On some norms for descriptor systems

Tatjana Stykel

Abstract—We present generalizations of the impulse and frequency responses as well as convolution and Hankel operators for continuous-time and discrete-time descriptor systems. Some norms for descriptor systems are introduced and their representations via different linear system concepts are considered.

Index Terms—Descriptor system, impulse response, frequency response, controllability and observability Gramians, convolution operator, Hankel operator, Hankel singular values, system norms.

I. INTRODUCTION

Consider a linear time-invariant descriptor system

\[ E(Dx(t)) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \]

where \( Dx(t) = \dot{x}(t), \ t \in \mathbb{R}, \) in the continuous-time case and \( Dx(t) = x_{t+1}, \ t \in \mathbb{Z}, \) in the discrete-time case. Here \( E, A \in \mathbb{R}^{n,n}, \)

\( B \in \mathbb{R}^{n,m}, \ C \in \mathbb{R}^{p,n}, \ x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \)

is the control input and \( y(t) \in \mathbb{R}^p \) is the output. Descriptor systems (or generalized state space systems) with singular \( E \) arise naturally in a variety of applications and have been investigated, e.g., in [5], [9], [10], [12]. We will assume that a pencil \( LE - A \) is regular, i.e., \( \det(LE - A) \neq 0 \) for some \( A \in \mathbb{C}. \) In this case a transfer function of (1) is given by \( G(\lambda) = C(LE - A)^{-1}B, \) where \( \lambda = s \) for the continuous-time system and \( \lambda = z \) for the discrete-time system. The transfer function \( G \) is proper if \( \lim_{\lambda \to \infty} G(\lambda) < \infty, \) and improper, otherwise. If \( \lim_{\lambda \to \infty} G(\lambda) = 0, \) then \( \hat{G} \) is said to be strictly proper.

Note that the improper transfer function can be additively decomposed as \( G(\lambda) = G_{sp}(\lambda) + p(\lambda), \) where \( G_{sp} \) is a strictly proper part and \( p \) is a polynomial part of \( G. \)

In many control problems such as model order reduction, robust control, system identification, we need to measure the dynamical systems. Consideration of system norms makes it possible to define the size of descriptor systems and distance between them. For various applications different norms are in use. If the transfer function \( G \) is (strictly) proper, then system norms [1], [6] known for standard state space systems \((E = I)\) can also be used for the descriptor system (1). However, to the author’s knowledge, norms for descriptor systems with the improper transfer function have not been considered in the literature so far. Such systems arise, for instance, in dynamical system inversion, PID-controller design, modeling of economic processes and mechanical systems with controlled constraints [4], [9], [10].

A possible approach to define the norm of improper \( G \) is to consider the norm of a weighted transfer function \( G_w(\lambda) = \frac{1}{p} G(\lambda) \), which is proper for \( k \geq d \) with \( d \) being the degree of the polynomial part of \( G. \) Since \( d \) is, in general, unknown, we may take \( k = n. \) In this case standard algorithms can be used to compute the norm of \( G_w. \)

It should be noted, however, that these algorithms employ usually state space representations, so the computation of the state space realization of \( G_w \) is required.

In this paper we consider different norms for descriptor systems that can be computed using given generalized state space representation (1) of \( G. \) We also give equivalent characterizations of these norms in terms of important linear system concepts like impulse and frequency responses, controllability and observability Gramians, convolution operators, Hankel operators and closely related Hankel singular values. Possible applications of considered system norms are \( H_2 \) and \( H_{\infty} \) control for descriptor systems as well as model reduction.

