BALANCED TRUNCATION MODEL REDUCTION OF SECOND-ORDER SYSTEMS

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Abstract. In this paper we consider structure-preserving model reduction of second-order systems using a balanced truncation approach. Several sets of singular values are introduced for such systems that lead to different concepts of balancing. We present two variants of the second-order balanced truncation method and compare their properties on numerical examples.

1. Introduction

Consider a linear time-invariant second-order system

$$\begin{aligned} M\,\ddot{q}(t) + D\,\dot{q}(t) + K\,q(t) &= B_2 u(t), \\ C_2\dot{q}(t) + C_1 q(t) &= y(t), \end{aligned} \tag{1}$$

where $M \in \mathbb{R}^{n,n}$ is a nonsingular mass matrix, $D \in \mathbb{R}^{n,n}$ is a damping matrix, $K \in \mathbb{R}^{n,n}$ is a stiffness matrix, $B_2 \in \mathbb{R}^{n,m}$, C_1, $C_2 \in \mathbb{R}^{p,n}$, $q(t) \in \mathbb{R}^n$ is a displacement vector, $u(t) \in \mathbb{R}^m$ is a control input and $y(t) \in \mathbb{R}^p$ is an output. Such systems arise frequently in many practical applications like mechanical systems, electrical circuits and large structures [5, 6, 14]. A *transfer function* of system (1) is given by

$$\boldsymbol{G}(s) = (sC_2 + C_1)(s^2 M + sD + K)^{-1} B_2. \tag{2}$$

The model reduction problem for (1) consists in an approximation of (1) by a reduced system

$$\begin{aligned} \widetilde{M}\,\ddot{\widetilde{q}}(t) + \widetilde{D}\,\dot{\widetilde{q}}(t) + \widetilde{K}\,\widetilde{q}(t) &= \widetilde{B}_2 u(t), \\ \widetilde{C}_2\dot{\widetilde{q}}(t) + \widetilde{C}_1\widetilde{q}(t) &= \widetilde{y}(t), \end{aligned} \tag{3}$$

where $\widetilde{M}$, $\widetilde{D}$, $\widetilde{K} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B}_2 \in \mathbb{R}^{\ell,m}$ and $\widetilde{C}_1$, $\widetilde{C}_2 \in \mathbb{R}^{p,\ell}$ with $\ell \ll n$. We require for the approximate system (3) to be stable and passive if (1) is stable and passive. Furthermore, it is desirable to have a small approximation error that can be measured by $\|\widetilde{y} - y\|$ in some norm.

The second-order system (3) can be rewritten as a first-order system

$$\begin{aligned} \mathcal{E}\,\dot{x}(t) &= \mathcal{A}\,x(t) + \mathcal{B}\,u(t), \\ y(t) &= \mathcal{C}\,x(t), \end{aligned} \tag{4}$$

where $x(t) = [\,q^T(t),\ \dot{q}^T(t)\,]^T$ and the matrices $\mathcal{E}$, $\mathcal{A} \in \mathbb{R}^{2n,2n}$, $\mathcal{B} \in \mathbb{R}^{2n,m}$ and $\mathcal{C} \in \mathbb{R}^{p,2n}$ have the following form

$$\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad \mathcal{C} = [\,C_1,\ C_2\,] \tag{5}$$

or

$$\mathcal{E} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \qquad \mathcal{C} = [\,C_1,\ C_2\,]. \tag{6}$$

Note that the matrices $\mathcal{E}$ and $\mathcal{A}$ in (6) are symmetric if M, D and K are symmetric. The transfer function of system (4) is given by $\boldsymbol{\mathcal{G}}(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$. One can show that systems (1) and (4) have the same transfer function, i.e., $\boldsymbol{G}(s) = \boldsymbol{\mathcal{G}}(s)$ for both forms of the matrix coefficients $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

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Applying any projection-based model reduction method to system (4), we obtain a reduced model

$$\begin{aligned} \widetilde{\mathcal{E}}\,\dot{\widetilde{x}}(t) &= \widetilde{\mathcal{A}}\,\widetilde{x}(t) + \widetilde{\mathcal{B}}\,u(t), \\ \widetilde{y}(t) &= \widetilde{\mathcal{C}}\,\widetilde{x}(t), \end{aligned} \tag{7}$$

where $\widetilde{\mathcal{E}} = W^T\mathcal{E}T$, $\widetilde{\mathcal{A}} = W^T\mathcal{A}T$, $\widetilde{\mathcal{B}} = W^T\mathcal{B}$, $\widetilde{\mathcal{C}} = \mathcal{C}T$ and $W,\ T \in \mathbb{R}^{2n,k}$ are projection matrices. Unfortunately, system (7) cannot, in general, be transformed into the second-order form (3), see [10, 12] for special cases when it can be done. Note that preservation of the second-order structure in the reduced system is important for physical interpretation.

