On some norms for descriptor systems

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***Abstract*— We present generalizations of the impulse and frequency responses as well as convolution and Hankel operators for continuous-time and discrete-time descriptor systems. Some norms for descriptor systems are introduced and their representations via the different linear system concepts are considered.**

***Index Terms*— Descriptor system, impulse response, frequency response, controllability and observability Gramians, convolution operator, Hankel operator, Hankel singular values, system norms.**

I. INTRODUCTION

Consider a linear time-invariant descriptor system

$$E(\mathcal{D}x(t)) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

where $\mathcal{D}x(t) = \dot{x}(t)$, $t \in \mathbb{R}$, in the continuous-time case and $\mathcal{D}x(t) = x_{t+1}$, $t \in \mathbb{Z}$, in the discrete-time case. Here E, $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input and $y(t) \in \mathbb{R}^p$ is the output. Descriptor systems (or *generalized state space systems*) with singular E arise naturally in a variety of applications and have been investigated, e.g., in [5], [9], [10], [12]. We will assume that a pencil $\lambda E - A$ is regular, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. In this case a *transfer function* of (1) is given by $\mathbf{G}(\lambda) = C(\lambda E - A)^{-1}B$, where $\lambda = s$ for the continuous-time system and $\lambda = z$ for the discrete-time system. The transfer function $\mathbf{G}$ is *proper* if $\lim_{\lambda\to\infty} \mathbf{G}(\lambda) < \infty$, and *improper*, otherwise. If $\lim_{\lambda\to\infty} \mathbf{G}(\lambda) = 0$, then $\mathbf{G}$ is said to be *strictly proper*. Note that the improper transfer function can be additively decomposed as $\mathbf{G}(\lambda) = \mathbf{G}_{sp}(\lambda) + \mathbf{P}(\lambda)$, where $\mathbf{G}_{sp}$ is a strictly proper part and $\mathbf{P}$ is a polynomial part of $\mathbf{G}$.

In many control problems such as model order reduction, robust control, system identification, we need to measure the dynamical systems. Consideration of system norms makes it possible to define the size of descriptor systems and distance between them. For various applications different norms are in use. If the transfer function $\mathbf{G}$ is (strictly) proper, then system norms [1], [6] known for standard state space systems ($E = I$) can also be used for the descriptor system (1). However, to the author's knowledge, norms for descriptor systems with the improper transfer function have not been considered in the literature so far. Such systems arise, for instance, in dynamical system inversion, PID-controller design, modeling of economic processes and mechanical systems with controlled constraints [4], [9], [10]. A possible approach to define the norm of improper $\mathbf{G}$ is to consider the norm of a weighted transfer function $\mathbf{G}_k(\lambda) = \frac{1}{\lambda^k}\mathbf{G}(\lambda)$ which is proper for $k \geq d$ with d being the degree of the polynomial part of $\mathbf{G}$. Since d is, in general, unknown, we may take $k = n$. In this case standard algorithms can be used to compute the norm of $\mathbf{G}_k$. It should be noted, however, that these algorithms employ usually state space representations, so the computation of the state space realization of $\mathbf{G}_k$ is required.

In this paper we consider different norms for descriptor systems that can be computed using given generalized state space representation (1) of $\mathbf{G}$. We also give equivalent characterizations of these norms in terms of important linear system concepts like impulse and frequency responses, controllability and observability Gramians, convolution operators, Hankel operators and closely related Hankel singular values. Possible applications of considered system norms are $\mathbb{H}_2$ and $\mathbb{H}_\infty$ control for descriptor systems as well as model reduction.

Throughout the paper we will denote by $\mathbb{Z}$ the set of integers, by $i\mathbb{R}$ the imaginary axis and by $\mathbb{T}$ the unit circle. The matrix A^T stands for the transpose of A. We will denote by $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$, respectively, eigenvalues and singular values of a matrix or a linear operator ordered decreasingly. The trace and the image of A are denoted by $\mathrm{tr}(A)$ and $\mathrm{Im}(A)$, respectively. We will denote by $\|A\|_2$ the spectral matrix norm and by $\|A\|_F$ the Frobenius matrix norm of $A \in \mathbb{R}^{n,m}$.

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II. DISCRETE-TIME DESCRIPTOR SYSTEMS

Since the results for the continuous-time case are partly related to the discrete-time case, we begin our discussion with the discrete-time descriptor system

$$Ex_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k. \tag{2}$$

A regular pencil $\lambda E - A$ can be reduced to the Weierstrass canonical form

$$E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T, \tag{3}$$

where W and T are nonsingular, I_m is an identity matrix of order m, J and N are in Jordan canonical form and N is nilpotent with index of nilpotence ν. The numbers n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The descriptor system (2) is called *asymptotically stable* if the pencil $\lambda E - A$ is *d-stable*, i.e., all the finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

Using (3), the transfer function $\mathbf{G}(z) = C(zE - A)^{-1}B$ of (2) can be expanded into a Laurent series around $z = \infty$ as

$$\mathbf{G}(z) = \sum_{k=-\infty}^{\infty} CF_{k-1}Bz^{-k},$$

where the matrices F_k have the form

$$F_k = T^{-1} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \qquad k \geq 0, \tag{4}$$

$$F_{-k} = T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{k-1} \end{bmatrix} W^{-1}, \quad k > 0. \tag{5}$$

A sequence $\{G_k\}_{k\in\mathbb{Z}}$ with $G_k = CF_{k-1}B$ defines an *impulse response* of the descriptor system (2). Observe that $G_k = 0$ for $k \leq -\nu$. As in the standard state space case, a *frequency response* of the discrete-time descriptor system (2) is given by the values of the transfer function on the unit circle $\mathbf{G}(e^{i\omega})$. We have

$$\mathbf{G}(e^{i\omega}) = \sum_{k=-\infty}^{\infty} G_k e^{-i\omega k}, \tag{6}$$

i.e., $\{G_k\}_{k\in\mathbb{Z}}$ is a sequence of the Fourier coefficients of the frequency response $\mathbf{G}(e^{i\omega})$.

