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On criteria for asymptotic stability of differential-algebraic equations

This paper discusses Lyapunov stability of the trivial solution of linear differential-algebraic equations. As a criterion for the asymptotic stability we propose numerical parameters characterizing the property of a regular matrix pencil $\lambda A - B$ to have all finite eigenvalues in the open left half-plane. Numerical aspects for computing these parameters are discussed.

Keywords: differential-algebraic equations, asymptotic stability, Lyapunov equation, matrix pencils, deflating subspaces, spectral projections

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1 Introduction

Differential-algebraic equations (DAEs)

$$A\dot{x}(t) = Bx(t) + f(t)$$

arise naturally in many applications, e.g., in control problems, electrical networks and constrained mechanical systems [4, 7, 8, 9, 32]. The theoretical analysis and the numerical solution of DAEs has been the subject of intense research for many years, see [4, 21, 22] and the references therein. In the study of differential-algebraic equations one is often interested in the existence of stationary solutions and their asymptotic behavior [9, 21, 31, 33, 37].

In this paper we propose an approach to study the asymptotic stability of the trivial solution of the equation $A\dot{x}(t) = Bx(t)$. The case of a nonsingular matrix A is well studied, asymptotic stability of the trivial solution is equivalent to the condition for the matrix $A^{-1}B$ to have all eigenvalues in the open left half-plane, e.g., [13]. If the matrix A is singular, then the investigation of the spectrum of the matrix pencil $\lambda A - B$ is necessary. However, it is well known that the (generalized) eigenvalue problem may be ill-conditioned in the sense that eigenvalues may change strongly even under small perturbations in A and B [36]. Recently the concept of ε -pseudospectra and spectral portrait [19, 38] was developed to better understand the influence of perturbations on the spectrum of matrices and matrix pencils. The application of the ε -pseudospectra in the study of the asymptotic stability of differential equations arising in Computational Fluid Dynamics can be found in [11, 12, 39].

Another possible approach to investigate the asymptotic behavior of solutions of linear ordinary differential equations without explicit computing the eigenvalues is the consideration of a dichotomy parameter that characterizes numerically the property of a matrix to have all eigenvalues in the open left half-plane and that is efficiently computable [5, 16, 18]. In this paper we generalize this parameter for DAEs and discuss the computation of deflating subspaces of the matrix pencil $\lambda A - B$ corresponding to the finite eigenvalues.

This paper is organized as follows. In Section 2 we recall fundamental characteristics of matrix pencils and some properties of DAEs. In Section 3 we introduce numerical parameters the boundness of those is equivalent to the property of the pencil $\lambda A - B$ to have all finite eigenvalues in the open left half-plane. Section 4 presents a generalized Lyapunov equation that can be used to compute these parameters. In Section 5 we describe an algorithm for computing the spectral projections onto the right and left deflating subspaces of $\lambda A - B$ corresponding to the finite eigenvalues and the solution of the generalized Lyapunov equation for the case of the pencil of index one. Section 6 presents a perturbation analysis for these projections. The sensitivity analysis for the generalized Lyapunov equation is presented in Section 7. Section 8 contains numerical examples.

Throughout the paper the space of complex matrices of size $n \times n$ is denoted by $\mathbb{C}^{n,n}$. The matrix A^* is the complex conjugate transpose of the matrix A , and $A^{-*} = (A^*)^{-1}$. The inner product of vectors x and y is defined as $\langle x, y \rangle = y^* x$, $\|\cdot\|$ denotes the spectral matrix norm and the Euclidean vector norm, $\text{cond}(A) = \|A\| \|A^{-1}\|$ is the condition number of A . We denote the nullspace of the matrix A by $\ker A$ and the range of A by $\text{im } A$.