In this paper we consider structure-preserving model reduction of second-order systems using balanced truncation. This method has been proved to be an efficient model reduction technique for first-order large-scale systems [1, 7, 11]. It is related to the *controllability Gramian* $\mathcal{P}$ and the *observability Gramian* $\mathcal{Q}$ of (4) that are defined as the unique symmetric, positive semidefinite solutions of the dual generalized Lyapunov equations

$$\mathcal{E}\,\mathcal{P}\mathcal{A}^T + \mathcal{A}\,\mathcal{P}\mathcal{E}^T = -\mathcal{B}\mathcal{B}^T, \qquad\qquad \mathcal{E}^T\mathcal{Q}\,\mathcal{A} + \mathcal{A}^T\mathcal{Q}\,\mathcal{E} = -\mathcal{C}^T\mathcal{C}$$

provided that the pencil $\lambda\mathcal{E} - \mathcal{A}$ is stable, i.e., all eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ have negative real part. The balanced truncation model reduction method consists in the state space transformation of (4) into a balanced form such that $\mathcal{P} = \mathcal{Q} = \mathrm{diag}(\xi_1, \ldots, \xi_{2n})$ with nonnegative entries which are called the *Hankel singular values*. Then the reduced system (7) is computed by truncating the states corresponding to the $2n - k$ smallest Hankel singular values. Important properties of the balanced truncation method for (4) are that the stability is preserved in the reduced system (4) and that there is a priori error bound [7].

The Gramians for the second-order system (1) have been considered in [3, 10, 13]. Using these Gramians we will define different concepts of balanced realizations and singular values for (1). The latter play a crucial role in identifying which states are important and which states can be truncated without changing the system properties significantly. We will compare different variants of second-order balanced truncation on numerical examples.

2. Singular values of second-order systems

Assume that a matrix polynomial $P(\lambda) = \lambda^2 M + \lambda D + K$ is *stable*, i.e., all zeros of $P(\lambda)$ have negative real part. In this case all eigenvalues of the pencil $\lambda\mathcal{E} - \mathcal{A}$ have also negative real part. Let the controllability and observability Gramians of the first-order system (4) be partitioned as

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_p & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_v \end{bmatrix}, \qquad\qquad \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_p & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_v \end{bmatrix},$$

where all blocks are of size $n \times n$. Then $\mathcal{P}_p$ and $\mathcal{P}_v$ are the *position* and *velocity controllability Gramians* of the second-order system (1) and $\mathcal{Q}_p$ and $\mathcal{Q}_v$ are the *position* and *velocity observability Gramians* of (1). An energy interpretation of these Gramians can be found in [3, 10].

Consider the block diagonal matrices

$$\mathcal{W} = \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix}, \qquad \mathcal{T} = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix}, \tag{8}$$

with nonsingular $W_j,\ T_j \in \mathbb{R}^{n,n}$ for $j = 1, 2$. Multiplying system (4), (5) by $\mathcal{W}^T$ from the left and setting $\mathcal{T}^{-1}x(t) = [\,x_1^T(t),\ x_2^T(t)\,]^T$, we obtain the equivalent system

$$\begin{aligned} W_1^T\,T_1\,\dot{x}_1(t) &= \qquad\qquad\qquad W_1^T\,T_2\,x_2(t), \\ W_2^T M T_2 \dot{x}_2(t) &= -W_2^T K T_1 x_1(t) - W_2^T D T_2 x_2(t) + W_2^T B_2 u(t), \\ y(t) &= \ C_1 T_1 x_1(t) + C_2 T_2 x_2(t). \end{aligned} \tag{9}$$

Since $W_1^T T_2$ is nonsingular, we have $x_2(t) = (W_1^T T_2)^{-1} W_1^T T_1 \dot{x}_1(t) = T_2^{-1} T_1 \dot{x}_1(t)$. Substituting this vector in the last two equations in (9) gives the transformed second-order system

$$\begin{aligned} \hat{M}\,\ddot{\hat{q}}(t) + \hat{D}\,\dot{\hat{q}}(t) + \hat{K}\,\hat{q}(t) &= \hat{B}_2 u(t), \\ \hat{C}_2\dot{\hat{q}}(t) + \hat{C}_1\hat{q}(t) &= y(t), \end{aligned} \tag{10}$$

where $\hat{q}(t) = x_1(t) = T_1^{-1}q(t)$, $\hat{M} = W_2^T M T_1$, $\hat{D} = W_2^T D T_1$, $\hat{K} = W_2^T K T_1$, $\hat{B}_2 = W_2^T B_2$, $\hat{C}_1 = C_1 T_1$ and $\hat{C}_2 = C_2 T_1$. Note that systems (1) and (10) have the same transfer function $\boldsymbol{G}(s)$. The position and velocity Gramians of (10) can be computed from the Gramians of (1) as

$$\hat{\mathcal{P}}_p = T_1^{-1}\mathcal{P}_p T_1^{-T}, \qquad \hat{\mathcal{P}}_v = T_2^{-1}\mathcal{P}_v T_2^{-T}, \qquad \hat{\mathcal{Q}}_p = W_1^{-1}\mathcal{Q}_p W_1^{-T}, \qquad \hat{\mathcal{Q}}_v = W_2^{-1}\mathcal{Q}_v W_2^{-T}.$$

