Q_l = I - P_l$. Using the Gramians, we can define the *proper* and *improper Hankel singular values* of system (2.1) which characterize the 'importance' of state variables in (2.1). Let n_f be the dimension of the deflating subspace corresponding to the finite eigenvalues of $\lambda E - A$. Then the proper Hankel singular values σ_j of system (2.1) are defined as the square roots of the largest n_f eigenvalues of the matrix $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$, and the improper Hankel singular values θ_j are defined as the square roots of the largest $n - n_f$ eigenvalues of the matrix $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$, i.e., $\sigma_j = \sqrt{\lambda_j(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E)}$ and $\theta_j = \sqrt{\lambda_j(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A)}$. We will assume that the proper and improper Hankel singular values are ordered decreasingly. A reduced-order system can be computed by truncation of the states corresponding to the small proper and zero improper Hankel singular values using the following algorithm.

ALGORITHM 4.1. *Generalized square root balanced truncation method.*
Given $\boldsymbol{G} = [\,E,\ A,\ B,\ C\,]$, *compute the reduced-order system* $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,]$.

1. *Compute the Cholesky factors* R_p, L_p, R_i, L_i *of the Gramians* $\mathcal{G}_{pc} = R_pR_p^T$, $\mathcal{G}_{po} = L_pL_p^T$, $\mathcal{G}_{ic} = R_iR_i^T$, $\mathcal{G}_{io} = L_iL_i^T$ *that satisfy* (4.1)–(4.4).
2. *Compute the singular value decompositions*

$$L_p^TER_p = [\,U_1,\ U_2\,]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}[\,V_1,\ V_2\,]^T,$$
$$L_i^TAR_i = U_3\Theta V_3^T,$$

where the matrices $[\,U_1,\ U_2\,]$, $[\,V_1,\ V_2\,]$, U_3 *and* V_3 *have orthonormal columns,* $\Sigma_1 = \mathrm{diag}(\sigma_1,\ldots,\sigma_d)$, $\Sigma_2 = \mathrm{diag}(\sigma_{d+1},\ldots,\sigma_{n_f})$ *and* $\Theta = \mathrm{diag}(\theta_1,\ldots,\theta_r)$ *with* $r = \mathrm{rank}(L_i^TAR_i)$.
3. *Compute the projection matrices*

$$W = [\,L_pU_1\Sigma_1^{-1/2},\ \ L_iU_3\Theta^{-1/2}\,], \quad T = [\,R_pV_1\Sigma_1^{-1/2},\ \ R_iV_3\Theta^{-1/2}\,].$$

4. *Compute the reduced-order system* $[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}] = [W^TET, W^TAT, W^TB, CT]$.

One can show that the reduced-order system $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,]$ computed by this method is stable, the error system $\widetilde{\boldsymbol{G}} - \boldsymbol{G}$ is proper, and the $\mathbb{H}_\infty$-norm error bound

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \le 2(\sigma_{d+1} + \ldots + \sigma_{n_f})$$

holds, where $\sigma_{d+1},\ldots,\sigma_{n_f}$ are the truncated proper Hankel singular values of system (2.1), see [27] for details. To solve the generalized Lyapunov equations (4.1)–(4.4) for the Cholesky factors without forming the Gramians explicitly, we can use the generalized Schur-Hammarling method [27] for problems of moderate size and the ADI method [18, 22, 28] for large-scale systems.

Now we apply the $\mathbb{H}_\infty$-norm estimates provided by the balanced truncation to the stability analysis and model reduction of the coupled system (1.1)–(1.3), where all subsystems are asymptotically stable. The following theorem gives a sufficient condition for the closed-loop reduced-order system (3.4) to be asymptotically stable.

THEOREM 4.1. *Consider a coupled system* (1.1)–(1.3) *with the stable pencils* $\lambda E_j - A_j$ *and the proper transfer functions* $\boldsymbol{G}_j(s) = C_j(sE_j - A_j)^{-1}B_j$ *for* $j = 1, \ldots, k$. *Let the reduced-order subsystems* (3.1) *be computed by Algorithm* 4.1 *applied to* (1.1) *and let*

$$\gamma = 2 \max_{1 \le j \le k} (\sigma_{d_j+1}^{(j)} + \ldots + \sigma_{n_{fj}}^{(j)}), \tag{4.5}$$

where $\sigma_{d_j+1}^{(j)}, \ldots, \sigma_{n_{fj}}^{(j)}$ *denote the truncated proper Hankel singular values of the* j*th subsystem* (1.1). *Assume that the Lyapunov equation* (2.8) *with* Ψ *as in* (2.7) *has the symmetric, positive definite solution* X. *If* $14\,\gamma\|\Phi_2\|_2\|X\|_2 < 1$ *with* $\Phi_2 = [\|K_{jl}\|_2]_{j,l=1}^k$, *then the reduced-order closed-loop system* (3.4) *is asymptotically stable.*

Proof. If the Lyapunov equation (2.8) has the symmetric, positive definite solution X, then all eigenvalues of the matrix Ψ lie inside the unit circle. In this case the solution of (2.8) is given by

$$X = \frac{1}{2\pi}\int_0^{2\pi} (e^{i\varphi}I - \Psi)^{-1}(e^{-i\varphi}I - \Psi^T)^{-1} d\varphi,$$

and we have the estimate $\max\limits_{\varphi\in[0,2\pi]} \|(e^{i\varphi}I - \Psi)^{-1}\|_2 \le 14\|X\|_2$, see [19]. Therefore,

$$\inf\left\{\|\Delta\|_2 \;:\; \det(e^{i\varphi}I - \Psi - \Delta) = 0\right\} = \frac{1}{\max\limits_{\varphi\in[0,2\pi]} \|(e^{i\varphi}I - \Psi)^{-1}\|_2} \ge \frac{1}{14\|X\|_2}.$$