2 Preliminaries

Let A and B be square complex matrices of order n . A matrix pencil $\lambda A - B$ is called *singular* if $\det(\lambda A - B) \equiv 0$ for all $\lambda \in \mathbb{C}$. Otherwise, the pencil $\lambda A - B$ is called *regular*. A complex value $\lambda \neq \infty$ is said to be a *finite eigenvalue* of $\lambda A - B$ if $\det(\lambda A - B) = 0$. The pencil $\lambda A - B$ has *infinite eigenvalue* if the matrix A is singular.

A regular pencil $\lambda A - B$ can be reduced to the Weierstrass canonical form [36], i.e., there exist nonsingular matrices W and T such that

$$A = W \begin{pmatrix} I_m & 0 \\ 0 & N \end{pmatrix} T \quad \text{and} \quad B = W \begin{pmatrix} J & 0 \\ 0 & I_{n-m} \end{pmatrix} T, \quad (2.1)$$

where I_m is the identity matrix of order m , J and N are matrices in Jordan canonical form and N is nilpotent with index of nilpotency k . The number k is called *index* of the pencil $\lambda A - B$. The block J corresponds to the finite eigenvalues, the block N corresponds to the infinite eigenvalues of $\lambda A - B$.

The representation (2.1) defines the decomposition of \mathbb{C}^n into complementary *deflating subspaces* of the pencil $\lambda A - B$ corresponding to its finite and infinite eigenvalues [36]. The matrices

$$P = T^{-1} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} T \quad \text{and} \quad \Pi = W \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \tag{2.2}$$

are the *spectral projections* onto the right and left deflating subspaces of the pencil $\lambda A - B$ corresponding to the finite eigenvalues. For simplicity, the deflating subspace of $\lambda A - B$ corresponding to the finite (infinite) eigenvalues we will call the *finite (infinite) deflating subspace*.

Consider the homogeneous differential-algebraic equation

$$A\dot{x}(t) = Bx(t), \tag{2.3}$$

where $A, B \in \mathbb{C}^{n,n}$. For equation (2.3) we may define a fundamental solution matrix as follows.

Definition 2.1. A matrix-valued function $\mathcal{F}(t) \equiv \mathcal{F}(t, A, B)$ is called *fundamental solution matrix* of equation (2.3) if it is continuously differentiable and satisfies the initial value problem

$$A\dot{\mathcal{F}}(t) = B\mathcal{F}(t), \quad \mathcal{F}(0) = P, \tag{2.4}$$

where P is the spectral projection onto the right finite deflating subspace of the pencil $\lambda A - B$.

The matrix $\mathcal{F}(t)$ is a generalization of the matrix exponential that is the fundamental solution matrix for linear time-invariant ordinary differential equations [18].

Theorem 2.1. Let $\lambda A - B$ be a regular pencil. There exists a unique fundamental solution matrix $\mathcal{F}(t)$ of equation (2.3). Moreover, the initial value problem

$$A\dot{x}(t) = Bx(t), \quad P(x(0) - x_0) = 0 \tag{2.5}$$

has a unique solution $x(t) \in \text{im } P$ for all $x_0 \in \mathbb{C}^n$. This solution is given by $x(t) = \mathcal{F}(t)x_0$.

Proof: Let the pencil $\lambda A - B$ be in Weierstrass canonical form (2.1). Then it is easy to verify that the matrix

$$\mathcal{F}(t) = T^{-1} \begin{pmatrix} e^{tJ} & 0 \\ 0 & 0 \end{pmatrix} T \tag{2.6}$$

satisfies the initial value problem (2.4).

Let us suppose that there exist two fundamental solution matrices $\mathcal{F}_1(t)$ and $\mathcal{F}_2(t)$. Then their difference $\mathcal{F}(t) = \mathcal{F}_1(t) - \mathcal{F}_2(t)$ satisfies the homogeneous system $A\dot{\mathcal{F}}(t) = B\mathcal{F}(t)$, $\mathcal{F}(0) = 0$. Using (2.1) we obtain that this system has only the trivial solution $\mathcal{F}(t) \equiv 0$. Hence, $\mathcal{F}_1(t) = \mathcal{F}_2(t)$.