Thus, from the equation

$$\hat{\mathcal{P}}_p \hat{M}^T \hat{\mathcal{Q}}_v \hat{M} = T_1^{-1}\mathcal{P}_p M^T \mathcal{Q}_v M T_1,$$

it follows that the eigenvalues of the matrix $\mathcal{P}_p M^T \mathcal{Q}_v M$ do not change under the system transformations $\mathcal{W}$ and $\mathcal{T}$. For the special choice of matrices W_j and T_j in (8) we obtain also the invariance of the eigenvalues of other matrices constructed from the position and velocity Gramians. Such matrices are collected in the following table.

$W_1 = T_1^{-T}$	$\hat{\mathcal{P}}_p \hat{\mathcal{Q}}_p = T_1^{-1}\mathcal{P}_p \mathcal{Q}_p T_1$
$W_2 = T_2^{-T}$	$\hat{\mathcal{P}}_v \hat{\mathcal{Q}}_v = T_2^{-1}\mathcal{P}_v \mathcal{Q}_v T_2$
$W_1 = T_2^{-T}$	$\hat{\mathcal{P}}_v \hat{\mathcal{Q}}_p = T_2^{-1}\mathcal{P}_v \mathcal{Q}_p T_2$
$W_2 = T_1^{-T}$	$\hat{\mathcal{P}}_p \hat{\mathcal{Q}}_v = T_1^{-1}\mathcal{P}_p \mathcal{Q}_v T_1$
$W_1 = W_2$	$\hat{\mathcal{P}}_p \hat{M}^T \hat{\mathcal{Q}}_p \hat{M} = T_1^{-1}\mathcal{P}_p M^T \mathcal{Q}_p M T_1$
$T_1 = T_2$	$\hat{\mathcal{P}}_v \hat{M}^T \hat{\mathcal{Q}}_v \hat{M} = T_1^{-1}\mathcal{P}_v M^T \mathcal{Q}_v M T_1$
$W_1 = W_2$, $T_1 = T_2$	$\hat{\mathcal{P}}_v \hat{M}^T \hat{\mathcal{Q}}_p \hat{M} = T_1^{-1}\mathcal{P}_v M^T \mathcal{Q}_p M T_1$

Using the position and velocity Gramians, we can define different sets of singular values for the second-order system (1).

Definition 2.1. Consider a second-order system (1) with a stable matrix polynomial $\lambda^2 M + \lambda D + K$.

1. The square roots of the eigenvalues of the matrix $\mathcal{P}_p \mathcal{Q}_p$, denoted by ξ_j^p, are called the *position singular values* of (1).
2. The square roots of the eigenvalues of the matrix $\mathcal{P}_v M^T \mathcal{Q}_v M$, denoted by ξ_j^v, are called the *velocity singular values* of (1).
3. The square roots of the eigenvalues of the matrix $\mathcal{P}_p M^T \mathcal{Q}_v M$, denoted by ξ_j^{pv}, are called the *position-velocity singular values* of (1).

We will assume that the position, velocity and position-velocity singular values of (1) are ordered decreasingly, i.e., $\xi_1^p \ge \ldots \ge \xi_n^p \ge 0$, $\xi_1^v \ge \ldots \ge \xi_n^v \ge 0$ and $\xi_1^{pv} \ge \ldots \ge \xi_n^{pv} \ge 0$. One can show that if the second-order system (1) is minimal, i.e., it is controllable and observable [8], then the controllability and observability Gramians $\mathcal{P}$ and $\mathcal{Q}$ of (4) are positive definite. In this case all the singular values of (1) are strictly positive. However, the positivity of ξ_j^p, ξ_j^v and ξ_j^{pv} does not imply that system (1) is minimal.

Definition 2.2. Consider a second-order system (1) with a stable matrix polynomial $\lambda^2 M + \lambda D + K$.

1. System (1) is called *position balanced* if $\mathcal{P}_p$ and $\mathcal{Q}_p$ are equal and diagonal.
2. System (1) is called *velocity balanced* if $\mathcal{P}_v$ and $\mathcal{Q}_v$ are equal and diagonal.
3. System (1) is called *position-velocity balanced* if $\mathcal{P}_p$, $\mathcal{Q}_p$, $\mathcal{P}_v$ and $\mathcal{Q}_v$ are equal and diagonal.