Let $\widetilde{\Psi} = \Phi_2 \operatorname{diag}(\|\widetilde{\boldsymbol{G}}_1\|_{\mathbb{H}_\infty}, \ldots, \|\widetilde{\boldsymbol{G}}_k\|_{\mathbb{H}_\infty})$. It follows from the bound

$$\|\widetilde{\boldsymbol{G}}_j\|_{\mathbb{H}_\infty} \le \|\boldsymbol{G}_j\|_{\mathbb{H}_\infty} + \|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty} \le \|\boldsymbol{G}_j\|_{\mathbb{H}_\infty} + \gamma$$

that $\widetilde{\Psi} \preceq \Psi + \gamma\Phi_2$. Thus, if $\gamma\|\Phi_2\|_2 < 1/(14\|X\|_2)$, then $\rho(\widetilde{\Psi}) \le \rho(\Psi + \gamma\Phi_2) < 1$. Hence, by Corollary 2.4 the reduced-order closed-loop system (3.4) is asymptotically stable. ∎

As a consequence of Theorem 3.1 we obtain the following error bounds for the closed-loop system computed by balanced truncation applied to the subsystems.

COROLLARY 4.2. *Consider the coupled system* (1.1)–(1.3) *and the reduced-order coupled system* (3.1)–(3.3) *computed by the balanced truncation method. Let* g_1, g_2 g_3 *and* g_4 *be as in Theorem* 3.1 *and* γ *be as in* (4.5).

1. *If* $\gamma g_1 < 1$, *then the* $\mathbb{H}_\infty$*-norm of the error* $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_2\big(\|H\|_2 + g_1\|\boldsymbol{G}H\|_{\mathbb{H}_\infty}\big)\gamma}{1 - \gamma g_1}. \tag{4.6}$$

2. *If* $\gamma g_3 < 1$, *then the* $\mathbb{H}_\infty$*-norm of the error* $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_4\big(\|R\|_2 + g_3\|R\,\boldsymbol{G}\|_{\mathbb{H}_\infty}\big)\gamma}{1 - \gamma g_3}. \tag{4.7}$$

These error bounds may be more conservative than those provided by the balanced truncation of the entire system (1.6). However, besides the advantage of interconnection structure preservation, the computational effort of the subsystem balanced truncation is lower. Moreover, all subsystems can be reduced simultaneously.

5. Extension to unstable subsystems. Up to now, the stability of all subsystems was required, i.e., $\boldsymbol{G}_j \in \mathbb{H}_\infty$ for $j = 1, \ldots, k$. As it can be seen from Example 2.3, the stability of each subsystem is not necessary for the stability of the closed-loop system and hence, the assumption that each subsystem is stable seems to be restrictive. In this section, we discuss model order reduction of coupled systems containing some unstable subsystems that is based on coprime factorization. We show that unstable descriptor systems can be represented as interconnection of stable systems. Hence, we can express a coupled system with some unstable $\boldsymbol{G}_j$ as an *extended* coupled system whose subsystems are all stable. In this case, model reduction as presented in the previous sections can be applied.

5.1. Coprime factor model reduction. In this subsection we generalize some results on coprime factor model reduction [20, 32] to descriptor systems.

Consider the transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ of the descriptor system (2.1) which is not necessary in $\mathbb{H}_\infty$. Such a transfer function can be represented in the form $\boldsymbol{G}(s) = \boldsymbol{N}(s)\boldsymbol{D}(s)^{-1}$, where $\boldsymbol{D} \in \mathbb{H}_\infty$ is square and $\boldsymbol{N} \in \mathbb{H}_\infty$ has the same matrix dimensions as $\boldsymbol{G}(s)$. If, additionally, there exist $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathbb{H}_\infty$ such that $\boldsymbol{X}(s)\boldsymbol{D}(s) + \boldsymbol{Y}(s)\boldsymbol{N}(s) = I$, then $\boldsymbol{D}$ and $\boldsymbol{N}$ are called *right coprime factors* of $\boldsymbol{G}$. If there exists a state feedback matrix $F \in \mathbb{R}^{m,n}$ such that the pencil $sE - A - BF$ is stable and of index at most one, then the factors $\boldsymbol{N}$ and $\boldsymbol{D}$ can be chosen as

$$\boldsymbol{N}(s) = C(sE - A - BF)^{-1}B, \qquad \boldsymbol{D}(s) = F(sE - A - BF)^{-1}B + I.$$

In this case the generalized state space representation of the extended transfer function $\boldsymbol{G}_0(s) = [\,\boldsymbol{N}(s)^T,\ \boldsymbol{D}(s)^T - I\,]^T$ is given by

$$\begin{aligned} E\dot{x}(t) &= (A + BF)x(t) + Bu_{\boldsymbol{ND}}(t), \\ \begin{bmatrix} y_{\boldsymbol{N}}(t) \\ y_{\boldsymbol{D}}(t) \end{bmatrix} &= \begin{bmatrix} C \\ F \end{bmatrix} x(t). \end{aligned} \tag{5.1}$$