A. Gramians and Hankel singular values

Assume that the pencil $\lambda E - A$ is d-stable. Then the *causal controllability* and *observability Gramians* of the descriptor system (2) are defined via

$$\mathcal{G}_{dcc} = \sum_{k=0}^{\infty} F_k BB^T F_k^T, \qquad \mathcal{G}_{dco} = \sum_{k=0}^{\infty} F_k^T C^T CF_k,$$

respectively, see [2], [12]. The matrices

$$\mathcal{G}_{dnc} = \sum_{k=-\nu}^{-1} F_k BB^T F_k^T, \qquad \mathcal{G}_{dno} = \sum_{k=-\nu}^{-1} F_k^T C^T CF_k \tag{7}$$

are the *non-causal controllability* and *observability Gramians* of (2). Note that these Gramians are, up to the sign, the same as in [2]. It has been shown in [12] that the Gramians are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations

$$A\mathcal{G}_{dcc}A^T - E\mathcal{G}_{dcc}E^T = -P_lBB^TP_l^T,\ \ P_r\mathcal{G}_{dcc}P_r^T = \mathcal{G}_{dcc}, \quad (8)$$

$$A^T\mathcal{G}_{dco}A - E^T\mathcal{G}_{dco}E = -P_r^TC^TCP_r,\ \ P_l^T\mathcal{G}_{dco}P_l = \mathcal{G}_{dco}, \quad (9)$$

$$A\mathcal{G}_{dnc}A^T - E\mathcal{G}_{dnc}E^T = Q_lBB^TQ_l^T,\ \ Q_r\mathcal{G}_{dnc}Q_r^T = \mathcal{G}_{dnc}, \quad (10)$$

$$A^T\mathcal{G}_{dno}A - E^T\mathcal{G}_{dno}E = Q_r^TC^TCQ_r,\ \ Q_l^T\mathcal{G}_{dno}Q_l = \mathcal{G}_{dno}, \quad (11)$$

where P_l and P_r are the spectral projectors onto the left and right deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues, $Q_l = I - P_l$ and $Q_r = I - P_r$.

Let $\Phi_d = \mathcal{G}_{dcc}E^T\mathcal{G}_{dco}E$ and $\Psi_d = \mathcal{G}_{dnc}A^T\mathcal{G}_{dno}A$. One can show that Φ_d and Ψ_d are simultaneously diagonalizable and all their eigenvalues are real and non-negative.

Definition 2.1: The square roots of the n_f largest eigenvalues of Φ_d, denoted by ς_j, are called the *causal Hankel singular values* of system (2). The square roots of the n_∞ largest eigenvalues of Ψ_d, denoted by θ_j, are called the *non-causal Hankel singular values* of (2).

Similarly to the continuous-time case [13], the causal and non-causal Gramians and Hankel singular values can be used in balanced truncation model reduction for discrete-time descriptor systems. Since the Gramians are symmetric and positive semidefinite, there exist full rank factorizations

$$\mathcal{G}_{dcc} = R_cR_c^T,\ \mathcal{G}_{dco} = L_c^TL_c,\ \mathcal{G}_{dnc} = R_nR_n^T,\ \mathcal{G}_{dno} = L_n^TL_n, \quad (12)$$

where R_c, L_c^T, R_n, L_n^T are full column rank factors. The following lemma gives a connection between the Hankel singular values and the singular values of the matrices L_cER_c and L_nAR_n.

Lemma 2.2: Let $\lambda E - A$ be d-stable. Consider the full rank factorizations (12). The non-zero causal Hankel singular values of system (2) are the singular values of the matrix L_cER_c, while the non-zero non-causal Hankel singular values of (2) are the singular values of L_nAR_n.

Proof: We have

$$\varsigma_j^2 = \lambda_j(R_cR_c^TE^TL_c^TL_cE) = \lambda_j(R_c^TE^TL_c^TL_cER_c) = \sigma_j^2(L_cER_c).$$

Similarly, we can show that $\theta_j = \sigma_j(L_nAR_n)$. ∎

B. System norms

In this subsection we generalize convolution and Hankel operators [1] to the discrete-time descriptor system (2). Moreover, we extend some known system norms [1], [6] to (2) and establish their connection with the Gramians, the matrices Φ_d and Ψ_d, the convolution and Hankel operators as well the Hankel singular values. In the following we will assume that the pencil $\lambda E - A$ is d-stable.

$\mathbb{L}_2^{p,m}(\Gamma)$*-norm:* Let $\mathbb{L}_2^{p,m}(\Gamma)$ be the Hilbert space of matrix-valued functions $\mathbf{F}:\Gamma \to \mathbb{C}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(\Gamma)$-norm

$$\|\mathbf{F}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \left(\frac{1}{2\pi}\int_0^{2\pi}\|\mathbf{F}(e^{i\omega})\|_F^2\,d\omega\right)^{1/2}. \quad (13)$$

Consider also the Hilbert space $\mathbb{l}_2^{p,m}(\mathbb{Z})$ of matrix-valued sequences $S = \{S_k\}_{k\in\mathbb{Z}}$, $S_k \in \mathbb{R}^{p,m}$, that have bounded $\mathbb{l}_2^{p,m}(\mathbb{Z})$-norm

$$\|S\|_{\mathbb{l}_2^{p,m}(\mathbb{Z})} = \left(\sum_{k=-\infty}^{\infty}\|S_k\|_F^2\right)^{1/2}.$$

By Parseval's identity [11] we find from relation (6) that $\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \|G\|_{\mathbb{l}_2^{p,m}(\mathbb{Z})}$, where $\mathbf{G}$ is the transfer function and $G = \{G_k\}_{k\in\mathbb{Z}}$ is the impulse response of (2). Furthermore, we get

$$\|\mathbf{G}\|^2_{\mathbb{L}_2^{p,m}(\Gamma)} = \mathrm{tr}\big(B^T(\mathcal{G}_{dco}+\mathcal{G}_{dno})B\big) = \mathrm{tr}\big(C(\mathcal{G}_{dcc}+\mathcal{G}_{dnc})C^T\big).$$

These relations lead to a simple numerical algorithm for computing the $\mathbb{L}_2^{p,m}(\Gamma)$-norm of the transfer function $\mathbf{G}$. Consider the full rank factorizations $\mathcal{G}_{dcc}+\mathcal{G}_{dnc} = RR^T$, $\mathcal{G}_{dco}+\mathcal{G}_{dno} = L^TL$. Then $\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \|LB\|_F = \|CR\|_F$. Note that the full rank factors R and L can be determined from the Lyapunov equations (8) – (11) without computing the Gramians explicitly, see [12].