Throughout the paper we will denote by \( \mathbb{Z} \) the set of integers, by \( \mathbb{i} \) the imaginary axis and by \( \mathbb{I} \) the unit circle. The matrix \( A^T \) stands for the transpose of \( A. \) We will denote by \( \lambda_j(\cdot) \) and \( \sigma_j(\cdot), \) respectively, eigenvalues and singular values of a matrix or a linear operator ordered decreasingly. The trace and the image of \( A \) are denoted by \( \text{tr}(A) \) and \( \text{Im}(A), \) respectively. We will denote by \( \|A\|_2 \) the spectral matrix norm and by \( \|A\|_{\text{Fro}} \) the Frobenius matrix norm of \( A \in \mathbb{R}^{m \times m}. \)

II. DISCRETE-TIME DESCRIPTOR SYSTEMS

Since the results for the continuous-time case are partly related to the discrete-time case, we begin our discussion with the discrete-time descriptor system

\[ E x_{k+1} = A x_k + B u_k, \quad y_k = C x_k. \]

A regular pencil \( LE - A \) can be reduced to the Weierstrass canonical form

\[ E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_s} \end{bmatrix} T, \]

where \( W \) and \( T \) are nonsingular, \( I_{n_f} \) an identity matrix of order \( m, \) \( J \) and \( N \) are in Jordan canonical form and \( N \) is nilpotent with index of nilpotence \( n_s. \) The numbers \( n_f \) and \( n_s \) are the dimensions of the deflating subspaces of \( LE - A \) corresponding to the finite and infinite eigenvalues, respectively. The descriptor system (2) is called asymptotically stable if the pencil \( LE - A \) is d-stable, i.e., all the finite eigenvalues of \( LE - A \) lie inside the unit circle.

Using (3), the transfer function \( G(z) = C(zE-A)^{-1}B \) of (2) can be expanded into a Laurent series around \( z = \infty \) as

\[ G(z) = \sum_{k=-\infty}^{\infty} C F_{k-1} B z^{-k}, \]

where the matrices \( F_k \) have the form

\[ F_{k} = T^{-1} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \quad k \geq 0, \]

\[ F_{-k} = T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{k-1} \end{bmatrix} W^{-1}, \quad k > 0. \]

A sequence \( \{G_k\}_{k \in \mathbb{Z}} \) with \( G_k = CF_{k-1}B \) defines an impulse response of the descriptor system (2). Observe that \( G_k = 0 \) for \( k \leq -\nu. \) As in the standard state space case, a frequency response of the discrete-time descriptor system (2) is given by the values of the transfer function on the unit circle \( G(e^{j\omega}). \) We have

\[ G(e^{j\omega}) = \sum_{k=-\infty}^{\infty} G_k e^{-j\omega k}, \]

i.e., \( \{G_k\}_{k \in \mathbb{Z}} \) is a sequence of the Fourier coefficients of the frequency response \( G(e^{j\omega}). \)

A. Gramians and Hankel singular values

Assume that the pencil \( LE - A \) is d-stable. Then the causal controllability and observability Gramians of the descriptor system (2) are defined via

\[ G_{drc} = \sum_{k=0}^{\infty} F_k B B^T F_k^T, \quad G_{dco} = \sum_{k=0}^{\infty} F_k^T C^T C F_k, \]

respectively, see [2], [12]. The matrices

\[ G_{dnc} = \sum_{k=-\nu}^{-1} F_k B B^T F_k^T, \quad G_{dno} = \sum_{k=-\nu}^{-1} F_k^T C^T C F_k \]

Institut für Mathematik, MA 3-3, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany, e-mail: stykel@math.tu-berlin.de. This work was supported by the DFG Research Center Matheon in Berlin.
are the non-causal controllability and observability Gramians of (2).

Note that these Gramians are, up to the sign, the same as in [2]. It has been shown in [12] that the Gramians are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations

\[ A_{\text{dec}}^T E^T - E_{\text{dec}} A^T - P_T B B^T P_T^T, \quad P_T A_{\text{dec}} P_T^T = A_{\text{dec}}, \]

\[ A_{\text{dec}}^T E^T - E_{\text{dec}} A^T - P_T C C^T P_T^T, \quad P_T A_{\text{dec}} P_T^T = A_{\text{dec}}, \]

\[ A_{\text{dec}}^T E^T - E_{\text{dec}} A^T - Q_{\text{dec}} B B^T Q_{\text{dec}}^T, \quad Q_{\text{dec}} A_{\text{dec}} Q_{\text{dec}}^T = A_{\text{dec}}, \]

\[ A_{\text{dec}}^T E^T - E_{\text{dec}} A^T = Q_T C C^T Q_T, \quad Q_T A_{\text{dec}} Q_T = A_{\text{dec}}, \]

where \( P_T \) and \( P_C \) are the spectral projectors onto the left and right deflating subspaces of the pencil \( A - E \) corresponding to the finite eigenvalues, \( Q_T = I - P_T \) and \( Q_C = I - P_C \).