In the following, we express the descriptor system (2.1) as an interconnection of (5.1) which itself is subjected to the interconnection relations

$$\begin{aligned} u_{\boldsymbol{ND}}(t) &= -y_{\boldsymbol{D}}(t) + u(t) = [\,0,\ -I\,] \begin{bmatrix} y_{\boldsymbol{N}}(t) \\ y_{\boldsymbol{D}}(t) \end{bmatrix} + u(t), \\ y(t) &= y_{\boldsymbol{N}}(t) = [\,I,\ 0\,] \begin{bmatrix} y_{\boldsymbol{N}}(t) \\ y_{\boldsymbol{D}}(t) \end{bmatrix}. \end{aligned} \tag{5.2}$$

This interconnection is shown in Figure 5.1. Indeed, according to the formula for the transfer function of the closed-loop system (1.5), we have

$$[\,I,\ 0\,]\big(I - \boldsymbol{G}_0(s)\,[\,0,\ -I\,]\big)^{-1}\boldsymbol{G}_0(s) = \boldsymbol{N}(s)\boldsymbol{D}(s)^{-1} = \mathbf{G}(s).$$

Thus, the descriptor system (2.1) and the coupled system (5.1), (5.2) have the same transfer function. Note that the state space dimension of (5.1) coincides with that of (2.1). Moreover, system (5.1) is asymptotically stable, and, hence, it can be approximated by a reduced-order system

$$\begin{aligned} W^T E T\,\dot{\tilde{x}}(t) &= W^T(A + BF)T\,\tilde{x}(t) + W^T B\,\tilde{u}_{\boldsymbol{ND}}(t), \\ \begin{bmatrix} \tilde{y}_{\boldsymbol{N}}(t) \\ \tilde{y}_{\boldsymbol{D}}(t) \end{bmatrix} &= \begin{bmatrix} CT \\ FT \end{bmatrix} \tilde{x}(t), \end{aligned} \tag{5.3}$$

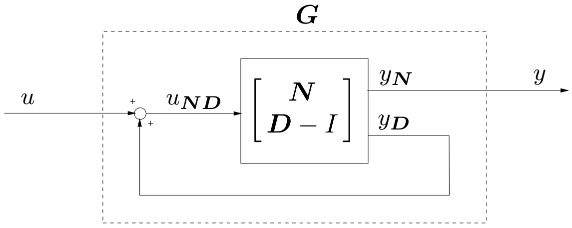


FIG. 5.1. *Coprime factorization as a coupled system.*

where the projection matrices $W,\, T \in \mathbb{R}^{n,\ell}$ are computed by the balanced truncation model reduction method described in Algorithm 4.1. The transfer function of (5.3) is given by $\widetilde{\boldsymbol{G}}_0(s) = [\, \widetilde{\boldsymbol{N}}(s)^T,\ \widetilde{\boldsymbol{D}}(s)^T - I\,]^T$ with

$$\begin{array}{rcl} \widetilde{\boldsymbol{N}}(s) & = & CT\left(W^T(sE-A-BF)T\right)^{-1} W^T B, \\ \widetilde{\boldsymbol{D}}(s) & = & FT\left(W^T(sE-A-BF)T\right)^{-1} W^T B + I, \end{array} \tag{5.4}$$

and we have the $\mathbb{H}_\infty$-norm error bound

$$\|\widetilde{\boldsymbol{G}}_0 - \boldsymbol{G}_0\|_{\mathbb{H}_\infty} = \left\| \begin{bmatrix} \widetilde{\boldsymbol{N}} - \boldsymbol{N} \\ \widetilde{\boldsymbol{D}} - \boldsymbol{D} \end{bmatrix} \right\|_{\mathbb{H}_\infty} \leq 2(\sigma^0_{d+1} + \ldots + \sigma^0_{n_f}),$$

where $\sigma^0_{d+1}, \ldots, \sigma^0_{n_f}$ denote the truncated proper Hankel singular values of (5.1).

Combining the reduced-order system (5.3) with the interconnection equations (5.2), where $u_{\boldsymbol{ND}}(t)$, $y_{\boldsymbol{N}}(t)$ and $y_{\boldsymbol{D}}(t)$ are replaced by $\widetilde{u}_{\boldsymbol{ND}}(t)$, $\widetilde{y}_{\boldsymbol{N}}(t)$ and $\widetilde{y}_{\boldsymbol{D}}(t)$, respectively, we obtain a reduced-order closed-loop system with the transfer function $\widetilde{\boldsymbol{G}}(s) = \widetilde{\boldsymbol{N}}(s)\widetilde{\boldsymbol{D}}(s)^{-1}$. The following theorem gives the generalized state space representation of $\widetilde{\boldsymbol{G}}$.

THEOREM 5.1. *Consider a descriptor system* (2.1)*, a coprime factor system* (5.1) *and a reduced-order system* (5.3). *The transfer function* $\widetilde{\boldsymbol{G}}(s) = \widetilde{\boldsymbol{N}}(s)\widetilde{\boldsymbol{D}}(s)^{-1}$ *with* $\widetilde{\boldsymbol{N}}(s)$ *and* $\widetilde{\boldsymbol{D}}(s)$ *as in* (5.4) *has a state space representation given by*

$$\begin{array}{rcl} \widetilde{E}\,\dot{\widetilde{x}}(t) & = & \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \\ \widetilde{y}(t) & = & \widetilde{C}\,\widetilde{x}(t), \end{array} \tag{5.5}$$

where $\widetilde{E} = W^T ET$, $\widetilde{A} = W^T AT$, $\widetilde{B} = W^T B$ *and* $\widetilde{C} = CT$.