The straightforward verification that the function $x(t) = \mathcal{F}(t)x_0$ is a unique solution of the initial value problem (2.5) concludes the proof. \square

Note that the initial condition $\mathcal{F}(0) = P$ can be replaced by the equivalent condition $\hat{P}(\mathcal{F}(0) - I) = 0$ with a certain projection \hat{P} along $\ker P$, see [21].

3 A numerical criterion for asymptotic stability

Asymptotic stability is an important property of differential equations. It is well known that the trivial solution of the differential-algebraic equation (2.3) is asymptotically stable if and only if all finite eigenvalues of the pencil $\lambda A - B$ have negative real part, e.g., [9, 21].

We consider now the problem to numerically check whether all finite eigenvalues of the pencil $\lambda A - B$ belong to the open left half-plane. This problem arises also in the study of the asymptotic properties of stationary solutions autonomous quasilinear and nonlinear DAEs [30, 37] and nonautonomous DAEs with constant coefficient linear part and small nonlinearity [31].

Set

$$\alpha(A, B) = 2\|(AP + B(I - P))^{-1}B\| \|A^*ZA\|,$$

where the matrix Z has the form

$$Z = (AP + B(I - P))^{-*} \left(\int_0^\infty \mathcal{F}^*(t)\mathcal{F}(t)dt \right) (AP + B(I - P))^{-1} \tag{3.1}$$

and $\mathcal{F}(t)$ is the fundamental solution matrix of (2.3). If $\lambda A - B$ is *stable*, i.e., all finite eigenvalues of $\lambda A - B$ lie in the open left half-plane, then $\Re \lambda_j(J) \leq -\eta < 0$ for all eigenvalues of J in (2.1). In this case we have the estimate

$$\|e^{tJ}\| \leq \theta(m) \left(\frac{\|J\|}{\eta} \right)^{m-1} e^{-t\eta/2},$$

where $\theta(m)$ is a constant that depends on m only [18]. Then it follows from (2.6) that

$$\|\mathcal{F}(t)\| \leq \|T^{-1}\| \|T\| \|e^{tJ}\| \leq \theta(m) \|T^{-1}\| \|T\| \left(\frac{\|J\|}{\eta} \right)^{m-1} e^{-t\eta/2}.$$

Hence, the integral in (3.1) is convergent. Moreover, we obtain from (2.1) and (2.2) that $AP + B(I - P) = WT$, i.e., the matrix $AP + B(I - P)$ is nonsingular. Thus, $\varkappa(A, B)$ is bounded if the pencil $\lambda A - B$ is stable. We set $\varkappa(A, B) = \infty$ if $\lambda A - B$ has at least one finite eigenvalue with nonnegative real part.

It is interesting that the parameter $\varkappa(A, B)$ can be used for pointwise estimation of the solution of the initial value problem (2.5). We will use a similar technique as in [19].

Theorem 3.1. *Let $x(t)$ be a solution of the initial value problem (2.5). Then*

$$\|x(t)\| \leq \sqrt{\varkappa(A, B)} e^{-t\|(AP+B(I-P))^{-1}B\|/\varkappa(A,B)} \|Px_0\|. \quad (3.2)$$

Proof: If $\varkappa(A, B) = \infty$ then inequality (3.2) is fulfilled. Assume that $\varkappa(A, B) < \infty$. Let us consider for $t \geq 0$ the matrix-valued function

$$Y(t) = \int_t^\infty \mathcal{F}^*(s) \mathcal{F}(s) ds.$$

Using the properties of the fundamental matrix $\mathcal{F}(t+s) = \mathcal{F}(t)\mathcal{F}(s) = \mathcal{F}(s)\mathcal{F}(t)$, we have

$$\begin{aligned} Y(t) &= \int_t^\infty \mathcal{F}^*(s) \mathcal{F}(s) ds = \mathcal{F}^*(t) \left\{ \int_0^\infty \mathcal{F}^*(s) \mathcal{F}(s) ds \right\} \mathcal{F}(t) = \\ &= \mathcal{F}^*(t) (AP + B(I - P))^* Z (AP + B(I - P)) \mathcal{F}(t) = \mathcal{F}^*(t) A^* Z A \mathcal{F}(t). \end{aligned}$$