Let \( \Phi_{\text{d}} = E_{\text{dec}} E^T \) and \( \Psi_{\text{d}} = A_{\text{dec}}^T E_{\text{dec}} A^T A_{\text{dec}} \). One can show that \( \Phi_{\text{d}} \) and \( \Psi_{\text{d}} \) are simultaneously diagonalizable and all their eigenvalues are real and non-negative.

**Definition 2.1:** The square roots of the \( n_f \) largest eigenvalues of \( \Phi_{\text{d}} \), denoted by \( \gamma_j \), are called the causal Hankel singular values of system (2). The square roots of the \( n_{\infty} \) largest eigenvalues of \( \Psi_{\text{d}} \), denoted by \( \theta_j \), are called the non-causal Hankel singular values of (2).

Similarly to the continuous-time case [13], the causal and non-causal Gramians and Hankel singular values can be used in balanced truncation model reduction for discrete-time descriptor systems. Since the Gramians are symmetric and positive semidefinite, there exist full rank factorizations

\[ G_{\text{dec}} = R^T C R, \quad G_{\text{dec}} = L^T \Lambda C L, \quad G_{\text{dec}} = R^T \Lambda R, \quad G_{\text{dec}} = L^T \Lambda L, \]

where \( R \), \( L \), \( L \), \( R \) are full column rank factors. The following lemma gives a connection between the Hankel singular values and the singular values of the matrices \( L, E, R, \) and \( L, A, R, \).

**Lemma 2.2:** Let \( A - E \) be d-stable. Consider the full rank factorizations (12). The non-zero causal Hankel singular values of system (2) are the singular values of the matrix \( L, E, R, \) while the non-zero non-causal Hankel singular values of (2) are the singular values of \( L, A, R, \).

**Proof:** We have

\[ \gamma_j^2 = \lambda_j(R^T C^T L^T R C E) = \lambda_j(R^T C^T L^T R C E) = \sigma_j^2(L, E, R), \]

Similarly, we can show that \( \theta_j = \sigma_j(L, A, R). \)

**B. System norms**

In this subsection we generalize convolution and Hankel operators [1] to the discrete-time descriptor system (2). Moreover, we extend some known system norms [1], [6] to (2) and establish their connection with the Gramians, the matrices \( \Phi_{\text{d}} \) and \( \Psi_{\text{d}} \), the convolution and Hankel operators as well as the discrete-time Hankel singular values. In the following we will assume that the pencil \( A - E \) is d-stable.

**L\(_2\)-norm:** Let \( C^p, m \) be the Hilbert space of matrix-valued functions \( F : \Gamma \to C^p, m \) that have bounded \( L_2(\Gamma) \)-norm

\[ \| F \|_{L_2(\Gamma)} = \left( \frac{1}{2\pi} \int_0^{2\pi} \| F(e^{i\omega}) \|^2_F \, d\omega \right)^{1/2}. \]

Consider also the Hilbert space \( L_2^p(\mathbb{Z}) \) of matrix-valued sequences \( S = \{ S_k \}_{k \in \mathbb{Z}} \), \( S_k \in \mathbb{R}^{p, m} \), that have bounded \( L_2^p(\mathbb{Z}) \)-norm

\[ \| S \|_{L_2^p(\mathbb{Z})} = \left( \sum_{k = -\infty}^{\infty} \| S_k \|^2_F \right)^{1/2}. \]

By Parseval’s identity [11] we find from relation (6) that \( |G|_{L_2^p(\mathbb{Z})} = |G|_{L_2^p(\mathbb{Z})} \), where \( G \) is the transfer function and \( G = (G_k)_{k \in \mathbb{Z}} \) is the impulse response of (2). Furthermore, we get

\[ \| G \|^2_{L_2^p(\mathbb{Z})} = \text{tr}(B^T (G_{\text{dec}} + G_{\text{dnc}}) B) = \text{tr}(C (G_{\text{dec}} + G_{\text{dnc}}) C^T). \]

These relations lead to a simple numerical algorithm for computing the \( L_2^p(\Gamma) \)-norm of the transfer function \( G \). Consider the full rank factorizations \( G_{\text{dec}} = R^T C \), \( G_{\text{dnc}} = L^T \). Then

\[ \| G \|^2_{L_2^p(\mathbb{Z})} = \| LR \|_F = \| CR \|_F \]

Note that the full rank factors \( R \) and \( L \) can be determined from the Lyapunov equations (8) – (11) without computing the Gramians explicitly, see [12].