Now we will show that if $P(\lambda) = \lambda^2 M + \lambda D + K$ is stable and system (1) is minimal, then there exist nonsingular matrices $\mathcal{W}$ and $\mathcal{T}$ as in (8) such that the Gramians of the transformed system (10) satisfy

$$\hat{\mathcal{P}}_p = \hat{\mathcal{Q}}_p = \Sigma_1, \qquad \hat{\mathcal{P}}_v = \hat{\mathcal{Q}}_v = \Sigma_2, \tag{11}$$

where Σ_1 and Σ_2 are diagonal. Consider the Cholesky factorizations of the position and velocity Gramians

$$\mathcal{P}_p = R_p R_p^T, \qquad \mathcal{P}_v = R_v R_v^T, \qquad \mathcal{Q}_p = L_p L_p^T, \qquad \mathcal{Q}_v = L_v L_v^T, \tag{12}$$

where R_p, R_v, L_p, $L_v \in \mathbb{R}^{n,n}$ are nonsingular lower triangular Cholesky factors. Then the position singular values of (1) can be computed as the classical singular values of the matrix $R_p^T L_p$. Indeed, we have

$$(\xi_j^p)^2 = \lambda_j(\mathcal{P}_p \mathcal{Q}_p) = \lambda_j(R_p R_p^T L_p L_p^T) = \lambda_j(L_p^T R_p R_p^T L_p) = \sigma_j^2(R_p^T L_p),$$

where $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, the eigenvalues and the singular values. Analogously we can show that the velocity and position-velocity singular values of (1) are the classical singular values of the matrices $R_v^T M^T L_v$ and $R_p^T M^T L_v$, respectively, i.e., $\xi_j^v = \sigma_j(R_v^T M^T L_v)$ and $\xi_j^{pv} = \sigma_j(R_p^T M^T L_v)$. Let

$$R_p^T L_p = U_p \Sigma_p V_p^T, \qquad R_v^T M^T L_v = U_v \Sigma_v V_v^T, \qquad R_p^T M^T L_v = U_{pv} \Sigma_{pv} V_{pv}^T$$

be singular value decompositions of $R_p^T L_p$, $R_v^T M^T L_v$ and $R_p^T M^T L_v$, where U_p, V_p, U_v, V_v, U_{pv} and V_{pv} are orthogonal, and

$$\Sigma_p = \mathrm{diag}(\xi_1^p, \ldots, \xi_n^p), \qquad \Sigma_v = \mathrm{diag}(\xi_1^v, \ldots, \xi_n^v), \qquad \Sigma_{pv} = \mathrm{diag}(\xi_1^{pv}, \ldots, \xi_n^{pv})$$

are nonsingular. Consider the matrices

$$W_1 = L_p V_p \Sigma_1^{-1/2}, \quad T_1 = R_p U_p \Sigma_1^{-1/2}, \quad W_2 = L_v V_v \Sigma_2^{-1/2}, \quad T_2 = R_v U_v \Sigma_2^{-1/2}. \tag{13}$$

One can show that these matrices are nonsingular and the Gramians of the transformed system (10) have the form

$$\begin{aligned} \hat{\mathcal{P}}_p = T_1^{-1}\,\mathcal{P}_p\,T_1^{-T} = \Sigma_1 = W_1^{-1}\mathcal{Q}_p W_1^{-T} = \hat{\mathcal{Q}}_p, \\ \hat{\mathcal{P}}_v = T_2^{-1}\,\mathcal{P}_v\,T_2^{-T} = \Sigma_2 = W_2^{-1}\mathcal{Q}_v W_2^{-T} = \hat{\mathcal{Q}}_v. \end{aligned}$$

Thus, system (10) is position balanced and velocity balanced. For $\Sigma_1 = \Sigma_2$, system (10) is also position-velocity balanced and has the equal position, velocity and position-velocity singular values that coincide with the position-velocity singular values of (1). If we additionally take $\Sigma_1 = \Sigma_p$, then $W_1 = T_1^{-T}$ and, hence, systems (1) and (10) have the same position singular values, but, in general, different velocity singular values. Note that the balancing transformation is not unique. For matrices W_1 and T_2 as in (13) and $T_1 = R_p U_{pv} \Sigma_1^{-1/2}$, $W_2 = L_v V_{pv} \Sigma_2^{-1/2}$, we obtain (11) again.

3. Balanced truncation

Similarly to balanced truncation model reduction of the first-order system (4), the approximate second-order model (3) can be computed by the transformation of system (1) into one of the balanced forms and truncation of the states corresponding to the small singular values.

In summary, we have the following algorithms that are generalizations of the *square root balanced truncation method* [9, 15] for the second-order system (1).

Algorithm 3.1. *Second-order square root method with position balancing.*
Input: $[\,M,\,D,\,K,\,B_2,\,C_1,\,C_2\,]$ *such that* $\lambda^2 M + \lambda D + K$ *is stable.*
Output: *A reduced order system* $[\,\widetilde{M},\,\widetilde{D},\,\widetilde{K},\,\widetilde{B}_2,\,\widetilde{C}_1,\,\widetilde{C}_2\,]$.