Proof. It follows from $\widetilde{\boldsymbol{D}}(s)^{-1} = -FT\left(W^T(sE-A)T\right)^{-1} W^T B + I$ that

$$\begin{aligned} \widetilde{\boldsymbol{G}}(s) &= CT\left(W^T(sE-A-BF)T\right)^{-1} W^T B \left(I - FT\left(W^T(sE-A)T\right)^{-1} W^T B\right) \\ &= \begin{bmatrix} CT & 0 \end{bmatrix} \begin{bmatrix} W^T(sE-A-BF)T & W^T BFT \\ 0 & W^T(sE-A)T \end{bmatrix}^{-1} \begin{bmatrix} W^T B \\ W^T B \end{bmatrix} \\ &= CT(sW^T ET - W^T AT)^{-1} W^T B. \end{aligned}$$

Thus, a state space representation of $\widetilde{\boldsymbol{G}}$ is given by (5.5). ◻

Theorem 5.1 shows that the state feedback matrix F does not appear explicitly in the reduced-order system (5.5). Nevertheless, the computation of F can not be avoided, since F is essential for the construction of the projection matrices W, T and for obtaining the error bounds. Computing the matrix F in a numerically efficient way for problems of moderate size has been considered in [5, 33]. If E is nonsingular and the triple (E, A, B) is stabilizable, then F can be taken as $F = -B^T Y E$, where Y is the solution of the generalized Riccati equation

$$E^T Y A + A^T Y E - E^T Y B B^T Y E + C^T C = 0,$$

see [2] for numerical algorithms for large-scale Riccati equations. The case of singular E requires further investigation.

We summarize the coprime factor model order reduction method for unstable descriptor systems in the following algorithm.

ALGORITHM 5.1. *Coprime factor balanced truncation method.*

Given $\boldsymbol{G} = [\,E,\, A,\, B,\, C\,]$, *compute the reduced-order system* $\widetilde{\boldsymbol{G}} = [\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C}\,]$.

1. *Compute a feedback matrix* F *such that the pencil* $\lambda E - A - BF$ *is stable and of index at most one.*
2. *Compute the projection matrices* W *and* T *by applying Algorithm* 4.1 *to the system* $\boldsymbol{G}_0 = \left[\,E,\, A + BF,\, B,\, [C^T,\, F^T]^T\,\right]$.
3. *Compute the reduced-order system* $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] = [W^T E T, W^T A T, W^T B, C T]$.

A similar procedure can be presented for the left coprime factorization given by $\boldsymbol{G}(s) = \boldsymbol{D}_l(s)^{-1}\boldsymbol{N}_l(s)$, where $\boldsymbol{D}_l(s)$ and $\boldsymbol{N}_l(s)$ are proper and stable.

5.2. Application to coupled systems. In this subsection, we apply the coprime factor model reduction to the coupled system (1.1)–(1.3) with unstable subsystems. Without loss of generality, we assume that only the first q subsystems are unstable and a right coprime factorization $\boldsymbol{G}_j(s) = \boldsymbol{N}_j(s)\boldsymbol{D}_j(s)^{-1}$, $j = 1, \ldots, q$, is available. Then we have the following interconnection equations

$$\begin{array}{rcll} u_{\boldsymbol{N}\boldsymbol{D}_j}(t) & = & -y_{\boldsymbol{D}_j}(t) + u_j(t) & 1 \le j \le q, \\ & = & K_{j1} y_{\boldsymbol{N}_1}(t) + \ldots + K_{jq} y_{\boldsymbol{N}_q}(t) - y_{\boldsymbol{D}_j}(t) & \\ & & +\, K_{j,q+1} y_{q+1}(t) + \ldots + K_{jk} y_k(t) + H_j u(t), & \\ u_j(t) & = & K_{j1} y_{\boldsymbol{N}_1}(t) + \ldots + K_{jq} y_{\boldsymbol{N}_q}(t) & q < j \le k, \\ & & +\, K_{j,q+1} y_{q+1}(t) + \ldots + K_{jk} y_k(t) + H_j u(t), & \\ y(t) & = & R_1 y_{\boldsymbol{N}_1}(t) + \ldots + R_q y_{\boldsymbol{N}_q}(t) + R_{q+1} y_{q+1}(t) + \ldots + R_k y_k(t). & \end{array}$$

In this case we obtain the extended interconnection matrices

$$\begin{aligned} K_0 &= K\,\mathrm{diag}\big([I_{p_1},\ 0], \ldots, [I_{p_q},\ 0],\ I\big) - \mathrm{diag}\big([0,\ I_{m_1}], \ldots, [0,\ I_{m_q}],\ 0\big)\,, \\ R_0 &= \left[\begin{array}{cccccccc} R_1 & 0 & \cdots & R_q & 0 & R_{q+1} & R_{q+2} & \cdots & R_k \end{array}\right], \end{aligned} \tag{5.6}$$

and an extended subsystem matrix

$$\boldsymbol{G}_0(s) = \mathrm{diag}\big(\boldsymbol{G}_{1,0}(s), \ldots, \boldsymbol{G}_{q,0}(s), \boldsymbol{G}_{q+1}(s), \ldots, \boldsymbol{G}_k(s)\big) \tag{5.7}$$

with $\boldsymbol{G}_{j,0}(s) = [\,\boldsymbol{N}_j(s)^T,\ \boldsymbol{D}_j(s)^T - I\,]^T$. The following theorem shows that the transfer function $\mathbf{G}_{cl}(s)$ of the closed-loop system (1.6), (1.7) coincides with the transfer function $R_0(I - \boldsymbol{G}_0(s)K_0)^{-1}\boldsymbol{G}_0(s)H$ of the extended closed-loop system.