Differentiating the matrix $Y(t)$, we obtain for an arbitrary vector $z \in \mathbb{C}^n$ that

$$\frac{d}{dt} \langle Y(t)z, z \rangle = -\langle \mathcal{F}(t)z, \mathcal{F}(t)z \rangle \leq -\frac{\langle A^* Z A \mathcal{F}(t)z, \mathcal{F}(t)z \rangle}{\|A^* Z A\|} = -\frac{\langle Y(t)z, z \rangle}{\|A^* Z A\|}$$

and, hence,

$$\frac{d}{dt} \left(e^{t/\|A^* Z A\|} \langle Y(t)z, z \rangle \right) \leq 0.$$

This estimate yields

$$\langle \mathcal{F}^*(t) A^* Z A \mathcal{F}(t)z, z \rangle = \langle Y(t)z, z \rangle \leq e^{-t/\|A^* Z A\|} \langle Y(0)z, z \rangle = e^{-t/\|A^* Z A\|} \langle A^* Z A Pz, Pz \rangle. \quad (3.3)$$

Furthermore, it is not difficult to verify that

$$\mathcal{F}(t) = e^{t(AP+B(I-P))^{-1}B} P = P e^{t(AP+B(I-P))^{-1}B}.$$

Then, taking into account that $\|e^{t(AP+B(I-P))^{-1}B} Pz\| \geq e^{-|t|\|(AP+B(I-P))^{-1}B\|} \|Pz\|$, see [18], we have

$$\begin{aligned} \langle A^* Z A Pz, Pz \rangle &= \langle (AP + B(I - P))^* Z (AP + B(I - P)) Pz, Pz \rangle = \int_0^\infty \|\mathcal{F}(t)Pz\|^2 dt \geq \\ &\geq \|Pz\|^2 \int_0^\infty e^{-2t\|(AP+B(I-P))^{-1}B\|} dt = \frac{\|Pz\|^2}{2\|(AP + B(I - P))^{-1}B\|}. \end{aligned} \quad (3.4)$$

Substituting in (3.4) the vector $z = \mathcal{F}(t)x_0$ we obtain

$$\|x(t)\|^2 = \|\mathcal{F}(t)x_0\|^2 \leq 2\|(AP + B(I - P))^{-1}B\| \langle A^* Z A \mathcal{F}(t)x_0, \mathcal{F}(t)x_0 \rangle.$$

Finally, using (3.3) with $z = Px_0$ we obtain estimate (3.2). \square

If $\varkappa(A, B)$ is bounded, then it follows from (3.2) that $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$, i.e., the trivial solution of equation (2.3) is asymptotically stable. On the other hand, the asymptotic stability implies that $\varkappa(A, B) < \infty$. Thus, the boundness of $\varkappa(A, B)$ is equivalent to the property of the trivial solution of (2.3) to be asymptotically stable.

The following example shows that estimate (3.2) is reached.

Example 1. Consider the differential-algebraic equation

$$A_\delta \dot{x}(t) = B_\delta x(t) \tag{3.5}$$

with

$$A_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_\delta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $0 < \delta$, the general solution of (3.5) is $x(t) = e^{-t} P x_0$ and, hence, the trivial solution of (3.5) is asymptotically stable. We have $\alpha(A_\delta, B_\delta) = 1$ and from (3.2) it follows that $\|x(t)\| \leq e^{-t} \|P x_0\|$. However, for $\delta = 0$ the pencil $\lambda A_\delta - B_\delta$ is singular, i.e., under a perturbation of norm δ the trivial solution of (3.5) is any no more asymptotically stable.