1. *Compute the Cholesky factors* R_p, R_v, L_p *and* L_v *of the position and velocity Gramians as in* (12).
2. *Compute the 'thin' singular value decompositions*

$$\begin{aligned} R_p^T L_p &= [\,U_{p1},\,U_{p2}\,]\begin{bmatrix} \Sigma_{p1} & 0 \\ 0 & \Sigma_{p2} \end{bmatrix}[\,V_{p1},\,V_{p2}\,]^T, \\ R_v^T M^T L_v &= [\,U_{v1},\,U_{v2}\,]\begin{bmatrix} \Sigma_{v1} & 0 \\ 0 & \Sigma_{v2} \end{bmatrix}[\,V_{v1},\,V_{v2}\,]^T \end{aligned}$$

where $[\,U_{p1},\,U_{p2}\,]$, $[\,V_{p1},\,V_{p2}\,]$, $[\,U_{v1},\,U_{v2}\,]$ *and* $[\,V_{v1},\,V_{v2}\,]$ *are orthogonal and*

$$\begin{aligned} \Sigma_{p1} = \mathrm{diag}(\xi_1^p, \ldots, \xi_\ell^p), \qquad \Sigma_{p2} = \mathrm{diag}(\xi_{\ell+1}^p, \ldots, \xi_n^p), \\ \Sigma_{v1} = \mathrm{diag}(\xi_1^v, \ldots, \xi_\ell^v), \qquad \Sigma_{v2} = \mathrm{diag}(\xi_{\ell+1}^v, \ldots, \xi_n^v). \end{aligned}$$

3. *Compute the reduced system*

$$\widetilde{M} = W^T M T, \quad \widetilde{D} = W^T D T, \quad \widetilde{K} = W^T K T, \quad \widetilde{B}_2 = W^T B_2, \quad \widetilde{C}_1 = C_1 T, \quad \widetilde{C}_2 = C_2 T$$

with the projection matrices $W = L_v V_{v1} \Sigma_{p1}^{-1/2}$ *and* $T = R_p U_{p1} \Sigma_{p1}^{-1/2}$.

Algorithm 3.2. *Second-order square root method with position-velocity balancing.*
Input: $[\, M,\, D,\, K,\, B_2,\, C_1,\, C_2\,]$ *such that* $\lambda^2 M + \lambda D + K$ *is stable.*
Output: *A reduced order system* $[\, \widetilde{M},\, \widetilde{D},\, \widetilde{K},\, \widetilde{B}_2,\, \widetilde{C}_1,\, \widetilde{C}_2\,]$.

1. *Compute the Cholesky factors* R_p *and* L_v *of the position controllability Gramians* $\mathcal{P}_p$ *and the velocity observability Gramian* $\mathcal{Q}_v$ *as in* (12).
2. *Compute the singular value decomposition*

$$R_p^T M^T L_v = [\, U_{pv,1},\, U_{pv,2}\,] \begin{bmatrix} \Sigma_{pv,1} & 0 \\ 0 & \Sigma_{pv,2} \end{bmatrix} [\, V_{pv,1},\, V_{pv,2}\,]^T,$$

where $[\, U_{pv,1},\, U_{pv,2}\,]$ *and* $[\, V_{pv,1},\, V_{pv,2}\,]$ *are orthogonal and*

$$\Sigma_{pv,1} = \mathrm{diag}(\xi_1^{pv}, \ldots, \xi_\ell^{pv}), \qquad \Sigma_{pv,2} = \mathrm{diag}(\xi_{\ell+1}^{pv}, \ldots, \xi_n^{pv}).$$

3. *Compute the reduced system*

$$\widetilde{M} = W^T M T, \quad \widetilde{D} = W^T D T, \quad \widetilde{K} = W^T K T, \quad \widetilde{B}_2 = W^T B_2, \quad \widetilde{C}_1 = C_1 T, \quad \widetilde{C}_2 = C_2 T$$

with the projection matrices $W = L_v V_{pv,1} \Sigma_{pv,1}^{-1/2}$ *and* $T = R_p U_{pv,1} \Sigma_{pv,1}^{-1/2}$.

Note that the projection matrices W and T in Algorithm 3.1 are up to the factor $\Sigma_{p1}^{-1/2}$ the same as those proposed in [2, Section 3].

4. Numerical examples

In this section we present numerical examples to demonstrate the properties of the presented balanced truncation model reduction methods for the second-order system (1). We consider three models[1]: the building model, the International Space Station (ISS) model and the clamped beam model, see [4] for detailed description. For every model, we compare the reduced first-order system (7) of dimension 2ℓ computed by the balanced truncation (BT) method applied to (4), (5) and the reduced second-order systems of the form (3) of dimension ℓ computed by Algorithm 3.1 (SOBTp) and Algorithm 3.2 (SOBTpv). For comparison, we present

(a) the Hankel singular values ξ_j of the first-order system (7), the position singular values ξ_j^p, the velocity singular values ξ_j^v and the position-velocity singular values ξ_j^{pv} of the second-order system (1);