THEOREM 5.2. *Let* K_0, R_0 *and* $\boldsymbol{G}_0$ *be as in* (5.6) *and* (5.7). *Then the transfer function* $\mathbf{G}_{cl}(s)$ *of system* (1.6), (1.7) *satisfies* $\mathbf{G}_{cl}(s) = R_0(I - \boldsymbol{G}_0(s)K_0)^{-1}\boldsymbol{G}_0(s)H$.

Proof. According to (1.4)–(1.7), the state space representation of the transfer function $R_0(I - \boldsymbol{G}_0(s)K_0)^{-1}\boldsymbol{G}_0(s)H$ is given by

$$\begin{array}{rcl} \mathcal{E}\,\dot{x}(t) & = & \mathcal{A}_0\,x(t) + \mathcal{B}\,u(t), \\ y(t) & = & \mathcal{C}_0\,x(t), \end{array} \tag{5.8}$$

where $\mathcal{E}$ and $\mathcal{B}$ are as in (1.7), $\mathcal{A}_0 = A_0 + BK_0C_0$ and $\mathcal{C}_0 = R_0C_0$ with

$$\begin{aligned} A_0 &= \operatorname{diag}(A_1 + B_1F_1, \dots, A_q + B_qF_q, A_{q+1}, \dots, A_k), \\ C_0 &= \operatorname{diag}\left(\begin{bmatrix} C_1 \\ F_1 \end{bmatrix}, \dots, \begin{bmatrix} C_q \\ F_q \end{bmatrix}, C_{q+1}, \dots, C_k\right). \end{aligned} \tag{5.9}$$

We have

$$\begin{aligned} A_0 &= A + \operatorname{diag}(B_1F_1, \dots, B_qF_q, 0, \dots, 0), \\ BK_0C_0 &= BKC - \operatorname{diag}(B_1F_1, \dots, B_qF_q, 0, \dots, 0). \end{aligned}$$

Then $\mathcal{A}_0 = A_0 + BK_0C_0 = A + BKC = \mathcal{A}$. Furthermore, we obtain from (1.7), (5.6) and (5.9) that $\mathcal{C}_0 = \mathcal{C}$. Thus,

$$\mathbf{G}_{cl}(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} = \mathcal{C}_0(s\mathcal{E} - \mathcal{A}_0)^{-1}\mathcal{B} = R_0(I - \boldsymbol{G}_0(s)K_0)^{-1}\boldsymbol{G}_0(s)H. \qquad \square$$

By making use of the artificial extension of the coupled system (1.1)–(1.3) with unstable subsystems, we are able to perform model order reduction with a priori error bounds. The following result is a consequence of Theorem 3.1 and gives the $\mathbb{H}_\infty$-norm error bounds for the reduced-order coupled system (3.1)–(3.3) computed by applying Algorithms 4.1 and 5.1 to the stable and unstable subsystems, respectively.

Corollary 5.3. *Consider the coupled system* (1.1)–(1.3). *Assume that the unstable subsystems* $\boldsymbol{G}_j = \boldsymbol{N}_j\boldsymbol{D}_j^{-1}$, $j = 1\dots, q$, *are approximated by the reduced-order systems* $\widetilde{\boldsymbol{G}}_j = \widetilde{\boldsymbol{N}}_j\widetilde{\boldsymbol{D}}_j^{-1}$ *computed by Algorithm* 5.1, *whereas the stable subsystems* $\boldsymbol{G}_j$, $j = q+1, \dots, k$, *are approximated by* $\widetilde{\boldsymbol{G}}_j$ *computed by Algorithm* 4.1. *Let* K_0, R_0 *and* $\boldsymbol{G}_0$ *be defined as in* (5.6) *and* (5.7) *and let*

$$\widetilde{\boldsymbol{G}}_0(s) = \operatorname{diag}\big(\widetilde{\boldsymbol{G}}_{1,0}(s), \dots, \widetilde{\boldsymbol{G}}_{q,0}(s), \widetilde{\boldsymbol{G}}_{q+1}(s), \dots, \widetilde{\boldsymbol{G}}_k(s)\big)$$

with $\widetilde{\boldsymbol{G}}_{j,0}(s) = [\,\widetilde{\boldsymbol{N}}_j(s)^T,\ \widetilde{\boldsymbol{D}}_j(s)^T - I\,]^T$. *Further, let* Π_l *and* Π_r *be the projectors such that* $\Pi_l\big(\widetilde{\boldsymbol{G}}_0(s) - \boldsymbol{G}_0(s)\big) = \big(\widetilde{\boldsymbol{G}}_0(s) - \boldsymbol{G}_0(s)\big)\Pi_r = \widetilde{\boldsymbol{G}}_0(s) - \boldsymbol{G}_0(s)$ *and let*

$$\begin{aligned} g_{1,0} &= \|\Pi_rK_0(I - \boldsymbol{G}_0K_0)^{-1}\|_{\mathbb{H}_\infty}, \quad g_{2,0} = \|R_0(I - \boldsymbol{G}_0K_0)^{-1}\Pi_l\|_{\mathbb{H}_\infty}, \\ g_{3,0} &= \|(I - K_0\boldsymbol{G}_0)^{-1}K_0\Pi_l\|_{\mathbb{H}_\infty}, \quad g_{4,0} = \|\Pi_r(I - K_0\boldsymbol{G}_0)^{-1}H\|_{\mathbb{H}_\infty}, \end{aligned}$$

$$\max\left\{\max_{1\le j\le q}\|\widetilde{\boldsymbol{G}}_{j,0} - \boldsymbol{G}_{j,0}\|_{\mathbb{H}_\infty}, \quad \max_{q<j\le k}\|\widetilde{\boldsymbol{G}}_j - \boldsymbol{G}_j\|_{\mathbb{H}_\infty}\right\} \le \gamma.$$

