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Model reduction and dynamic iteration for coupled nonlinear systems

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The dynamical behavior of coupled systems is determined by different interconnected subsystems that are usually governed by entirely different physical laws and often act in different time and space scales. We discuss the simulation of coupled nonlinear systems using dynamic iteration combined with model order reduction. We also study the convergence of this approach and derive error estimates for approximate solutions.

1 Introduction

A wide variety of physical and technical processes can be modeled by coupled systems of differential equations. Application areas of coupled systems include circuit-device simulation and micro-electro-mechanical systems. The dynamical behavior of coupled systems is characterized by different properties of the interacting subsystems that often describe different physical effects in the system. The increasing complexity of mathematical models requires the development of new simulation techniques for large-scale coupled systems.

Dynamic iteration (known also as waveform relaxation), e.g., [1], has proven to be a useful tool for simulation of coupled systems since at every iteration, the decoupled subsystems can be solved separately. Such a modular approach allows us to use different time steps and to employ most appropriate integration methods for the different subsystems.

Another approach for dealing with complex dynamical systems is model reduction. The aim of model reduction is to approximate a large-scale system $\dot{x} = f(x)$, $x(T_0) = x_0$ with $x \in \mathbb{R}^n$ by a reduced-order model

$$\dot{\hat{x}} = \hat{f}(\hat{x}), \qquad \hat{x}(T_0) = \hat{x}_0 \tag{1}$$

with $\hat{x} \in \mathbb{R}^r$ and $r \ll n$, which nearly approximates the dynamical behavior of the original system. A most popular model reduction method for nonlinear systems is proper orthogonal decomposition (POD), e.g., [5]. It is based on determining a snapshot matrix $X = [x(t_1), \ldots, x(t_q)]$ and computing a singular value decomposition $X = [V, V_0]\,\mathrm{diag}(\Sigma, \Sigma_0)\,[W, W_0]^T$, where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\Sigma_0 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_q)$ with $\sigma_1 \leq \ldots \leq \sigma_r < \sigma_{r+1} \leq \ldots \leq \sigma_q$, and the matrices $[V, V_0]$ and $[W, W_0]$ are orthogonal. Then x is approximated by $V\hat{x}$, where $\hat{x}$ solves the reduced model (1) with $\hat{f}(\hat{x}) = V^T f(V\hat{x})$ and $\hat{x}_0 = V^T x_0$. Though this system is of low dimension r, the evaluation of the nonlinearity $V^T f(V\hat{x})$ still has a computational complexity of n. To overcome this difficulty, the discrete empirical interpolation method (DEIM) has been developed in [2] which provides an approximate model

$$\dot{\hat{x}} = V^T U (P^T U)^{-1} P^T f(V\hat{x}), \qquad \hat{x}(T_0) = \hat{x}_0,$$

where $U \in \mathbb{R}^{n\times m}$ is a POD basis matrix obtained from the snapshot matrix $[f(x(t_1)), \ldots, f(x(t_q))]$ and $P = [e_{p_1}, \ldots, e_{p_m}]$ is a selector matrix constructed from U by a Greedy algorithm. Here e_{p_j} denotes the p_j-th column of the identity matrix. Note that $P^T f(V\hat{x})$ needs only m function evaluations. For a posteriori error estimates for POD-DEIM reduced models, we refer to [6].

In [4], dynamic iteration was combined with reduced-order models resulting in a DIRM method. This method involves successive simulation of each unreduced subsystem coupled with other reduced-order subsystems. Unfortunately, the convergence analysis carried out in [4] is restricted to coupled linear time-invariant systems with a week coupling. In this paper, we extend these results to coupled nonlinear systems and present a posteriori error estimation for the DIRM iteration.

2 Dynamic iteration using reduced-order models

Consider a coupled system of nonlinear differential equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad x_1(T_0) = x_1^0, \tag{2}$$

$$\dot{x}_2 = f_2(x_1, x_2), \quad x_2(T_0) = x_2^0, \tag{3}$$

where $x_j : \mathbb{I} \to \mathbb{R}^{n_j}$, $\mathbb{I} = [T_0, T_e]$ and $f_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_j}$ for $j = 1, 2$. We assume that f_j are continuously differentiable and system (2), (3) is solvable. For a better readability, we restrict ourselves here to the coupling of two autonomous subsystems, although all results can easily be extended to more general systems with an arbitrary number of subsystems.

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We apply now the DIRM method from [4] to the coupled system (2), (3) which is based on a combination of dynamic iteration combined and POD-DEIM model order reduction. First, the time interval $[T_0, T_e]$ is split into windows $[T_l, T_{l+1}]$ with a time grid $T_0 < T_1 < \ldots < T_L = T_e$. Then a k-th iteration in the window $[T_l, T_{l+1}]$ is determined from the Jacobi-type equations