(b) the eigenvalues of systems (1) and (3);

(c) the spectral norms $\|\boldsymbol{G}(i\omega)\|$, $\|\widetilde{\boldsymbol{G}}(i\omega)\|$ and $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega)\|$ of the frequency responses

$$\begin{aligned} \boldsymbol{G}(i\omega) &= (sC_2 + C_1)(s^2 M + sD + K)^{-1} B_2 = \mathcal{C}(s\,\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}, \\ \widetilde{\boldsymbol{G}}(i\omega) &= (s\widetilde{C}_2 + \widetilde{C}_1)(s^2 \widetilde{M} + s\widetilde{D} + \widetilde{K})^{-1} \widetilde{B}_2, \qquad \widetilde{\boldsymbol{\mathcal{G}}}(i\omega) = \widetilde{\mathcal{C}}(s\,\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}})^{-1}\widetilde{\mathcal{B}} \end{aligned}$$

for the frequency range $\omega \in [\, \omega_{\min},\, \omega_{\max}\,]$;

(d) the absolute errors $\|\widetilde{\boldsymbol{G}}(i\omega) - \boldsymbol{G}(i\omega)\|$ and $\|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega) - \boldsymbol{G}(i\omega)\|$ for the same frequency range and the error bound

$$\|\widetilde{\boldsymbol{\mathcal{G}}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|\widetilde{\boldsymbol{\mathcal{G}}}(i\omega) - \boldsymbol{G}(i\omega)\| \le 2(\xi_{2\ell+1} + \ldots + \xi_{2n}) \tag{14}$$

for the first-order balanced truncation method. Here $\xi_{2\ell+1}, \ldots, \xi_{2n}$ are the truncated Hankel singular values of (4).

[1] Available from `http://www.win.tue.nl/niconet/niconet.html`

Example 4.1. Building model: $n = 24$, $m = 1$, $p = 1$, $\ell = 4$
Figures 1(c) and 1(d) show that for low frequencies the both reduced second-order systems have the better approximate properties than the reduced first-order system, whereas for higher frequencies, all three approximations are about the same. If we compare the reduced second-order systems, we see that Algorithm 3.1 provides only slightly better approximation.