1. *If* $\gamma g_{1,0} < 1$, *then the error* $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_{2,0}\big(\|H\|_2 + g_{1,0}\|\boldsymbol{G}_0H\|_{\mathbb{H}_\infty}\big)\,\gamma}{1 - \gamma g_{1,0}}. \tag{5.10}$$

2. *If* $\gamma g_{3,0} < 1$, *then the error* $\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}$ *can be bounded as*

$$\|\widetilde{\boldsymbol{G}}_{cl} - \boldsymbol{G}_{cl}\|_{\mathbb{H}_\infty} \le \frac{g_{4,0}\big(\|R_0\|_2 + g_{3,0}\|R_0\boldsymbol{G}_0\|_{\mathbb{H}_\infty}\big)\,\gamma}{1 - \gamma g_{3,0}}. \tag{5.11}$$

Proof. We apply Theorem 3.1 to the extended closed-loop system (5.8). $\square$

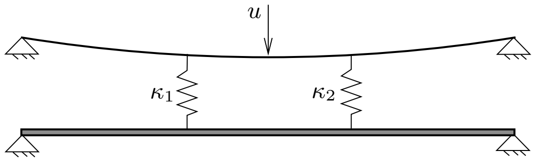


FIG. 6.1. *A coupled string-beam system.*

6. Numerical examples. In this section we present two numerical examples to demonstrate the reliability of the discussed model reduction approach for coupled systems and the quality of the error bounds (4.6), (4.7), (5.10) and (5.11). The computations were performed using MATLAB 7.

Example 6.1. Consider a uniform string and a uniform beam coupled by two springs as shown in Figure 6.1. We assume that both the string and the beam are simply-supported and have the same length l. Ignoring the displacement in lateral direction, the vertical displacements $d_1(t,z)$ and $d_2(t,z)$ of the string and the beam, respectively, are described by the equations

$$\varrho_1\alpha_1\frac{\partial^2 d_1}{\partial t^2}(t,z)+\beta_1\frac{\partial d_1}{\partial t}(t,z)-\tau\frac{\partial^2 d_1}{\partial z^2}(t,z)= f_1(t,z)+f_2(t,z)+f(t,z), \tag{6.1}$$

$$\varrho_2\alpha_2\frac{\partial^2 d_2}{\partial t^2}(t,z)+\beta_2\frac{\partial d_2}{\partial t}(t,z)+\epsilon\iota\frac{\partial^4 d_2}{\partial z^4}(t,z)=-f_1(t,z)-f_2(t,z), \tag{6.2}$$

with the boundary conditions

$$\begin{array}{ll} d_1(t,0)=d_1(t,l)=0, & \\ d_2(t,0)=d_2(t,l)=0, & \dfrac{\partial^2 d_2}{\partial z^2}(t,0)=\dfrac{\partial^2 d_2}{\partial z^2}(t,l)=0. \end{array}$$

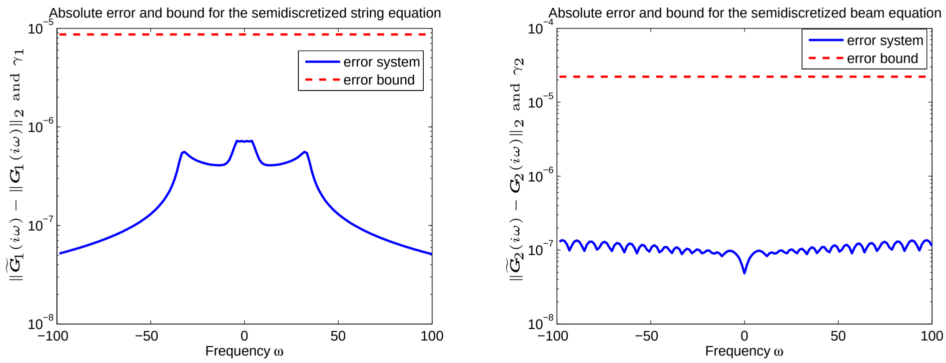
Here $\varrho_1\alpha_1$ and $\varrho_2\alpha_2$ are the masses pro unit length of the string and the beam, respectively, β_1 and β_2 are the damping parameters, τ is the tension of the spring, $\epsilon\iota$ is the bending stiffness of the beam. The loading forces are given by

$$\begin{array}{rcl} f(t,z) &=& \delta(z-l/2)u(t),\\ f_j(t,z) &=& \delta(z-jl/3)\,(d_1(z,t)-d_2(z,t))\,\kappa_j, \quad j=1,2, \end{array}$$

where $\delta(z)$ denotes the Dirac distribution, $u(t)$ is the external force and κ_j are the spring constants.