$\mathbb{L}_\infty^{p,m}(\Gamma)$*-norm:* Let $\mathbb{L}_\infty^{p,m}(\Gamma)$ be the Banach space of matrix-valued functions that are (essentially) bounded on Γ. The $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm of $\mathbf{G}$ is defined by $\|\mathbf{G}\|_{\mathbb{L}_\infty^{p,m}(\Gamma)} = \operatorname*{ess\,sup}_{\omega\in[0,2\pi]}\|\mathbf{G}(e^{i\omega})\|_2$. Consider a *convolution operator* $\mathcal{K}_d : \mathbb{l}_2^m(\mathbb{Z}) \to \mathbb{l}_2^p(\mathbb{Z})$ for the discrete-time descriptor system (2) that maps the inputs u_k into the outputs y_k. This operator is defined via

$$y_k = (\mathcal{K}_du)_k = \sum_{j=-\infty}^{k+\nu-1} G_{k-j}u_j.$$

For the column vectors $y = \big[\cdots,\ y_{-1}^T,\ y_0^T,\ y_1^T,\ \cdots\big]^T$ and $u = \big[\cdots,\ u_{-1}^T,\ u_0^T,\ u_1^T,\ \cdots\big]^T$, this relation can be rewritten as a linear system $y = \mathcal{K}_du$, where

$$\mathcal{K}_d = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & G_0 & G_{-1} & G_{-2} & \cdots \\ \cdots & G_1 & G_0 & G_{-1} & \cdots \\ \cdots & G_2 & G_1 & G_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is the matrix representation of the convolution operator. We see that the operator $\mathcal{K}_d$ has block Toeplitz structure and gives an input-output relationship in the time domain. The spectral norm of $\mathcal{K}_d$ is given by $\|\mathcal{K}_d\|_2 = \sup_{u\neq 0}\|\mathcal{K}_du\|_{\mathbb{l}_2^p(\mathbb{Z})}/\|u\|_{\mathbb{l}_2^m(\mathbb{Z})}$. By Parseval's identity [11] we have $\|\mathbf{G}\|_{\mathbb{L}_\infty^{p,m}(\Gamma)} = \|\mathcal{K}_d\|_2$. Thus, the $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm of $\mathbf{G}$ can be interpreted as a ratio of the output energy to the input energy of the descriptor system (2). For computing the $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm of $\mathbf{G}$ we can use an algorithm from [7], [8].

The Hilbert-Schmidt norm and the Hankel norm: Let $\mathbb{Z}^-$ and $\mathbb{Z}_0^+$ denote the sets of negative and non-negative integers, respectively. A *causal Hankel operator* $\mathcal{H}_c : \mathbb{l}_2^m(\mathbb{Z}^-) \to \mathbb{l}_2^p(\mathbb{Z}_0^+)$ for the descriptor system (2) is defined via

$$y_k = (\mathcal{H}_cu)_k = \sum_{j=-\infty}^{-1} G_{k-j}u_j, \qquad k \ge 0. \quad (14)$$

A *non-causal Hankel operator* $\mathcal{H}_n : \mathbb{l}_2^p(\mathbb{Z}_0^+) \to \mathbb{l}_2^m(\mathbb{Z}^-)$ for (2) is given by

$$y_k = (\mathcal{H}_nu)_k = \sum_{j=0}^{\infty} G_{k-j+1}u_j, \qquad k < 0. \quad (15)$$

For the vectors $y_+ = \big[\,y_0^T,\ y_1^T,\ \ldots\big]^T$, $y_- = \big[\,\ldots,\ y_{-2}^T,\ y_{-1}^T\big]^T$, $u_+ = \big[\,\ldots,\ u_1^T,\ u_0^T\big]^T$ and $u_- = \big[\,u_{-1}^T,\ u_{-2}^T,\ \ldots\big]^T$, relations (14) and (15) can be written as the linear systems $y_+ = \mathcal{H}_cu_-$ and $y_- = \mathcal{H}_nu_+$, respectively, where the Hankel matrices

$$\mathcal{H}_c = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad \mathcal{H}_n = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & G_{-4} & G_{-3} & G_{-2} \\ \cdots & G_{-3} & G_{-2} & G_{-1} \\ \cdots & G_{-2} & G_{-1} & G_0 \end{bmatrix} \quad (16)$$

TABLE I
GENERALIZED NORMS FOR ASYMPTOTICALLY STABLE DISCRETE-TIME DESCRIPTOR SYSTEMS

$\mathbf{G}(z)=C(zE-A)^{-1}B$	$\Vert\mathbf{G}\Vert_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}$	$\Vert\mathbf{G}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}$
$\mathbf{G}(e^{i\omega})$	$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\Vert\mathbf{G}(e^{i\omega})\Vert_F^2\,d\omega\right)^{\frac{1}{2}}$	$\sup\limits_{\omega\in\mathbb{R}}\Vert\mathbf{G}(e^{i\omega})\Vert_2$
G_k	$\left(\sum\limits_{k=-\infty}^{\infty}\Vert G_k\Vert_F^2\right)^{\frac{1}{2}}$	
$\mathcal{G}_{dcc}+\mathcal{G}_{dnc}=RR^T$	$\sqrt{\mathrm{tr}\big(C(\mathcal{G}_{dcc}+\mathcal{G}_{dcc})C^T\big)}=\Vert CR\Vert_F$	
$\mathcal{G}_{dco}+\mathcal{G}_{dno}=L^TL$	$\sqrt{\mathrm{tr}\big(B^T(\mathcal{G}_{dco}+\mathcal{G}_{dno})B\big)}=\Vert LB\Vert_F$	
$\mathcal{K}_d$		$\Vert\mathcal{K}_d\Vert_2$
$\mathbf{G}(z)=C(zE-A)^{-1}B$	$\Vert\mathbf{G}\Vert_{HS}$	$\Vert\mathbf{G}\Vert_H$
G_k	$\left(\sum\limits_{k=1}^{\infty}k\left(\Vert G_k\Vert_F^2+\Vert G_{-k+1}\Vert_F^2\right)\right)^{\frac{1}{2}}$	
$\mathcal{H}_c,\quad\mathcal{H}_n$	$\sqrt{\Vert\mathcal{H}_c\Vert_F^2+\Vert\mathcal{H}_n\Vert_F^2}$	$\max(\Vert\mathcal{H}_c\Vert_2,\Vert\mathcal{H}_n\Vert_2)$
$\mathcal{G}_{dcc}=R_cR_c^T,\ \mathcal{G}_{dco}=L_c^TL_c$ $\mathcal{G}_{dnc}=R_nR_n^T,\ \mathcal{G}_{dno}=L_n^TL_n$	$\Vert[L_cER_c,\ L_nAR_n]\Vert_F$	$\max(\Vert L_cER_c\Vert_2,\Vert L_nAR_n\Vert_2)$
$\Phi_d,\quad\Psi_d$	$\sqrt{\mathrm{tr}(\Phi_d+\Psi_d)}$	$\sqrt{\lambda_{\max}(\Phi_d+\Psi_d)}$
$\varsigma_1\geq\ldots\geq\varsigma_{n_f},\quad\theta_1\geq\ldots\geq\theta_{n_\infty}$	$\sqrt{\varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2}$	$\max(\varsigma_1,\,\theta_1)$