It is possible to derive from (3.2) a weaker bound for the solution $x(t)$. Indeed, using $\|A^* Z A\| \leq \|A\|^2 \|Z\|$ we obtain the estimate

$$\begin{aligned} \|x(t)\| &\leq \sqrt{2\|A\|^2 \|Z\| \| (AP + B(I - P))^{-1} \| \|B\|} e^{-t/(2\|A\|^2 \|Z\|)} \|P x_0\| = \\ &= \sqrt{\kappa(A, B) \|A\| \| (AP + B(I - P))^{-1} \|} e^{-t\|B\|/(\|A\| \kappa(A, B))} \|P x_0\|, \end{aligned} \tag{3.6}$$

where $\kappa(A, B) = 2\|A\| \|B\| \|Z\|$.

In spite of the fact that bound (3.6) may overestimate the solution $x(t)$ of (2.5), the parameter $\kappa(A, B)$ also characterizes the behavior of $x(t)$ at infinity. Moreover, $\kappa(A, B)$ in contrast with $\alpha(A, B)$ may be more useful to estimate the robustness for the asymptotic stability. We see in Example 1 that $\kappa(A_\delta, B_\delta) = \delta^{-2} \rightarrow \infty$ as $\delta \rightarrow 0$ and, hence, (3.5) approaches to an unstable system.

Note that for $A = I$ both the parameters $\alpha(A, B)$ and $\kappa(A, B)$ coincide with the parameter $\alpha(B)$ introduced in [5, 18] to study the asymptotic stability of linear ordinary differential equations.

4 Generalized Lyapunov equations

It is well known that the study of the asymptotic behavior of solutions of ordinary differential equations is directly related to the analysis of Lyapunov matrix equations, see [13, 18]. In this section we present a generalized Lyapunov equation that can be used to investigate the asymptotic stability of the differential-algebraic equation (2.3).

Consider the generalized Lyapunov equation

$$A^* Z B + B^* Z A = -P^* C P, \tag{4.1}$$

where $A, B, C \in \mathbb{C}^{n,n}$ are given matrices, P is the spectral projection onto the right finite deflating subspace of $\lambda A - B$ and $Z \in \mathbb{C}^{n,n}$ is the unknown matrix. If A is nonsingular, then $P = I$ and (4.1) is equivalent to the standard Lyapunov equation $Z B A^{-1} + (B A^{-1})^* Z = -A^{-*} C A^{-1}$ that has a unique Hermitian, positive definite solution Z for every Hermitian, positive definite matrix C if and only if all eigenvalues of the matrix $B A^{-1}$ lie in the open left half-plane [13].

For a singular matrix A the solvability of (4.1) depends only on the structure of the finite spectrum of $\lambda A - B$. The following theorem gives a necessary and sufficient condition for the existence of solutions of the generalized Lyapunov equation (4.1).

Theorem 4.1. *Let $\lambda A - B$ be a regular pencil and let P, Π be the spectral projections onto the right and left finite deflating subspaces of $\lambda A - B$. There exists an Hermitian, positive semidefinite matrix Z satisfying the generalized Lyapunov equation (4.1) with an Hermitian, positive definite matrix C if and only if the pencil $\lambda A - B$ is stable. Moreover, if the solution of (4.1) satisfies $Z = Z \Pi$, then it is unique and given by*

$$Z = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* C P (i\xi A - B)^{-1} d\xi. \tag{4.2}$$

