Control problems for differential-algebraic equations

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In this report we briefly discuss stability, passivity and model order reduction of linear time-invariant control systems described by differential-algebraic equations (DAEs)

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
\tag{1}
$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the output. Such equations arise in a variety of applications including multibody dynamics and circuit simulation.

It is well known that the stability properties of system (1) can be characterized in terms of the eigenvalues of the pencil $\lambda E - A$. System (1) with $u(t) \equiv 0$ is *asymptotically stable* if and only if $\lambda E - A$ is stable, i.e., all the finite eigenvalues of $\lambda E - A$ have negative real part. Note, however, that the eigenvalues of $\lambda E - A$ may be very ill-conditioned in the sense that they may change largely even for small perturbations in E and A. Hence, eigenvalues that are computed numerically in finite precision arithmetic, may not always provide the correct information on the stability of dynamical systems. As an alternative to the use of eigenvalues in the stability analysis, one can employ spectral parameters based on projected Lyapunov equations [5, 6]. One can show that the pencil $\lambda E - A$ is stable if and only if the projected generalized continuous-time Lyapunov equation

$$
A^T H E + E^T H A = -P_r^T P_r, \qquad H = P_l^T H P_l
\tag{2}
$$

has a unique symmetric, positive semidefinite solution H. Here P_r and P_l are the spectral projectors onto the right and left deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues. The parameter $\kappa(E, A) = 2\|E\|\|A\|\|H\|$, where $\|\cdot\|$ denotes the spectral matrix norm, can be used to characterize the stability of $\lambda E - A$ and also the sensitivity of its eigenvalues to perturbations in the matrices E and A, see [5].

Passivity is an important concept in circuit simulation. System (1) is *passive* if and only if its transfer function $\mathbf{G}(s) = C(sE - A)^{-1}B + D$ is positive real, i.e., $\mathbf{G}(s)$ is analytic in $\mathbb{C}^+ = \{\, s \in \mathbb{C} \;:\; \mathrm{Re}(s) > 0 \,\}$ and the matrix $\mathbf{G}(s) + \mathbf{G}^T(\bar{s})$ is positive semidefinite for all $s \in \mathbb{C}^+$. We have the following result.

Proposition. *Let $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where $\mathbf{G}_{sp}(s)$ is the strictly proper part and $\mathbf{P}(s) = P_0 + sP_1 + \ldots + s^q P_q$ is the polynomial part of $\mathbf{G}(s)$.*

1. *If P_1 is symmetric, positive semidefinite, $P_j = 0$ for $j \geq 2$ and if the projected generalized Lur'e equation*

$$
\begin{aligned}
A^T \mathcal{Y} E + E^T \mathcal{Y} A = -P_r^T L^T L P_r, &\quad \mathcal{Y} = P_l^T \mathcal{Y} P_l, \\
B^T \mathcal{Y} E - C P_r = -K^T L P_r, &\quad K^T K = P_0 + P_0^T
\end{aligned}
\tag{3}
$$

has the solution $\mathcal{Y}$, L, K, where $\mathcal{Y}$ is symmetric and positive semidefinite, then $\mathbf{G}(s)$ is positive real.

2. *If* $\mathbf{G}(s)$ *is positive real and if system* (1) *is minimal, then the projected generalized Lur'e equation* (3) *has the solution* $\mathcal{Y}$, L *and* K.

If $R = P_0 + P_0^T$ is nonsingular, then the projected Lur'e equation (3) is equivalent to the projected generalized Riccati equation

$$A^T\mathcal{Y}E + E^T\mathcal{Y}A + (B^T\mathcal{Y}E - CP_r)^T R^{-1}(B^T\mathcal{Y}E - CP_r) = 0, \quad \mathcal{Y} = P_l^T\mathcal{Y}P_l.$$

Modelling of complex physical and technical processes such as VLSI chip design and control of fluid flow often leads to linear DAE control systems of very large order n, while the number m of inputs and the number p of outputs are typically small compared to n. Despite the ever increasing computational speed, simulation, optimization or real-time controller design for such large-scale systems is difficult because of large storage requirements and computation time. In this context, *model order reduction* is of crucial importance. A general idea of model reduction is to approximate the large-scale system (1) by a reduced-order model that preserves essential properties of (1) like stability and passivity and that has a small approximation error.

Balanced truncation is one of the most effective and well studied model reduction approaches for standard state space systems [2, 4]. This approach has been extended to DAE systems in [7]. An important property of the balanced truncation model reduction methods is that the asymptotic stability is preserved in the reduced-order system. Moreover, the existence of computable error bounds allows an adaptive choice of the state space dimension of the approximate model. The balanced truncation methods are closely related to the proper and improper controllability and observability Gramians of system (1) that are defined by the solutions of the two dual continuous-time and two dual discrete-time projected generalized Lyapunov equations.