**L\(_2\)-norm:** Let \( L_2^p(\mathbb{Z}) \) be the Banach space of matrix-valued functions that (essentially) bounded on \( \Gamma \). The \( L_2^p(\Gamma) \)-norm of \( G \) is defined by \( |G|_{L_2^p(\Gamma)} = \sup_{\omega \in [0, 2\pi]} \| G(e^{i\omega}) \|_2 \). Consider a convolution operator \( K_c : L_2^p(\mathbb{Z}) \to L_2^p(\mathbb{Z}) \) for the discrete-time descriptor system (2) that maps the inputs \( u \) into the outputs \( y \). This operator is defined via

\[ y_k = (K_c u)_k = \sum_{j = -\infty}^{k-1} G_{k-j} u_j. \]

For the column vectors \( y = [\cdots, y_{-1}, y_0, y_1, \cdots]^T \) and \( u = [\cdots, u_{-1}^T, u_0^T, u_1^T, \cdots]^T \), this relation can be rewritten as a linear system \( y = K_c u \), where

\[ K_c = \begin{bmatrix} \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix} \]

is the matrix representation of the convolution operator. We see that the operator \( K_c \) has block Toeplitz structure and gives an input-output relationship in the time domain. The spectral norm of \( K_c \) is given by \( \| K_c \|_2 = \sup_{\omega \in [0, 2\pi]} \| K_c u \|_2 / \| u \|_2 \). By Parseval’s identity [11] we have \( |G|_{L_2^p(\Gamma)} = \| K_c \|_2 \). Thus, the \( L_2^p(\Gamma) \)-norm of \( G \) can be interpreted as a ratio of the output energy to the input energy of the descriptor system (2). For computing the \( L_2^p(\Gamma) \)-norm of \( G \) we can use an algorithm from [7], [8].

The Hilbert-Schmidt norm and the Hankel norm: Let \( Z \) and \( Z \) denote the sets of negative and non-negative integers, respectively. A causal Hankel operator \( \mathcal{H}_c : L_2^p(Z^-) \to L_2^p(Z_0) \) for the descriptor system (2) is defined via

\[ y_k = (\mathcal{H}_c u)_k = \sum_{j = -\infty}^{k-1} G_{k-j} u_j, \quad k \geq 0. \]

A non-causal Hankel operator \( \mathcal{H}_n : L_2^p(Z_0) \to L_2^p(Z^-) \) for (2) is given by

\[ y_k = (\mathcal{H}_n u)_k = \sum_{j = 0}^{k} G_{j} u_{j+1}, \quad k < 0. \]

For the vectors \( y = [y_0, y_1, \cdots]^T \) and \( u = [u_0^T, u_1^T, \cdots]^T \), relations (14) and (15) can be written as the linear systems \( y_k = \mathcal{H}_c u_k \) and \( y_k = \mathcal{H}_n u_k \), respectively, where the Hankel matrices

\[ \mathcal{H}_c = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

and \( \mathcal{H}_n = \begin{bmatrix} \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix} \)
are the matrix representations of the causal and non-causal Hankel operators. The operator $\mathcal{H}_c$ maps past inputs ($u_k = 0$, $k \leq 0$) to present and future outputs ($y_k = 0$, $k < 0$), whereas the operator $\mathcal{H}_n$ maps present and future inputs ($u_k = 0$, $k < 0$) to past outputs ($y_k = 0$, $k \geq 0$).

We will now establish a connection between the singular values of the Hankel operators $\mathcal{H}_c$, $\mathcal{H}_n$ and the Hankel singular values of system (2).