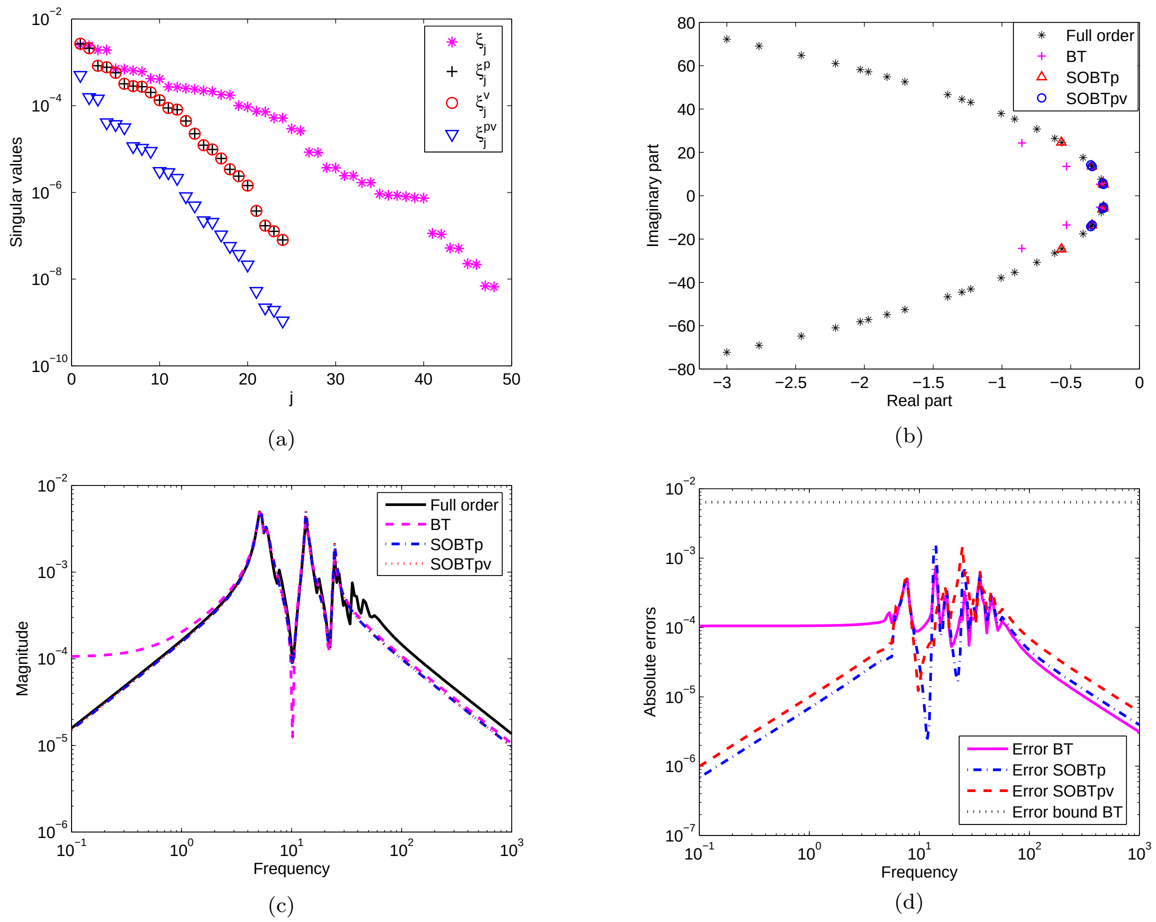


Figure 1: Building model: (a) the Hankel singular values of the first-order system, the position, velocity and position-velocity singular values of the second-order system; (b) the eigenvalues of the full and reduced systems; (c) the frequency responses; (d) the absolute errors and the error bound.

Example 4.2. ISS model: $n = 135$, $m = 3$, $p = 3$, $\ell = 13$
Figure 2(d) demonstrates that the reduced first-order system and the reduced second-order system computed by Algorithm 3.1 have almost the same errors that are smaller for high frequencies than the error for the system computed by Algorithm 3.2. The latter system provides a better approximation for low frequencies.

Example 4.3. Clamped beam model: $n = 174$, $m = 1$, $p = 1$, $\ell = 17$
Figures 4(c) and 4(d) show that the reduced first-order system has better approximation properties than the reduced second-order systems. We also see that the approximation error for the system computed by Algorithm 3.1 is smaller than the error for the system computed by Algorithm 3.2. Finally, this example shows that the error bound (14) does not hold anymore for the second-order balanced truncation methods.

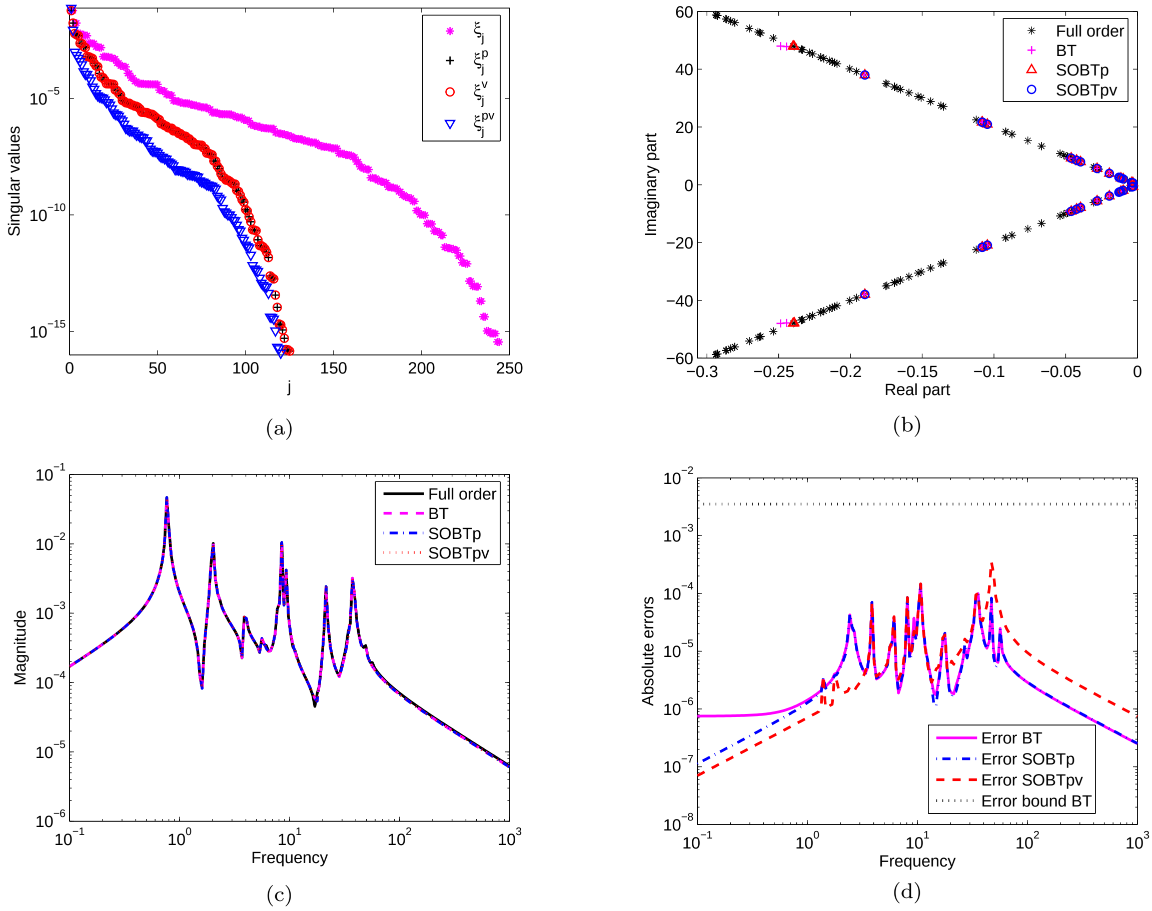


Figure 2: ISS model: (a) the Hankel singular values of the first-order system, the position, velocity and position-velocity singular values of the second-order system; (b) the eigenvalues of the full and reduced systems; (c) the frequency responses; (d) the absolute errors and the error bound.