Note that Lyapunov-based balanced truncation, in general, does not preserve passivity in the reduced-order system. In a passivity-preserving model reduction approach, known as positive real balanced truncation, instead of the continuous-time projected Lyapunov equations we have to solve the projected generalized Riccati equations. For the DAE control system (1) that is not necessarily minimal but that has the proper transfer function $\mathbf{G}(s)$, we have the following algorithm.

Algorithm. Positive real balanced truncation for DAE systems.
Given $\mathbf{G} = [\,E,\, A,\, B,\, C,\, D\,]$, compute the reduced-order system $[\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D}\,]$.

1. Compute the Cholesky factors R_i and L_i of the improper controllability and observability Gramians $\mathcal{G}_{ic} = R_iR_i^T$ and $\mathcal{G}_{io} = L_iL_i^T$ by solving the projected generalized discrete-time Lyapunov equations
$$\begin{aligned} A\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T &= Q_lBB^TQ_l^T, \qquad & \mathcal{G}_{ic} &= Q_r\mathcal{G}_{ic}Q_r^T, \\ A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E &= Q_r^TC^TCQ_r, \qquad & \mathcal{G}_{io} &= Q_l^T\mathcal{G}_{io}Q_l, \end{aligned}$$
with $Q_r = I - P_r$ and $Q_l = I - P_l$.
2. Compute the skinny singular value decomposition $L_i^TAR_i = U\Theta V^T$, where U and V have orthonormal columns and Θ is nonsingular.
3. Compute $W_2 = L_iU\Theta^{-1/2}$, $T_2 = R_iV\Theta^{-1/2}$, $P_0 = D - CT_2W_2^TB$, $R = P_0 + P_0^T$.

4. Compute the Cholesky factors R and L of the solutions $\mathcal{X} = RR^T$ and $\mathcal{Y} = LL^T$ of the projected generalized Riccati equations

$$\begin{aligned} &A\mathcal{X}E^T + E\mathcal{X}A^T + (E\mathcal{X}C^T - P_l B)R^{-1}(E\mathcal{X}C^T - P_l B)^T = 0, \qquad \mathcal{X} = P_r \mathcal{X} P_r^T, \\ &A^T\mathcal{Y}E + E^T\mathcal{Y}A + (B^T\mathcal{Y}E - CP_r)^T R^{-1}(B^T\mathcal{Y}E - CP_r) = 0, \qquad \mathcal{Y} = P_l^T \mathcal{Y} P_l. \end{aligned}$$

5. Compute the skinny singular value decomposition

$$L^T E R = [\,U_1,\, U_2\,] \begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix} [\,V_1,\, V_2\,]^T,$$

where $[\,U_1,\, U_2\,]$ and $[\,V_1,\, V_2\,]$ have orthonormal columns, $\Pi_1 = \mathrm{diag}(\pi_1, \ldots, \pi_\ell)$ and $\Pi_2 = \mathrm{diag}(\pi_{\ell+1}, \ldots, \pi_r)$ with $\pi_1 \ge \ldots \ge \pi_\ell \gg \pi_{\ell+1} \ge \ldots \ge \pi_r > 0$.

6. Compute the reduced-order system

$$[\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C},\, \widetilde{D}\,] = [\,W_1^T E T_1,\, W_1^T A T_1,\, W_1^T B,\, C T_1,\, P_0\,]$$

with $W_1 = L U_1 \Pi_1^{-1/2}$ and $T_1 = R V_1 \Pi_1^{-1/2}$.

Similarly to the standard state space case [3], one can show that the reduced-order system with the transfer function $\widetilde{\mathbf{G}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}$ is passive, and the $\mathbb{H}_\infty$-norm error bound

$$\|\widetilde{\mathbf{G}} - \mathbf{G}\|_{\mathbb{H}_\infty} \le 2\|R^{-1}\|^2 \|\mathbf{G} + \widetilde{D}^T\|_{\mathbb{H}_\infty} \|\widetilde{\mathbf{G}} + \widetilde{D}^T\|_{\mathbb{H}_\infty} \sum_{j=\ell+1}^{r} \pi_j$$

holds where $\|\mathbf{G}\|_{\mathbb{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega)\|$ denotes the $\mathbb{H}_\infty$-norm of $\mathbf{G}$.

A major difficulty in the numerical solution of the projected Lyapunov and Riccati equations with large matrix coefficients is that the spectral projectors onto the left and right deflating subspaces corresponding to the finite and infinite eigenvalues of the pencil $\lambda E - A$ are required. However, in many applications such as control of fluid flow, electrical circuit simulation and constrained multibody systems, the matrices E and A have some special block structure. This structure can be used to construct the projectors in explicit form [1, 8].

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