are the matrix representations of the causal and non-causal Hankel operators. The operator $\mathcal{H}_c$ maps past inputs ($u_k=0,\ k\geq 0$) to present and future outputs ($y_k=0,\ k<0$), whereas the operator $\mathcal{H}_n$ maps present and future inputs ($u_k=0,\ k<0$) to past outputs ($y_k=0,\ k\geq 0$).

We will now establish a connection between the singular values of the Hankel operators $\mathcal{H}_c$, $\mathcal{H}_n$ and the Hankel singular values of system (2).

Theorem 2.3: Consider a discrete-time descriptor system (2), where a pencil $\lambda E-A$ is d-stable. The causal and non-causal Hankel operators $\mathcal{H}_c$ and $\mathcal{H}_n$ as in (16) have the finite set of non-zero singular values that coincide with the non-zero causal and non-causal Hankel singular values of (2), respectively.

Proof: Using (3) and (4), we obtain that $F_jEF_k=F_{j+k}$ for all $j,k\geq 0$. Then the causal Hankel operator can be represented as $\mathcal{H}_c=\mathbf{O}_+E\mathbf{C}_+$, where $\mathbf{C}_+=[\,F_0B,\,\ldots,\,F_kB\,\ldots]$ and $\mathbf{O}_+=[\,F_0^TC^T,\,\ldots,\,F_k^TC^T,\,\ldots]^T$. Hence, $\varsigma_j^2=\sigma_j^2(\mathbf{O}_+E\mathbf{C}_+)=\sigma_j^2(\mathcal{H}_c)$. Similarly, we can prove that $\theta_j=\sigma_j(\mathcal{H}_n)$. ∎

A *Hilbert-Schmidt norm* (*HS-norm*) of the transfer function $\mathbf{G}$ is defined via

$$\begin{aligned}\Vert\mathbf{G}\Vert_{HS}^2&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\Vert G_{j+k+1}\Vert_F^2+\Vert G_{-j-k}\Vert_F^2\right)\\&=\sum_{k=1}^{\infty}k\left(\Vert G_k\Vert_F^2+\Vert G_{-k+1}\Vert_F^2\right).\end{aligned}\tag{17}$$

It follows from (16) and Theorem 2.3 that

$$\begin{aligned}\Vert\mathbf{G}\Vert_{HS}^2&=\Vert\mathcal{H}_c\Vert_F^2+\Vert\mathcal{H}_n\Vert_F^2=\varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2\\&=\mathrm{tr}(\Phi_d+\Psi_d).\end{aligned}\tag{18}$$

A *Hankel norm* of the transfer function $\mathbf{G}$ is defined via

$$\Vert\mathbf{G}\Vert_H=\max(\Vert\mathcal{H}_c\Vert_2,\Vert\mathcal{H}_n\Vert_2)=\max(\varsigma_1,\,\theta_1),\tag{19}$$

where ς_1 and θ_1 are the largest causal and non-causal Hankel singular values of (2), respectively. We have $\Vert\mathbf{G}\Vert_H=\sqrt{\lambda_{\max}(\Phi_d+\Psi_d)}$.

To compute the HS-norm and the Hankel norm of the transfer function $\mathbf{G}$ we can solve the generalized Lyapunov equations (8) – (11) for the full rank factors R_c, L_c, R_n and L_n as in (12) using the generalized Schur-Hammarling method [12]. Then by Lemma 2.2 we find that $\Vert\mathbf{G}\Vert_{HS}=\Vert[\,L_cER_c,\,L_nAR_n\,]\Vert_F$ and $\Vert\mathbf{G}\Vert_H=\max(\Vert L_cER_c\Vert_2,\Vert L_nAR_n\Vert_2)$.

We summarize the considered norms for the asymptotically stable discrete-time descriptor system (2) in Table I.

In the remainder of this section we establish a connection among different system norms. It follows from (17)–(19) that $\Vert\mathbf{G}\Vert_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}\leq\Vert\mathbf{G}\Vert_{HS}$ and $\Vert\mathbf{G}\Vert_H\leq\Vert\mathbf{G}\Vert_{HS}\leq\sqrt{n}\Vert\mathbf{G}\Vert_H$. Furthermore, taking into account the matrix representations of the convolution operator and the Hankel operators, we get

$$\Vert\mathbf{G}\Vert_H\leq\Vert\mathbf{G}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}\leq\Vert\mathbf{G}_{sp}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}+\Vert\mathbf{P}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})},$$

where $\mathbf{G}_{sp}(z)=\sum\limits_{k=1}^{\infty}G_kz^{-k}$ and $\mathbf{P}(z)=\sum\limits_{k=0}^{\nu-1}G_{-k}z^k$ are the strictly proper and polynomial parts of $\mathbf{G}$. As in the standard state space case [6], we have an estimate $\Vert\mathbf{G}_{sp}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}\leq 2(\varsigma_1+\ldots+\varsigma_{n_f})$. Furthermore, a transfer function $\mathbf{G}_0(z)=-\frac{1}{z}\mathbf{P}(\frac{1}{z})$ is strictly proper and has only zero poles. Clearly, $\mathbf{G}_0$ and $\mathbf{P}$ have the same Hankel singular values that are just the improper Hankel singular values θ_j of (2). Then $\Vert\mathbf{P}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}=\Vert\mathbf{G}_0\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}\leq 2(\theta_1+\ldots+\theta_{n_\infty})$. Hence, $\Vert\mathbf{G}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}\leq 2(n_f\varsigma_1+n_\infty\theta_1)\leq 2n\Vert\mathbf{G}\Vert_H$. Thus, the $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm, the HS-norm and the Hankel norm of the asymptotically stable discrete-time descriptor system (2) are equivalent.