**Theorem 2.3:** Consider a discrete-time descriptor system (2), where a pencil $\lambda E - A$ is stable. The causal and non-causal Hankel operators $\mathcal{H}_c$ and $\mathcal{H}_n$ as in (16) have the finite set of non-zero singular values that coincide with the non-causal and non-causal Hankel singular values of (2), respectively.

**Proof:** Using (3) and (4), we obtain that $F_j E F_k = F_{j+k}$ for all $j$, $k \geq 0$. Then the causal Hankel operator can be represented as $\mathcal{H}_c = [C_+ E C_+^T]$, where $C_+ = [F_0 B, \ldots, F_k B \ldots]$ and $O_+ = [F_0^T C^T, \ldots, F_k^T C^T, \ldots]^T$. Hence, $\varsigma_1^2 = \sigma_1^2(\mathcal{H}_c)$. Similarly, we can prove that $\theta_1 = \sigma_1(\mathcal{H}_n)$.

A Hilbert-Schmidt norm (HS-norm) of the transfer function $G$ is defined via

\[ \|G\|_{HS}^2 = \sum_{k=1}^{\infty} \|G_k\|^2_F + \|G_{-k+1}\|^2_F. \]

It follows from (16) and Theorem 2.3 that

\[ \|G\|_{HS}^2 = \|H_c\|^2_F + \|H_n\|^2_F = \varsigma_1^2 + \ldots + \varsigma_k^2 + \theta_1^2 + \ldots + \theta_{\infty}^2. \]

**A Hankel norm of the transfer function $G$ is defined via**

\[ \|G\|_H = \max(\|H_c\|_2, \|H_n\|_2) = \max(\varsigma_1, \theta_1), \]

where $\varsigma_1$ and $\theta_1$ are the largest causal and non-causal Hankel singular values of (2), respectively. We have $\|G\|_H = \sqrt{\lambda_{\max}(\Phi_d + \Psi_d)}$.

To compute the HS-norm and the Hankel norm of the transfer function $G$ we can solve the generalized Lyapunov equations (8) – (11) for the full rank factors $R_c$, $L_c$, $R_n$ and $L_n$ as in (12) using the generalized Schur-Hammarling method [12]. Then by Lemma 2.2 we find that

\[ \|G\|_{HS} = \max(\|L_c E R_c\|_2, \|L_n A R_n\|_2) \quad \text{and} \quad \|G\|_H = \max(\|L_c E R_c\|_2, \|L_n A R_n\|_2). \]

We summarize the considered norms for the asymptotically stable discrete-time descriptor system (2) in Table I.

In the remainder of this section we establish a connection among different system norms. It follows from (17)–(19) that $\|G\|_{L_2,\infty} = \max(|\varsigma_1|, |\theta_1|)$ is defined via

\[ \|G\|_{L_2,\infty} = \max(\|H_c\|_2, \|H_n\|_2). \]

Furthermore, taking into account the matrix representations of the convolution operator and the Hankel operators, we get

\[ \|G\|_H \leq \|G\|_{L_2,\infty} \leq \|G\|_{L_2,\infty} \leq \|G\|_{H} \leq \|G\|_L \leq \|G\|_2. \]

**III. CONTINUOUS-TIME DESCRIPTOR SYSTEMS**

In this section we consider the continuous-time descriptor system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t). \]

Although there are differences between the continuous-time and discrete-time descriptor systems, some linear system concepts are similar. Therefore, to avoid repetition, results for (20) are only listed without proof unless necessary.

The continuous-time descriptor system (20) is called asymptotically stable if the pencil $\lambda E - A$ is c-stable, that is, all the finite eigenvalues of $\lambda E - A$ have negative real part. An impulse response of the continuous-time descriptor system (20) is defined via

\[ G(t) = CF(t)B + \sum_{k=0}^{\infty} CF_{-k}B\delta(t), \quad t \geq 0, \]

where the matrices $F_k$ are as in (5), $\delta(t)$ is the delta function and $F(t)$ is the fundamental solution matrix of (20) given by

\[ F(t) = T^{-1} \begin{bmatrix} e^{t\Omega} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}. \]
A frequency response of the continuous-time descriptor system (20) is given by \( G(\lambda\omega) \), i.e., the values of \( G(s) = C(sE - A)^{-1}B \) on the imaginary axis. From (21) we obtain that \( G(\lambda\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} G(t) \, dt \).