$$\begin{aligned} \dot{x}_1^{[k]} &= f_1(x_1^{[k]}, V_2^{[k]}\hat{x}_2^{[k]}), \quad x_1^{[k]}(T_l) = x_1^{[k_{l-1}]}(T_l), && (4)\\ \dot{\hat{x}}_2^{[k]} &= W_2^{[k]} f_2(x_1^{[k]}, V_2^{[k]}\hat{x}_2^{[k]}), \quad \hat{x}_2^{[k]}(T_l) = \hat{x}_2^{[k_{l-1}]}(T_l), \qquad k = 1, \ldots, k_l, && (5) \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{x}}_1^{[k]} &= W_1^{[k]} f_1(V_1^{[k]}\hat{x}_1^{[k]}, x_2^{[k]}), \quad \hat{x}_1^{[k]}(T_l) = \hat{x}_1^{[k_{l-1}]}(T_l), && (6)\\ \dot{x}_2^{[k]} &= f_2(V_1^{[k]}\hat{x}_1^{[k]}, x_2^{[k]}), \quad x_2^{[k]}(T_l) = x_2^{[k_{l-1}]}(T_l), \qquad k = 1, \ldots, k_l, && (7) \end{aligned}$$

where $W_j^{[k]} = (V_j^{[k]})^T U_j^{[k]} \big((P_j^{[k]})^T U_j^{[k]}\big)^{-1} (P_j^{[k]})^T$, $j = 1, 2$, with the POD basis matrices $V_j^{[k]}$, $U_1^{[k]}$ and $U_2^{[k]}$ calculated from the snapshots $\Big\{x_j^{[k-1]}(t_{il})\Big\}_{i=1}^{q_l}$, $\Big\{f_1\big(x_1^{[k-1]}(t_{il}), V_2^{[k-1]}\hat{x}_2^{[k-1]}(t_{il})\big)\Big\}_{i=1}^{q_l}$ and $\Big\{f_2\big(V_1^{[k-1]}\hat{x}_1^{[k-1]}(t_{il}), x_2^{[k-1]}(t_{il})\big)\Big\}_{i=1}^{q_l}$, respectively, on the time window $[T_l, T_{l+1}]$, and $P_j^{[k]}$ are the DEIM selector matrices. Note that if (5) and (6) have low dimensions, then solving systems (4), (5) and (6), (7) is only slightly expensive than solving the subsystems (2) and (3).

In order to analyze the convergence of the DIRM iteration, we have to study the errors caused by approximate initial conditions and model reduction in each window $[T_l, T_{l+1}]$ and the error propagation from one window to the next one.

Proposition *Let $L[J_f]\big(x^{[k]}(t)\big) = \lambda_{\max}\Big(J_f\big(x^{[k]}(t)\big) + J_f^T\big(x^{[k]}(t)\big)\Big)/2$ be a logarithmic norm of the Jacobian $J_f\big(x^{[k]}(t)\big)$ of $f = \big[f_1^T, f_2^T\big]^T$ at $x^{[k]}(t) = \big[(x_1^{[k]})^T(t), (x_2^{[k]})^T(t)\big]^T$. Then the error in the k-th DIRM iteration is estimated as*

$$\|e(t)\| := \left\| \begin{bmatrix} x_1(t) - x_1^{[k]}(t) \\ x_2(t) - x_2^{[k]}(t) \end{bmatrix} \right\| \le \int_{T_l}^{t} \beta(s) \exp\Big(\int_s^t \alpha(\tau)d\tau\Big) ds + \|e(T_l)\| \exp\Big(\int_{T_l}^t \alpha(\tau)d\tau\Big), \quad t \in [T_l, T_{l+1}], \tag{8}$$

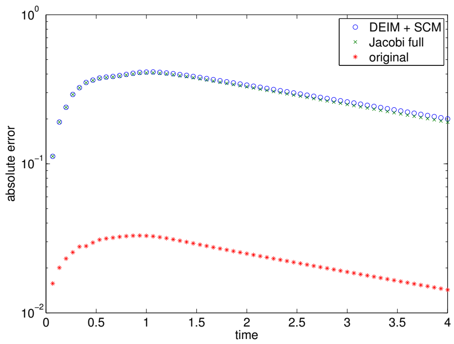
where

$$\alpha(t) = L[J_f]\big(x^{[k]}(t)\big), \qquad \beta(t) = \begin{bmatrix} f_1(x_1^{[k]}(t), x_2^{[k]}(t)) - f_1(x_1^{[k]}(t), V_2^{[k]}\hat{x}_2^{[k]}(t)) \\ f_2(x_1^{[k]}(t), x_2^{[k]}(t)) - f_2(V_1^{[k]}\hat{x}_1^{[k]}(t), x_2^{[k]}(t)) \end{bmatrix}.$$

The logarithmic norm of the Jacobian can be computed efficiently by a successive constraint method (SCM), see [3], if we first find a low-dimensional approximation $J_f(x) \approx \sum_{j=1}^{n_J} \theta_j(x) J_j$, where $J_j \in \mathbb{R}^{n\times n}$, $\theta : \mathbb{R}^n \to \mathbb{R}$ and $n_J \ll n = n_1 + n_2$. Such an approximation can be determined using a matrix DEIM approach [6].

Example Consider the nonlinear equation

$$\begin{aligned} \partial_t x &= \Delta x + x\,\partial_\xi x, && (t,\xi) \in (0,1)\times(0,T_e),\\ x(t,0) &= x(t,1) = 0, && t \in (0,T_e),\\ x(0,\xi) &= x_0(\xi), && \xi \in (0,1). \end{aligned}$$



We discretize first this equation in space with 100 equally spaced grid points and then split the obtained system into 10 subsystems. These subsystems were approximated by reduced models of order 2. The DIRM iteration was terminated after 3 steps in all time windows. The figure shows the DIRM error $\|e(t)\|$, the error estimator (8) and the approximated error calculated employing the matrix DEIM and SCM.

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