III. Continuous-time descriptor systems

In this section we consider the continuous-time descriptor system

$$E\dot{x}(t)=Ax(t)+Bu(t),\qquad y(t)=Cx(t).\tag{20}$$

Although there are differences between the continuous-time and discrete-time descriptor systems, some linear system concepts are similar. Therefore, to avoid repetition, results for (20) are only listed without proof unless necessary.

The continuous-time descriptor system (20) is called *asymptotically stable* if the pencil $\lambda E-A$ is *c-stable*, that is, all the finite eigenvalues of $\lambda E-A$ have negative real part. An *impulse response* of the continuous-time descriptor system (20) is defined via

$$G(t)=C\mathcal{F}(t)B+\sum_{k=0}^{\nu-1}CF_{-k-1}B\delta^{(k)}(t),\qquad t\geq 0,\tag{21}$$

where the matrices F_k are as in (5), $\delta(t)$ is the delta function and $\mathcal{F}(t)$ is the *fundamental solution matrix* of (20) given by

$$\mathcal{F}(t)=T^{-1}\begin{bmatrix}e^{tJ}&0\\0&0\end{bmatrix}W^{-1}.$$

A *frequency response* of the continuous-time descriptor system (20) is given by $\mathbf{G}(i\omega)$, i.e., the values of $\mathbf{G}(s)=C(sE-A)^{-1}B$ on the imaginary axis. From (21) we obtain that $\mathbf{G}(i\omega)=\int_{-\infty}^{\infty} e^{-i\omega t}G(t)dt$. Therefore, the frequency response $\mathbf{G}(i\omega)$ is just the Fourier transform of the impulse response $G(t)$.

A. Gramians and Hankel singular values

Assume that the pencil $\lambda E - A$ is c-stable. Then the *proper controllability* and *observability Gramians* of the continuous-time descriptor system (20) are defined via

$$\mathcal{G}_{cpc} = \int_0^\infty \mathcal{F}(t)BB^T\mathcal{F}^T(t)\,dt, \quad \mathcal{G}_{cpo} = \int_0^\infty \mathcal{F}^T(t)C^TC\mathcal{F}(t)\,dt.$$

It has been shown in [12] that the proper Gramians are the unique symmetric, positive semidefinite solutions of the projected generalized continuous-time Lyapunov equations

$$\begin{aligned} E\mathcal{G}_{cpc}A^T + A\mathcal{G}_{cpc}E^T &= -P_lBB^TP_l^T, \quad \mathcal{G}_{cpc} = P_r\mathcal{G}_{cpc}P_r^T,\\ E^T\mathcal{G}_{cpo}A + A^T\mathcal{G}_{cpo}E &= -P_r^TC^TCP_r, \quad \mathcal{G}_{cpo} = P_l^T\mathcal{G}_{cpo}P_l. \end{aligned}$$

The *improper controllability Gramian* $\mathcal{G}_{cic}$ and the *improper observability Gramian* $\mathcal{G}_{cio}$ of the continuous-time system (20) coincide with the non-causal controllability and observability Gramians of the discrete-time system (2) given in (7).

Similarly to the discrete-time case, the *proper* and *improper Hankel singular values* of system (20) are defined via $\varsigma_j = \sqrt{\lambda_j(\Phi_c)}$, $j = 1,\ldots,n_f$, and $\theta_j = \sqrt{\lambda_j(\Psi_c)}$, $j = 1,\ldots,n_\infty$, respectively, where $\Phi_c = \mathcal{G}_{cpc}E^T\mathcal{G}_{cpo}E$ and $\Psi_c = \mathcal{G}_{cic}A^T\mathcal{G}_{cio}A$. The proper and improper Gramians and Hankel singular values play an important role in balanced truncation model reduction for continuous-time descriptor systems, see [13] for details.

B. System norms

In this subsection we introduce convolution and Hankel operators for the continuous-time descriptor system (20). We also consider system norms for (20) and establish their connection with the frequency response $G(t)$, the Gramians, the matrices Φ_c and Ψ_c, the convolution and Hankel operators and the Hankel singular values of (20).

$\mathbb{H}_2$*-norm and* $\mathbb{HL}_2$*-norm:* Let $\mathbb{L}_2^{p,m}(i\mathbb{R})$ be the Hilbert space of matrix-valued functions $\mathbf{F}: i\mathbb{R} \to \mathbb{C}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(i\mathbb{R})$-norm

$$\|\mathbf{F}\|_{\mathbb{L}_2^{p,m}(i\mathbb{R})} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathbf{F}(i\omega)\|_F^2\,d\omega\right)^{1/2}.$$

The subspace $\mathbb{H}_2$ of $\mathbb{L}_2^{p,m}(i\mathbb{R})$ consists of all strictly proper rational functions that are analytic in the closed right half-plane. The $\mathbb{H}_2$*-norm* of the transfer function $\mathbf{G}$ of (20) coincides with the $\mathbb{L}_2^{p,m}(i\mathbb{R})$-norm. If the pencil $\lambda E - A$ is c-stable and $\mathbf{G}$ is strictly proper, then $\mathbf{G} \in \mathbb{H}_2$. However, the condition $\mathbf{G} \in \mathbb{H}_2$ does not imply that $\lambda E - A$ is c-stable. Note that improper $\mathbf{G}$ does not belong to $\mathbb{L}_2^{p,m}(i\mathbb{R})$ even if the pencil $\lambda E - A$ is c-stable.

Consider an additive decomposition of the transfer function $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where

$$\mathbf{G}_{sp}(s) = \sum_{k=1}^{\infty} G_k s^{-k} \qquad \text{and} \qquad \mathbf{P}(s) = \sum_{k=0}^{\nu-1} G_{-k}s^k \tag{22}$$

are, respectively, the *strictly proper part* and the *polynomial part* of $\mathbf{G}$, and $G_k = CF_{k-1}B$ are the *Markov parameters* of the descriptor system (20). We denote by $\mathbb{HL}_2$ the space of transfer functions $\mathbf{G}$ such that $\mathbf{G}_{sp}(s) \in \mathbb{H}_2$. The $\mathbb{HL}_2$*-norm* of $\mathbf{G}$ is defined via

$$\|\mathbf{G}\|_{\mathbb{HL}_2} = \sqrt{\|\mathbf{G}_{sp}\|_{\mathbb{H}_2}^2 + \|\mathbf{P}\|_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}^2},$$

where $\|\cdot\|_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}$ is as in (13).