Therefore, the frequency response \( G(\lambda\omega) \) is just the Fourier transform of the impulse response \( G(t) \).

### A. Gramians and Hankel singular values

Assume that the pencil \( \lambda E - A \) is c-stable. Then the proper controllability and observability Gramians of the continuous-time descriptor system (20) are defined via

\[
G_{cpc} = \int_0^\infty F(t) B B^T F(t) \, dt, \quad G_{cpo} = \int_0^\infty F^T(t) C^T C F(t) \, dt.
\]

It has been shown in [12] that the proper Gramians are the unique symmetric, positive semidefinite solutions of the projected generalized continuous-time Lyapunov equations

\[
E G_{cpc} A^T + A G_{cpc} E = - P_1 B B^T P_1^T, \quad G_{cpc} = P_1 G_{cpc} P_1^T, \quad E^T G_{cpc} A + A^T G_{cpc} E = - P_1^T C^T C P_1, \quad G_{cpo} = P_1^T G_{cpo} P_1.
\]

The improper controllability Gramian \( G_{isc} \) and the improper observability Gramian \( G_{isc} \) of the continuous-time system (20) coincide with the non-causal controllability and observability Gramians of the discrete-time system (2) given in (7).

Similarly to the discrete-time case, the proper and improper Hankel singular values of system (20) are defined via \( c_j = \sqrt{\lambda_j} \Phi_j(\lambda_j) \), \( j = 1, \ldots, n_f \), and \( \theta_j = \sqrt{\lambda_j} \Psi_j(\lambda_j) \), \( j = 1, \ldots, n_w \), respectively, where \( \Phi_j = G_{cpc} E \Psi_j E \) and \( \Psi_j = G_{isc} A \).

### B. System norms

In this subsection we introduce convolution and Hankel operators for the continuous-time descriptor system (20). We also consider system norms for (20) and establish their connection with the frequency response \( G(t) \), the Gramians, the convolution and Hankel operators and the Hankel singular values of (20).

**\( H_2 \)-norm and \( H_\infty \)-norm:** Let \( L_{p,m}^2(\mathbb{C}) \) be the space of transfer functions \( F : i\mathbb{R} \rightarrow \mathbb{C}^p \) that have bounded \( L_{p,m}^2(\mathbb{C}) \)-norm

\[
\|F\|_{L_{p,m}^2(\mathbb{C})} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|F(i\omega)\|^2 \, d\omega \right)^{1/2}.
\]

The subspace \( H_{\infty}^2 \) of \( L_{p,m}^2(\mathbb{C}) \) consists of all strictly proper rational functions that are analytic in the closed right half-plane. The \( H_\infty \)-norm of the transfer function \( G \) is defined via

\[
\|G\|_{H_\infty} = \sup_{u \in \mathbb{R}} \|G(u)\|_{L_{p,m}^2(\mathbb{C})}.
\]

The following lemma gives an upper bound on the \( H_\infty \)-norm of \( G \).

**Lemma 3.1:** Consider a transfer function \( G(s) = G_{sp}(s) + P(s) \), where \( G_{sp} \) and \( P \) are as in (22). Let \( G_{sp}(s) = G_{sp}(s) + G_0 \in H_{\infty} \). Let \( L_{p,m}^2(\mathbb{C}) \) be the space of vector-valued functions \( F : i\mathbb{R} \rightarrow \mathbb{C}^m \) that have bounded \( L_{p,m}^2(\mathbb{C}) \)-norm

\[
\|F\|_{L_{p,m}^2(\mathbb{C})} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=0}^{\nu-1} \|G_{-k}\|^2 \, d\omega \right)^{1/2}.
\]

The subspace \( H_{\infty}^2 \) of \( L_{p,m}^2(\mathbb{C}) \) consists of all strictly proper rational functions that are analytic in the closed right half-plane. The \( H_\infty \)-norm of the transfer function \( G \) is defined via

\[
\|G\|_{H_\infty} = \sup_{u \in \mathbb{R}} \|G(u)\|_{L_{p,m}^2(\mathbb{C})}.
\]

The following lemma gives an upper bound on the \( H_\infty \)-norm of \( G \).