Proof: Let the matrix pencil $\lambda A - B$ be in Weierstrass canonical form (2.1), where all eigenvalues of J have negative real part. Let the matrices

$$T^{-*} C T^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^* & T_{22} \end{pmatrix} \quad \text{and} \quad W^* Z W = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \tag{4.3}$$

be partitioned in blocks accordingly to A and B . We have from (4.1) the decoupled system of equations

$$Z_{11} J + J^* Z_{11} = -T_{11}, \tag{4.4}$$

$$Z_{12} + J^* Z_{12} N = 0, \tag{4.5}$$

$$N^* Z_{21} J + Z_{21} = 0, \tag{4.6}$$

$$N^* Z_{22} + Z_{22} N = 0. \tag{4.7}$$

Since all eigenvalues of J lie in the open half-plane, the Lyapunov equation (4.4) with the Hermitian, positive definite T_{11} has a unique Hermitian, positive definite solution which is given by

$$Z_{11} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi I - J)^{-*} T_{11} (i\xi I - J)^{-1} d\xi,$$

see [18]. Because the matrices J^{-*} and $-N$ have disjoint spectra, equations (4.5), (4.6) are uniquely solvable [25] and have the trivial solutions $Z_{12} = 0$ and $Z_{21} = 0$. Equation (4.7) has not unique solution [25]. From $Z = Z\Pi$ we obtain that $Z_{22} = 0$ which also satisfies (4.7). Thus, the generalized Lyapunov equation (4.1) together with $Z = Z\Pi$ has the unique Hermitian, positive semidefinite solution Z given by (4.2).

Assume now that an Hermitian, positive semidefinite matrix Z as in (4.3) satisfies (4.1) with an Hermitian, positive definite matrix C . Then equations (4.4)-(4.7) are fulfilled. Since T_{11} is positive definite and Z_{11} is positive semidefinite, we have from (4.4) that the eigenvalues of J have negative real part, i.e., all finite eigenvalues of the pencil $\lambda A - B$ lie in the open left half-plane. \square

Remark 4.1. Note that the assertion of Theorem 4.1 remains valid if the matrix C is positive definite only on the subspace $\text{im } P$, i.e., $\langle Cz, z \rangle > 0$ for all nonzero $z \in \text{im } P$, since in this case the property of the matrix T_{11} in (4.3) to be positive definite is preserved.

We will now establish a connection between the solution of the generalized Lyapunov equation and the differential-algebraic equation (2.3). This connection is well known for the standard Lyapunov equation ($A = I$) and the linear ordinary differential equation, see [18, 24].

Let the pencil $\lambda A - B$ be stable and let Z be the Hermitian solution of the generalized Lyapunov equation

$$A^* Z B + B^* Z A = -P^* P \tag{4.8}$$

together with $Z = Z\Pi$. For all nonzero solution $x(t) \in \text{im } P$ of the differential-algebraic equation (2.3) we have

$$\frac{d}{dt} \langle A^* Z A x(t), x(t) \rangle = \langle (A^* Z B + B^* Z A) x(t), x(t) \rangle = -\langle P x(t), P x(t) \rangle = -\langle x(t), x(t) \rangle.$$

The quadratic form $\langle A^* Z A x, x \rangle$ is an extension of the Lyapunov function for ordinary differential equations [18] to differential-algebraic equations. Furthermore, taking into account the relation

$$e^{tJ} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} (i\xi I - J)^{-1} d\xi,$$

see [18], we obtain from (2.6) that

$$\mathcal{F}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} P (i\xi A - B)^{-1} A d\xi, \tag{4.9}$$

i.e., the entries of the matrix $P(i\xi A - B)^{-1} A$ are the Fourier transformations of the entries of the fundamental solution matrix $\mathcal{F}(t)$. Then it follows from (4.2) with $C = I$ and Parseval's identity [35] that

$$A^* Z A = \frac{1}{2\pi} \int_{-\infty}^{\infty} A^* (i\xi A - B)^{-*} P^* P (i\xi A - B)^{-1} A d\xi = \int_0^{\infty} \mathcal{F}^*(t) \mathcal{F}(t) dt.$$

Thus, the solution of the generalized Lyapunov equation (4.8) may be used to compute the parameters $\alpha(A, B)$ and $\kappa(A, B)$ that characterize the asymptotic stability of the differential-algebraic equation (2.3).