Let $\mathbb{I}$ denote either $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}_0^+ = [0,\infty)$. Consider the Hilbert space $\mathbb{L}_2^{p,m}(\mathbb{I})$ of matrix-valued functions $F: \mathbb{I} \to \mathbb{R}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(\mathbb{I})$-norm

$$\|F\|_{\mathbb{L}_2^{p,m}(\mathbb{I})} = \left(\int_{\mathbb{I}} \|F(t)\|_F^2\,dt\right)^{1/2}.$$

Using Parseval's identity [11] in the continuous-time and discrete-time case, we get

$$\|\mathbf{G}\|_{\mathbb{HL}_2}^2 = \int_0^\infty \|G_{sp}(t)\|_F^2\,dt + \sum_{k=0}^{\nu-1}\|G_{-k}\|_F^2.$$

Moreover, just as in the discrete-time case, we have

$$\begin{aligned} \|\mathbf{G}_{sp}\|_{\mathbb{H}_2}^2 &= \mathrm{tr}\big(B^T\mathcal{G}_{cpo}B\big) = \mathrm{tr}\big(C\mathcal{G}_{cpc}C^T\big),\\ \|\mathbf{P}\|_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}^2 &= \mathrm{tr}\big(B^T\mathcal{G}_{cio}B\big) = \mathrm{tr}\big(C\mathcal{G}_{cic}C^T\big) \end{aligned}$$

and, hence,

$$\begin{aligned} \|\mathbf{G}\|_{\mathbb{HL}_2}^2 &= \mathrm{tr}\big(B^T(\mathcal{G}_{cpo}+\mathcal{G}_{cio})B\big) = \mathrm{tr}\big(C(\mathcal{G}_{cpc}+\mathcal{G}_{cic})C^T\big)\\ &= \|LB\|_F^2 = \|CR\|_F^2, \end{aligned}$$

where R and L are the full rank factors of $\mathcal{G}_{cpo} + \mathcal{G}_{cio} = RR^T$ and $\mathcal{G}_{cpc} + \mathcal{G}_{cic} = L^TL$.

$\mathbb{H}_\infty$*-norm and* $\mathbb{HL}_\infty$*-norm:* Let $\mathbb{L}_\infty^{p,m}(i\mathbb{R})$ be the Banach space of matrix-valued functions that are (essentially) bounded on $i\mathbb{R}$. The subspace of $\mathbb{L}_\infty^{p,m}(i\mathbb{R})$, denoted by $\mathbb{H}_\infty$, consists of all proper rational functions that are analytic and bounded in the closed right half-plane. The $\mathbb{H}_\infty$*-norm* of the proper transfer function $\mathbf{G}$ is defined via

$$\|\mathbf{G}\|_{\mathbb{H}_\infty} = \sup_{\mathbf{u}\neq 0}\frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\|\mathbf{u}\|_{\mathbb{L}_2^m(i\mathbb{R})}} = \sup_{\omega\in\mathbb{R}}\|\mathbf{G}(i\omega)\|_2.$$

Let $\mathbb{HL}_\infty$ denote a space of transfer functions $\mathbf{G}(s) = \mathbf{G}_{sp}(s)+\mathbf{P}(s)$ with the proper part $\mathbf{G}_p(s) = \mathbf{G}_{sp}(s) + G_0 \in \mathbb{H}_\infty$. Let $\mathbb{L}_{2,l}^m(i\mathbb{R})$ be the space of vector-valued functions $\mathbf{f}: i\mathbb{R} \to \mathbb{C}^m$ that have bounded $\mathbb{L}_{2,l}^m(i\mathbb{R})$-norm

$$\|\mathbf{f}\|_{\mathbb{L}_{2,l}^m(i\mathbb{R})} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sum_{k=0}^{l}|\omega|^{2k}\Big)\|\mathbf{f}(i\omega)\|^2 d\omega\right)^{1/2}.$$

The $\mathbb{HL}_\infty$*-norm* of the transfer function $\mathbf{G}$ is defined via

$$\|\mathbf{G}\|_{\mathbb{HL}_\infty} = \sup_{\mathbf{u}\neq 0}\frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\|\mathbf{u}\|_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}}.$$

The following lemma gives an upper bound on the $\mathbb{HL}_\infty$-norm of $\mathbf{G}$.

Lemma 3.1: Consider a transfer function $\mathbf{G}(s) = \mathbf{G}_{sp}(s)+\mathbf{P}(s)$, where $\mathbf{G}_{sp}$ and $\mathbf{P}$ are as in (22). Let $\mathbf{G}_p(s) = \mathbf{G}_{sp}(s) + G_0$ be the proper part of $\mathbf{G}$. We have

$$\|\mathbf{G}\|_{\mathbb{HL}_\infty} \le \left(\|\mathbf{G}_p\|_{\mathbb{H}_\infty}^2 + \sum_{k=1}^{\nu-1}\|G_{-k}\|_2^2\right)^{1/2}. \tag{23}$$

Proof: For any $\mathbf{u} \in \mathbb{L}_{2,\nu-1}^m(i\mathbb{R})$, we obtain

$$\begin{aligned} \|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}^2 &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\|\mathbf{G}_p(i\omega)\|_2^2\sum_{k=0}^{\nu-1}|\omega|^{2k}\|\mathbf{u}(i\omega)\|^2 d\omega\\ &\quad+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{k=1}^{\nu-1}\|G_{-k}\|_2^2\right)\sum_{k=0}^{\nu-1}|\omega|^{2k}\|\mathbf{u}(i\omega)\|^2 d\omega\\ &\le \left(\|\mathbf{G}_p\|_{\mathbb{H}_\infty}^2 + \sum_{k=1}^{\nu-1}\|G_{-k}\|_2^2\right)\|\mathbf{u}\|_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}^2. \end{aligned}$$

Thus, estimate (23) holds. ■

Note that if the transfer function $\mathbf{G}(s) = \mathbf{G}_p(s)$ is proper, then the equality in (23) holds.