**Lemma 3.1:** Consider a transfer function \( G(s) = G_{sp}(s) + P(s) \), where \( G_{sp} \) and \( P \) are as in (22). Let \( G_{sp}(s) = G_{sp}(s) + G_0 \) be the proper part of \( G \). We have

\[
\|G\|_{H_\infty} \leq \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=0}^{\nu-1} \|G_{-k}\|^2 \, d\omega \right)^{1/2}.
\]

Proof: For any \( u \in L_{p,m}^2(-1,1) \), we obtain

\[
\|G(u)\|_{L_{p,m}^2(\mathbb{C})} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=0}^{\nu-1} |\omega|^{2k} \|u(\omega)\|^2 \, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=0}^{\nu-1} |\omega|^{2k} \|u(\omega)\|^2 \, d\omega \leq \left( \|G_{sp}\|_{H_{\infty}} + \sum_{k=0}^{\nu-1} \|G_{-k}\|^2 + \|P\|_{L_{p,m}^2(\mathbb{C})} \right)^{1/2}.
\]
Thus, estimate (23) holds.

Note that if the transfer function \( G(s) = G_u(s) \) is proper, then the equality in (23) holds.

For the continuous-time descriptor system (20), we consider a convolution operator \( K_c \) that maps the input \( u(t) \) into the output \( y(t) \). This operator is defined via

\[
y(t) = (K_c u)(t) = \int_{-\infty}^{\infty} G(t - \tau) u(\tau) \, d\tau.
\]

It describes the input-output behavior of the descriptor system (20) in the time domain. Substituting (21) in (24), we find that

\[
(K_c u)(t) = \int_{-\infty}^{\infty} C F(t - \tau) B u(\tau) \, d\tau + \sum_{k=0}^{n-1} C F_{-k-1} B u(k),
\]

where \( u(k) \) are the distributional derivatives.

Let \( \mathbb{L}_{2,\infty}^n(\mathbb{R}) \) denote the Sobolev space consisting of vector-valued functions \( f : \mathbb{R} \to \mathbb{R}^m \) such that \( j f(k) \in \mathbb{L}_{2,\infty}^n(\mathbb{R}) \) with \( k = 0, 1, \ldots, l \). The \( \mathbb{H}_\infty \)-norm of the improper transfer function \( G \) can be computed by the method proposed in [3]. Computing the \( \mathbb{H}_\infty \)-norm of the improper transfer function \( G \) is still an open problem.

The Hilbert-Schmidt norm and the Hankel norm: For system (20), we define a proper Hankel operator \( \mathcal{H}_p \) transforming the past inputs \( u_-(t) \) into the present and future outputs \( y_+(t) \) through the state \( x(0) \) in \( \text{Im}(P_c) \) via

\[
y_+(t) = (\mathcal{H}_p u_-(t)) = \int_{0}^{\infty} G_{sp}(t - \tau) u_-(\tau) \, d\tau, \quad t \geq 0.
\]

If \( \Lambda E - A \) is c-stable, then \( \mathcal{H}_p \) acts from \( \mathbb{L}_{2,\infty}^n(\mathbb{R}^-) \) into \( \mathbb{L}_2(\mathbb{R}_+^m) \).

Theorem 3.2: Consider system (20), where \( \Lambda E - A \) is c-stable. The non-zero proper Hankel singular values of (20) are the non-zero Hankel matrix \( \mathcal{H}_{\infty} \), given in (16).

Remark 3.3: Note that the proper Hankel singular values of the continuous-time descriptor system (20) are not equal to the singular values of the causal Hankel matrix \( \mathcal{H}_c \) as in (16). However, as in the discrete-time case, the non-zero improper Hankel singular values coincide with the classical non-zero singular values of the non-causal Hankel matrix \( \mathcal{H}_c \), defined in (16).