The numerical solution of the standard Lyapunov equation has been studied in numerous publications, see, e.g., [2, 23] and the references therein. Numerical methods for the generalized Lyapunov equation with nonsingular A have been considered in [3, 14, 15, 34]. However, the case of singular A is more complicated, since the solution of the generalized Lyapunov equation is not unique. We need the special solution Z of (4.8), namely, such that $Z = Z\Pi$. In the next section we present an algorithm for computing the projections P , Π and the desired matrix Z for the matrix pencil $\lambda A - B$ with index at most one.

5 Computing projections and the matrix Z

We now assume that the matrix pencil $\lambda A - B$ is regular of index at most one. Recall that $\lambda A - B$ has index one if and only if the matrix $A + BQ$ is nonsingular for any projection Q onto the nullspace of A [21]. In this case the spectral projections P and Π onto the right and left finite deflating subspaces of $\lambda A - B$ can be represented as

$$P = I - Q(A + BQ)^{-1} B, \quad \Pi = I - BQ(A + BQ)^{-1}, \tag{5.1}$$

see [21]. The norms of these projections characterize the conditioning of the deflating subspaces of $\lambda A - B$ associated with the finite and infinite eigenvalues and the property of $\lambda A - B$ to be regular of index one. A large value of $\|P\|$ or

$\|\Pi\|$ indicates that the problem to find the finite deflating subspace of the pencil $\lambda A - B$ with index one is ill-conditioned, i.e., either the finite and infinite eigenvalues are hard to be separated from each other, or $\lambda A - B$ is nearly a pencil of index greater than one or a singular pencil.

We now describe an algorithm for computing the projections P and Π for the matrix pencil $\lambda A - B$ of index one. Let $r = \text{rank } A$ and let

$$A = V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^* \quad (5.2)$$

be the singular value decomposition of A [17, 20], where U and V are unitary matrices and Σ is a nonsingular diagonal $(r \times r)$ -matrix with positive diagonal elements $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_{r-1}(A) \geq \sigma_r(A) > 0$ which are the nonzero singular values of A . Then

$$Q^\perp = U \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} U^*$$

is the orthogonal projection onto $\ker A$. Let the matrix

$$V^* B U = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad (5.3)$$

be partitioned in blocks in accordance with $V^* A U$. Then from (5.1) with $Q = Q^\perp$ we obtain

$$P = U \begin{pmatrix} I & 0 \\ -B_{22}^{-1} B_{21} & 0 \end{pmatrix} U^*, \quad \Pi = V \begin{pmatrix} I & -B_{12} B_{22}^{-1} \\ 0 & 0 \end{pmatrix} V^*. \quad (5.4)$$

The accuracy in the computation of the projections P and Π will clearly depend on the condition number of B_{22} with respect to inversion. But it also depends on the condition number of Σ as is shown in the following example.

Example 2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \delta \end{pmatrix}.$$

For $\varepsilon \neq 0$ we have $B_{22} = \delta$. If δ is small, then the pencil $\lambda A - B$ is nearly a pencil of index two. If $\varepsilon = 0$, then the 2×2 -matrix B_{22} is well-conditioned for δ not too large. But for $\varepsilon = 0$ the dimension of the finite deflating subspace changes.

If the matrices Σ and B_{22} are well-conditioned, then we can easily compute the Weierstrass canonical form (2.1). The transformation matrices W and T are given by

$$W = V \begin{pmatrix} \Sigma & B_{12} \\ 0 & B_{22} \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ B_{22}^{-1} B_{21} & I \end{pmatrix} U^*,$$

and the block J associated with the finite eigenvalues has the form $J = \Sigma^{-1}(B_{11} - B_{12} B_{22}^{-1} B_{21})$. Note that a transformation of J to Jordan form is not necessary.