For the continuous-time descriptor system (20), we consider a *convolution operator* $\mathcal{K}_c$ that maps the input $u(t)$ into the output $y(t)$. This operator is defined via

$$y(t) = (\mathcal{K}_c u)(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)\,d\tau. \qquad (24)$$

It describes the input-output behavior of the descriptor system (20) in the time domain. Substituting (21) in (24), we find that

$$(\mathcal{K}_c u)(t) = \int_{-\infty}^{t} C\mathcal{F}(t-\tau)Bu(\tau)\,d\tau + \sum_{k=0}^{\nu-1} CF_{-k-1}Bu^{(k)}(t),$$

where $u^{(k)}(t)$ are the distributional derivatives.

Let $\mathbb{L}_{2,l}^m(\mathbb{R})$ denote the Sobolev space consisting of vector-valued functions $f:\mathbb{R}\to\mathbb{R}^m$ such that $f^{(k)}(t)\in\mathbb{L}_2^m(\mathbb{R})$, $k=0,1,\ldots,l$. The $\mathbb{L}_{2,l}^m(\mathbb{R})$-norm is defined via

$$\|f\|_{\mathbb{L}_{2,l}^m(\mathbb{R})} = \left(\sum_{k=0}^{l} \|f^{(k)}\|^2_{\mathbb{L}_2^m(\mathbb{R})}\right)^{1/2}.$$

If $\lambda E - A$ is c-stable, then $\mathcal{K}_c$ is the bounded operator mapping $\mathbb{L}_{2,\nu-1}^m(\mathbb{R})$ into $\mathbb{L}_2^p(\mathbb{R})$. In this case the spectral norm of $\mathcal{K}_c$ is given by $\|\mathcal{K}_c\|_2 = \sup_{u\neq 0} \|\mathcal{K}_c u\|_{\mathbb{L}_2^p(\mathbb{R})}/\|u\|_{\mathbb{L}_{2,\nu-1}^m(\mathbb{R})}$. Using the Fourier transform, the time domain relation $y(t) = (\mathcal{K}_c u)(t)$ is expressed in the frequency domain via $\mathbf{y}(i\omega) = \mathbf{G}(i\omega)\mathbf{u}(i\omega)$. Since the Fourier transform gives an isometric isomorphism between $\mathbb{L}_{2,\nu-1}^m(\mathbb{R})$ and $\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})$, we obtain by Parseval's identity that $\|\mathcal{K}_c\|_2 = \|\mathbf{G}\|_{\mathbb{HL}_\infty}$.

The $\mathbb{H}_\infty$-norm of the proper transfer function $\mathbf{G}$ can be computed by the method proposed in [3]. Computing the $\mathbb{HL}_\infty$-norm of the improper transfer function $\mathbf{G}$ is still an open problem.

The Hilbert-Schmidt norm and the Hankel norm: For system (20), we define a *proper Hankel operator* $\mathcal{H}_p$ transforming the past inputs $u_-(t)$ ($u_-(t)=0$ for $t\geq 0$) into the present and future outputs $y_+(t)$ ($y_+(t)=0$ for $t<0$) through the state $x(0)\in\operatorname{Im}(P_r)$ via

$$y_+(t) = (\mathcal{H}_p u_-)(t) = \int_{-\infty}^{0} G_{sp}(t-\tau)u_-(\tau)\,d\tau, \quad t\geq 0. \qquad (25)$$

If $\lambda E - A$ is c-stable, then $\mathcal{H}_p$ acts from $\mathbb{L}_2^m(\mathbb{R}^-)$ into $\mathbb{L}_2^p(\mathbb{R}_0^+)$.

Theorem 3.2: Consider system (20), where $\lambda E - A$ is c-stable. The non-zero proper Hankel singular values of (20) are the non-zero singular values of the proper Hankel operator $\mathcal{H}_p$.

Proof: Consider an adjoint operator $\mathcal{H}_p^*$ of the proper Hankel operator $\mathcal{H}_p$ that has the form

$$(\mathcal{H}_p^* y)(\tau) = \int_{0}^{\infty} B^T\mathcal{F}^T(t-\tau)C^T y(t)\,dt.$$

Let $\sigma\neq 0$ be a singular value of $\mathcal{H}_p$ and let $u\in\mathbb{L}_2^m(\mathbb{R}^-)$ be a corresponding right singular vector, i.e., $\sigma^2 u(t) = (\mathcal{H}_p^*\mathcal{H}_p u)(t)$. Then

$$\sigma^2 u(t) = \int_{0}^{\infty}\int_{-\infty}^{0} B^T\mathcal{F}^T(\tau-t)C^TC\mathcal{F}(\tau-\xi)Bu(\xi)\,d\xi\,d\tau$$
$$= \int_{0}^{\infty}\int_{-\infty}^{0} B^T\mathcal{F}^T(-t)E^T\mathcal{F}^T(\tau)C^TC\mathcal{F}(\tau)E\mathcal{F}(-\xi)Bu(\xi)\,d\xi\,d\tau. \qquad (26)$$

It follows from (26) that $v = \int_{-\infty}^{0}\mathcal{F}(-\xi)Bu(\xi)\,d\xi \neq 0$ and

$$\sigma^2 v = \mathcal{G}_{cpc}E^T\mathcal{G}_{cpo}Ev = \Phi_c v, \qquad (27)$$

i.e., v is an eigenvector of the matrix Φ_c corresponding to the eigenvalue σ^2.

On the other hand, consider an eigenvalue $\sigma^2\neq 0$ of Φ_c with an eigenvector v. Then from (27) we have $\sigma^2 u(\tau) = (\mathcal{H}_p^*\mathcal{H}_p u)(\tau)$ with $u(\tau) = \int_0^\infty B^T\mathcal{F}^T(\xi-\tau)C^TC\mathcal{F}(\xi)Evd\xi$. Since the proper Hankel operator of the asymptotically stable system (20) is the Hilbert-Schmidt operator, it is compact. In this case $\mathcal{H}_p$ has a discrete set of non-zero singular values and they coincide with the non-zero proper Hankel singular values. ■

Remark 3.3: Note that the proper Hankel singular values of the continuous-time descriptor system (20) are not equal to the singular values of the causal Hankel matrix $\mathcal{H}_c$ as in (16). However, as in the discrete-time case, the non-zero improper Hankel singular values coincide with the classical non-zero singular values of the non-causal Hankel matrix $\mathcal{H}_n$ given in (16).