A Hilbert-Schmidt norm (HS-norm) of the transfer function \( G \) is defined by

\[
\| G \|_{HS} = \left( \int_{0}^{\infty} \int_{0}^{\infty} \| G_{sp}(t + \tau) \|^2 \, dt \, d\tau + \sum_{k=0}^{n-1} \| G_{-k-1} \|^2 \right)^{1/2},
\]

where \( G_{sp}(t) = C F(t) B, G_{-k-1} = C F_{-k-1} B. \) Taking into account that \( F(t + \tau) = F(t) E F(\tau) \) and \( F_{-k-1} = - F_{-k-1} A F_{-k-1} \) for \( j, k \geq 0 \), we obtain that

\[
\int_{0}^{\infty} \int_{0}^{\infty} \| G_{sp}(t + \tau) \|^2 \, dt \, d\tau = \text{tr}(G_{sp}E^T G_{sp} E) = \text{tr}(\Phi_e),
\]

where \( \text{tr}(\Phi_e) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \| G_{-j-k} \|^2 \),

Hence,

\[
\| G \|_H = \| \Phi_e + \Psi_e \| = \sum_{j=0}^{\infty} \| \Phi^{(j)}_e \|^2 = \sum_{j=0}^{\infty} \| \Phi^{(j)}_e \|^2 + \sum_{j=0}^{\infty} \| \Phi^{(j)}_e \|^2,
\]

As a consequence of Theorem 3.2 and Remark 3.3 we have

\[
\| G \|_H = \| \mathcal{H}_p \|_F + \| \mathcal{N}_n \|^2 = \| L_p R_p \|_F + \| L_1 A R_1 \|_F^2,
\]

where \( L_p, L_1, R_p, R_1 \) are the full rank factors of the Gramians \( G_{pc} = R_p^T L_p, G_{cpo} = L_p^T R_p, G_{cie} = R_1^T L_1, G_{ci} = L_1^T L_1 \).

A Hankel norm of the transfer function \( G \) is defined by

\[
\| G \|_H = \max(\| \mathcal{H}_p \|_2, \| \mathcal{N}_n \|_2) = \max(\varsigma_1, \theta_1),
\]

where \( \varsigma_1 \) and \( \theta_1 \) are the largest proper and improper Hankel singular values of the descriptor system (20). From the definition of the Hankel singular values we find that

\[
\| G \|_H = \sqrt{\lambda_{\max}(\Phi_e + \Psi_e)} = \max(\| L_p R_p \|_2, \| L_1 A R_1 \|_2).
\]

We summarize system norms for the asymptotically stable continuous-time descriptor system (20) in Table II.

REFERENCES

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 51, NO. 5, 842–847, MAY 2006

TABLE II
GENERALIZED NORMS FOR ASYMPTOTICALLY STABLE CONTINUOUS-TIME DESCRIPTOR SYSTEMS

| G(s) = C(sE − A)−1B | |G|| [H∞] | |G|| ||H|| |
|-------------------------|-----------------|-----------------|-----------------|
| G(ω) = Gsp(ω) + P(ω) | \( \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G_{sp}(\omega)\|^2 d\omega + \frac{1}{2\pi} \int_{0}^{2\pi} \|P(e^{j\omega})\|^2 d\omega \right)^{1/2} \) | \( \sup_{\omega \in \Gamma} \|G\|_{L_p}^{1/2} \) | \( \|K\|_{2} \) |
| Gsp(t), Gk | \( \left( \int_{0}^{\infty} \|G_{sp}(t)\|^2 dt + \sum_{k=1}^{n} \|G_{-k+1}\|^2 \right)^{1/2} \) | \( \|G\|_{H^\infty} \) | \( \max(\|H_{p}\|_{2}, \|H_{n}\|_{2}) \) |
| Gsp, Gc, \( \theta \geq \theta_{n} \) | | | |
| \( \sqrt{\|H_{p}\|^2 + \|H_{n}\|^2} \) | \( \max(\|L_{p}E_{p}\|_{2}, \|L_{n}A_{n}\|_{2}) \) | \( \sqrt{\lambda_{\max}(\Phi_{c} + \Psi_{c})} \) | \( \max(\theta_{1}, \theta_{n}) \) |