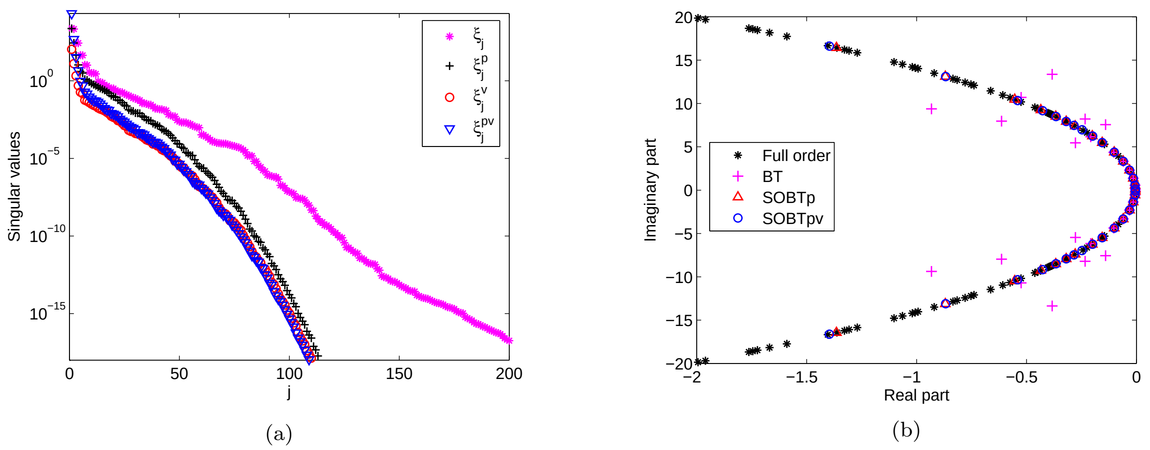
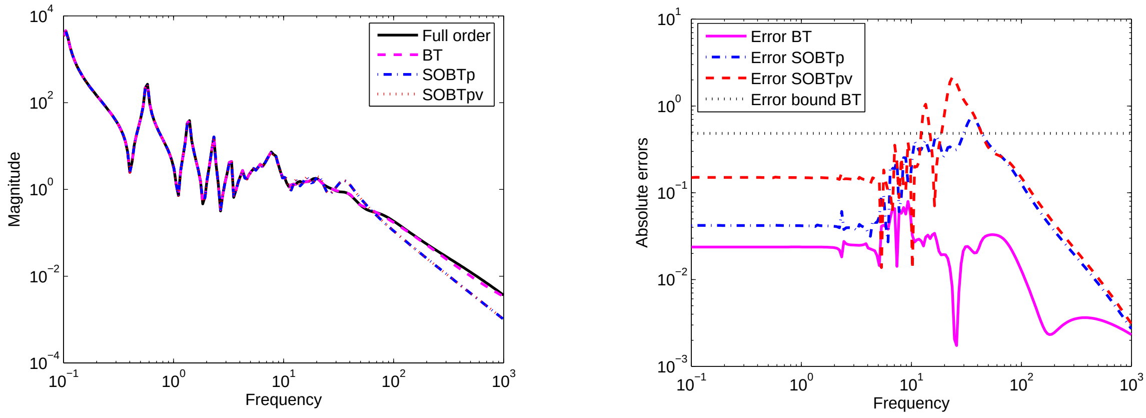


Figure 3: Clamped beam model: (a) the Hankel singular values of the first-order system, the position, velocity and position-velocity singular values of the second-order system; (b) the eigenvalues of the full and reduced systems.



(c) (d)

Figure 4: Clamped beam model: (c) the frequency responses; (d) the absolute errors and the error bound.

5. Conclusions

In this paper we have considered structure-preserving model reduction of second-order systems based on balanced truncation. Using the pairs $(\mathcal{P}_p, \mathcal{Q}_p)$ and $(\mathcal{P}_v, \mathcal{Q}_v)$ of the position and velocity Gramians from [3, 10, 13], we have introduced the position, velocity and position-velocity singular values that can be used to characterize the importance of the position and velocity components. It has been shown that all four Gramians can be diagonalized simultaneously by the coordinate transformation. We have presented two structure-preserving balanced truncation model reduction methods for second-order systems. The questions whether the stability is preserved in the reduced system and whether there exists a global error bound for these methods remain open.

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