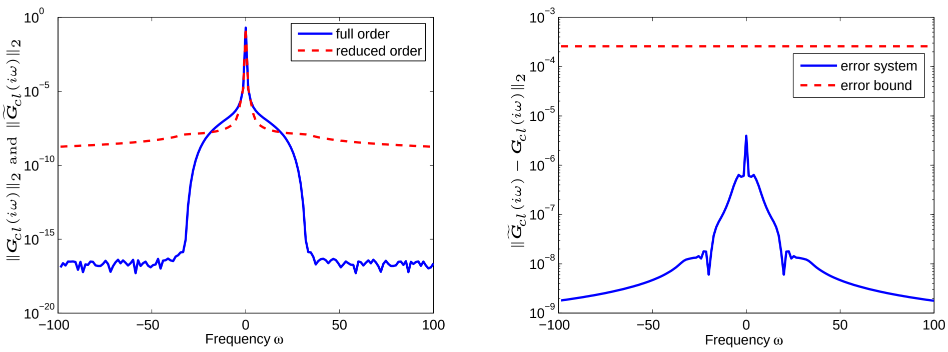
Using a finite difference method for spatial discretization of (6.1) and (6.2) with, respectively, η_1+1 and η_2+1 equidistant grid points we obtain two second order subsystems that can be rewritten as the first order subsystems (1.1) of the state space dimension $n_1=2\eta_1$ and $n_2=2\eta_2$. The interconnection equations are given by $u_1(t)=[\,y_2^T(t),\ u^T(t)]^T$ and $u_2(t)=y_1(t)$. The output equation of the system is $y(t)=y_1(t)$. We use the parameters $l=1$, $\varrho_1\alpha_1=1$, $\varrho_2\alpha_2=50$, $\beta_1=4$, $\beta_2=20$, $\tau=0.01$, $\epsilon\iota=0.01$, $\kappa_1=\kappa_2=20$ and $n_1=n_2=1006$. Note that both the subsystems are asymptotically stable and we have $\rho(\Psi)=0.9157$ with Ψ as in (2.7). Thus, by Corollary 2.4 the closed-loop system is also asymptotically stable.

Using the balanced truncation model reduction method, the semidiscretized subsystems for the spring and the beam have been approximated by the reduced models of order $\ell_1=40$ and $\ell_2=24$, respectively. We do not present the frequency responses $\boldsymbol{G}_j(i\omega)$ and $\widetilde{\boldsymbol{G}}_j(i\omega)$ of the original and the reduced-order subsystems, since they were



(a) (b)

FIG. 6.2. *Example 6.1: (a) the absolute error* $\|\widetilde{\boldsymbol{G}}_1(i\omega) - \boldsymbol{G}_1(i\omega)\|_2$ *and the error bound* γ_1*; (b) the absolute error* $\|\widetilde{\boldsymbol{G}}_2(i\omega) - \boldsymbol{G}_2(i\omega)\|_2$ *and the error bound* γ_2*.*



(a) (b)

FIG. 6.3. *Example 6.1: (a) the spectral norms of the frequency responses* $\boldsymbol{G}_{cl}(i\omega)$ *and* $\widetilde{\boldsymbol{G}}_{cl}(i\omega)$*; (b) the absolute error* $\|\widetilde{\boldsymbol{G}}_{cl}(i\omega) - \boldsymbol{G}_{cl}(i\omega)\|_2$ *and the a posteriori error bound.*

impossible to distinguish. Figure 6.2 shows the absolute errors $\|\widetilde{\boldsymbol{G}}_j(i\omega) - \boldsymbol{G}_j(i\omega)\|_2$ for the frequency range $\omega \in [-10^2, 10^2]$ and the error bounds $\gamma_j = 2(\sigma_{\ell_j+1}^{(j)} + \ldots + \sigma_{n_j}^{(j)})$ for $j = 1, 2$. In Figure 6.3 (a) we present the frequency responses $\boldsymbol{G}_{cl}(i\omega)$ and $\widetilde{\boldsymbol{G}}_{cl}(i\omega)$ of the original and the reduced-order closed-loop systems. The absolute error $\|\widetilde{\boldsymbol{G}}_{cl}(i\omega) - \boldsymbol{G}_{cl}(i\omega)\|_2$ and the a posteriori error bound computed as a minimum of bounds (4.6) and (4.7), where $\boldsymbol{G}$ is replaced by $\widetilde{\boldsymbol{G}}$, are presented in Figure 6.2 (b).

One can see that the reduced-order closed-loop system matches the peak at $\omega = 0$, but the approximation is less accurate for large frequencies. Nevertheless, the error $\|\widetilde{\boldsymbol{G}}_{cl}(i\omega) - \boldsymbol{G}_{cl}(i\omega)\|_2$ remains below the level 10^{-5}. Note that the Hankel singular values of both subsystems decay quite slowly. This results in a large difference between the errors and corresponding error bounds for the subsystems and, as a consequence, for the closed-loop system.

Example 6.2. Consider a heated beam whose temperature is steered by a PI-controller as shown in Figure 6.4. The transfer function of the PI-controller is given by

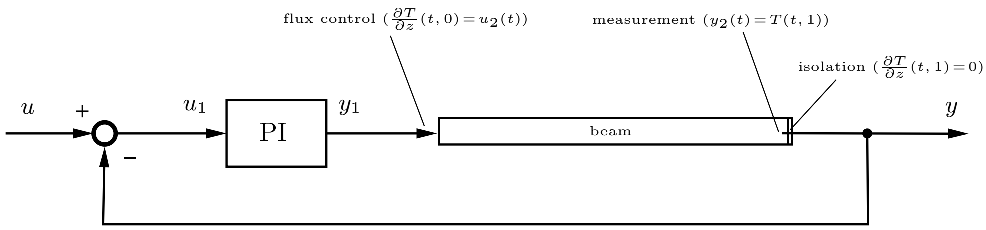


FIG. 6.4. *A heated beam with a PI-controller.*

$\boldsymbol{G}_1(s) = k_I s^{-1} + k_P$ and it is realized by the descriptor system