A *Hilbert-Schmidt norm* (*HS-norm*) of the transfer function $\mathbf{G}$ is defined by

$$\|\mathbf{G}\|_{HS} = \left(\int_0^\infty\int_0^\infty \|G_{sp}(t+\tau)\|_F^2\,dt\,d\tau + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|G_{-j-k}\|_F^2\right)^{1/2},$$

where $G_{sp}(t) = C\mathcal{F}(t)B$, $G_{-k} = CF_{-k-1}B$. Taking into account that $\mathcal{F}(t+\tau) = \mathcal{F}(t)E\mathcal{F}(\tau)$ and $F_{-j-k-1} = -F_{-j-1}AF_{-k-1}$ for $j,k\geq 0$, we obtain that

$$\int_0^\infty\int_0^\infty \|G_{sp}(t+\tau)\|_F^2\,dt\,d\tau = \operatorname{tr}\big(\mathcal{G}_{cpc}E^T\mathcal{G}_{cpo}E\big) = \operatorname{tr}(\Phi_c),$$
$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|G_{-j-k}\|_F^2 = \operatorname{tr}\big(\mathcal{G}_{cic}A^T\mathcal{G}_{cio}A\big) = \operatorname{tr}(\Psi_c).$$

Hence, $\|\mathbf{G}\|_{HS}^2 = \operatorname{tr}(\Phi_c+\Psi_c) = \varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2$. As a consequence of Theorem 3.2 and Remark 3.3 we have

$$\|\mathbf{G}\|_{HS}^2 = \|\mathcal{H}_p\|_F^2 + \|\mathcal{H}_n\|_F^2 = \|[\,L_pER_p,\ L_iAR_i\,]\|_F^2,$$

where R_p, L_p, R_i and L_i are the full rank factors of the Gramians $\mathcal{G}_{cpc} = R_pR_p^T$, $\mathcal{G}_{cpo} = L_p^TL_p$, $\mathcal{G}_{cic} = R_iR_i^T$ and $\mathcal{G}_{cio} = L_i^TL_i$.

A *Hankel norm* of the transfer function $\mathbf{G}$ is defined by

$$\|\mathbf{G}\|_H = \max(\|\mathcal{H}_p\|_2, \|\mathcal{H}_n\|_2) = \max(\varsigma_1, \theta_1),$$

where ς_1 and θ_1 are the largest proper and improper Hankel singular values of the descriptor system (20). From the definition of the Hankel singular values we find that

$$\|\mathbf{G}\|_H = \sqrt{\lambda_{\max}(\Phi_c+\Psi_c)} = \max(\|L_pER_p\|_2, \|L_iAR_i\|_2).$$

We summarize system norms for the asymptotically stable continuous-time descriptor system (20) in Table II.

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TABLE II

GENERALIZED NORMS FOR ASYMPTOTICALLY STABLE CONTINUOUS-TIME DESCRIPTOR SYSTEMS

$\mathbf{G}(s) = C(sE-A)^{-1}B$	$\\|\mathbf{G}\\|_{\mathbb{HL}_2}$	$\\|\mathbf{G}\\|_{\mathbb{HL}_\infty}$
$\mathbf{G}(i\omega) = \mathbf{G}_{sp}(i\omega) + \mathbf{P}(i\omega)$	$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\\|\mathbf{G}_{sp}(i\omega)\\|_F^2 d\omega + \frac{1}{2\pi}\int_0^{2\pi}\\|\mathbf{P}(e^{i\omega})\\|_F^2 d\omega\right)^{\frac{1}{2}}$	$\sup\limits_{\mathbf{u}\neq 0} \frac{\\|\mathbf{Gu}\\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\\|\mathbf{u}\\|_{\mathbb{L}^m_{2,\nu-1}(i\mathbb{R})}}$
$G_{sp}(t), \quad G_k$	$\left(\int_0^\infty \\|G_{sp}(t)\\|_F^2 dt + \sum_{k=1}^{\nu}\\|G_{-k+1}\\|_F^2\right)^{\frac{1}{2}}$	
$\mathcal{G}_{cpc} + \mathcal{G}_{cic} = RR^T$	$\sqrt{\mathrm{tr}\big(C(\mathcal{G}_{cpc}+\mathcal{G}_{cic})C^T\big)} = \\|CR\\|_F$	
$\mathcal{G}_{cpo} + \mathcal{G}_{cio} = L^TL$	$\sqrt{\mathrm{tr}\big(B^T(\mathcal{G}_{cpo}+\mathcal{G}_{cio})B\big)} = \\|LB\\|_F$	
$\mathcal{K}_c$		$\\|\mathcal{K}_c\\|_2$

$\mathbf{G}(s) = C(sE-A)^{-1}B$	$\\|\mathbf{G}\\|_{HS}$	$\\|\mathbf{G}\\|_{H}$
$G_{sp}(t), \quad G_k$	$\left(\int_0^\infty\int_0^\infty \\|G_{sp}(t+\tau)\\|_F^2 dt d\tau + \sum_{k=1}^{\nu} k\\|G_{-k+1}\\|_F^2\right)^{\frac{1}{2}}$	
$\mathcal{H}_p, \quad \mathcal{H}_n$	$\sqrt{\\|\mathcal{H}_p\\|_F^2 + \\|\mathcal{H}_n\\|_F^2}$	$\max(\\|\mathcal{H}_p\\|_2, \\|\mathcal{H}_n\\|_2)$
$\mathcal{G}_{cpc} = R_pR_p^T,\ \mathcal{G}_{cpo} = L_p^TL_p$ $\mathcal{G}_{cic} = R_iR_i^T,\ \mathcal{G}_{cio} = L_i^TL_i$	$\\|[\, L_pER_p,\ L_iAR_i\,]\\|_F$	$\max(\\|L_pER_p\\|_2, \\|L_iAR_i\\|_2)$
$\Phi_c, \quad \Psi_c$	$\sqrt{\mathrm{tr}(\Phi_c + \Psi_c)}$	$\sqrt{\lambda_{\max}(\Phi_c + \Psi_c)}$
$\varsigma_1 \geq \ldots \geq \varsigma_{n_f},\, \theta_1 \geq \ldots \geq \theta_{n_\infty}$	$\sqrt{\varsigma_1^2 + \ldots + \varsigma_{n_f}^2 + \theta_1^2 + \ldots + \theta_{n_\infty}^2}$	$\max(\varsigma_1, \theta_1)$

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