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}_1(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} k_I \\ -k_P \end{bmatrix} u_1(t), \\
y_1(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_1(t).
\end{aligned} \tag{6.3}
$$

The heat transfer along the 1D beam of length 1 is described by

$$\frac{\partial T}{\partial t}(t,z) = \kappa \frac{\partial^2 T}{\partial z^2}(t,z),$$

where $t > 0$ is the time, $z \in [0,1]$ is the position, $T(t,z)$ is the temperature distribution and κ is the heat conductivity of the material. On the left-hand side of the beam, the temperature flux is controlled by an input $u_2(t)$, whereas the beam is assumed to be perfectly isolated on the right-hand side. From this, we get the boundary conditions

$$\frac{\partial T}{\partial z}(t,0) = u_2(t), \qquad \frac{\partial T}{\partial z}(t,1) = 0.$$

The temperature is measured at $z = 1$ and it forms the output of the system, i.e., $y_2(t) = T(t,1)$ and $y(t) = y_2(t)$. By a spatial discretization of the beam with $n_2 + 1$ equidistant grid points, we obtain the system

$$
\begin{aligned}
E_2 \dot{x}_2(t) &= A_2 x_2(t) + B_2 u_2(t), \\
y_2(t) &= C_2 x_2(t),
\end{aligned} \tag{6.4}
$$

where $E_2 = I_{n_2}$ and

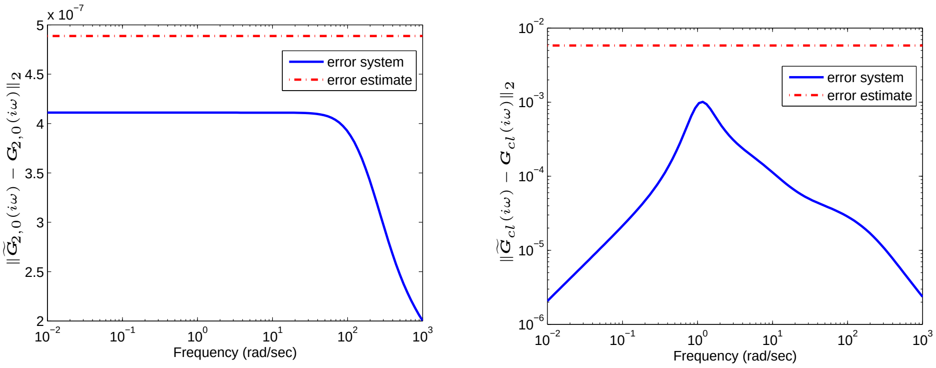
$$
A_2 = \kappa(n_2+1)^2 \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} \kappa(n_2+1) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T.
$$

The interconnection of the PI-controller and the beam is expressed by the relations

$$u_1(t) = u(t) - y_2(t), \quad u_2(t) = y_1(t).$$

Note that both the subsystems (6.3) and (6.4) are not asymptotically stable, since their transfer functions $\boldsymbol{G}_1(s) = C_1(sE_1 - A_1)^{-1}B_1$ and $\boldsymbol{G}_2(s) = C_2(sE_2 - A_2)^{-1}B_2$ have a pole at the origin. The stabilizing state feedback matrices can be chosen as

$$F_1 = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -n_2 - 1 & 0 & \cdots & 0 \end{bmatrix}.$$



(a) (b)

FIG. 6.5. *Example 6.2: (a) the absolute error* $\|\widetilde{\boldsymbol{G}}_{2,0}(i\omega) - \boldsymbol{G}_{2,0}(i\omega)\|_2$ *and the error bound* γ*; (b) the absolute error* $\|\widetilde{\boldsymbol{G}}_{cl}(i\omega) - \boldsymbol{G}_{cl}(i\omega)\|_2$ *and the a posteriori error bound.*

In our experiments, we took the numerical values $k_P = k_I = \kappa = 1$, $n_2 = 1000$. The second subsystem (6.4) has been approximated by a reduced model of order $\ell_2 = 20$ computed by the coprime factor balanced truncation method. In Figure 6.5 (a) we present the absolute error $\|\widetilde{\boldsymbol{G}}_{2,0}(i\omega) - \boldsymbol{G}_{2,0}(i\omega)\|_2$ and the error bound γ that is twice the sum of the truncated Hankel singular values. Figure 6.5 (b) shows the spectral norm of the error system $\widetilde{\boldsymbol{G}}_{cl}(i\omega) - \boldsymbol{G}_{cl}(i\omega)$ and the a posteriori error bound computed as a minimum of bounds (5.10) and (5.11), where $\boldsymbol{G}_0$ is replaced by $\widetilde{\boldsymbol{G}}_0$. Comparing the approximation errors, we see that due the coupling the error in the closed-loop system is larger than the error in the subsystem. Moreover, Figure 6.5 shows that the error bound has the same magnitude order as the actual error in the closed-loop system.

7. Conclusion. In this paper we have considered the stability and model order reduction for coupled systems. We have presented sufficient criteria for the stability of such systems. Furthermore, we have discussed (coprime factor) balanced truncation subsystem model reduction and obtained the $\mathbb{H}_\infty$-norm error bounds for the reduced-order closed-loop system.

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