Consider now the generalized Lyapunov equation (4.8). Let the matrix

$$Z = V \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} V^* \quad (5.5)$$

partitioned conformally with A in (5.2) be the solution of (4.8). Substituting (5.2)-(5.5) in (4.8) and $Z = Z\Pi$, we obtain

$$\Sigma^* X_{11} B_{11} + \Sigma^* X_{12} B_{21} + B_{11}^* X_{11} \Sigma + B_{21}^* X_{12}^* \Sigma = -(I + B_{21}^* B_{22}^{-1} B_{21}), \quad (5.6)$$

$$\Sigma^* X_{11} B_{12} + \Sigma^* X_{12} B_{22} = 0, \quad (5.7)$$

$$X_{22} = -X_{12}^* B_{12} B_{22}^{-1}.$$

It follows from (5.7) that $X_{12} = -X_{11} B_{12} B_{22}^{-1}$. Inserting X_{12} in (5.6) we have

$$\Sigma^* X_{11} (B_{11} - B_{12} B_{22}^{-1} B_{21}) + (B_{11} - B_{12} B_{22}^{-1} B_{21})^* X_{11} \Sigma = -(I + B_{21}^* B_{22}^{-1} B_{21}). \quad (5.8)$$

Thus, the matrix

$$Z = V \begin{pmatrix} X_{11} & -X_{11} B_{12} B_{22}^{-1} \\ -B_{22}^{-1} B_{21}^* X_{11} & B_{22}^{-1} B_{21}^* X_{11} B_{12} B_{22}^{-1} \end{pmatrix} V^*,$$

where X_{11} is the solution of (5.8), satisfies (4.8) and $Z = Z\Pi$.

We rewrite the generalized Lyapunov equation (5.8) as

$$\Sigma^* X_{11} F + F^* X_{11} \Sigma = -D, \tag{5.9}$$

where $F = B_{11} - B_{12} B_{22}^{-1} B_{21}$ and $D = I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21}$. This equation with nonsingular Σ can be solved using the generalized Bartels-Stewart algorithm [14, 15, 34] or the sign function method [3].

Note that for computing the matrix D we have to multiply the matrices $B_{21}^* B_{22}^{-*}$ and $B_{22}^{-1} B_{21}$. This may lead to a larger sensitivity, in the worst case the condition number may be squared. In fact, this multiplication is not necessary. The matrix D can be represented as

$$D = I + B_{21}^* B_{22}^{-*} B_{22}^{-1} B_{21} = \begin{bmatrix} I \\ B_{22}^{-1} B_{21} \end{bmatrix}^* \begin{bmatrix} I \\ B_{22}^{-1} B_{21} \end{bmatrix}.$$

Then using the generalized Hammarling method [34] we compute the solution of (5.9) in factored form $X_{11} = Y^* Y$. In this case the solution Z of (4.8) can be written as

$$Z = V \begin{bmatrix} I & -B_{12} B_{22}^{-1} \end{bmatrix}^* Y^* Y \begin{bmatrix} I & -B_{12} B_{22}^{-1} \end{bmatrix} V^*.$$

Hence, $\|Z\| = \|Y \begin{bmatrix} I & -B_{12} B_{22}^{-1} \end{bmatrix}\|^2$ and $\|A^* Z A\| = \|Y \Sigma\|^2$.

The described methods to compute the projections P, Π and to solve the generalized Lyapunov equation (4.8) can be used if the matrices Σ and B_{22} are well-conditioned. Since

$$\text{cond}(\Sigma) \leq \text{cond}(A + BQ^\perp) \quad \text{and} \quad \text{cond}(B_{22}) \leq \text{cond}(A + BQ^\perp),$$

we can take the condition number of the matrix $A + BQ^\perp$ as a measure for the sensitivity of the projection onto the finite deflating subspace of the pencil $\lambda A - B$ of index at most one.

6 Perturbation analysis for the projections

The numerical computation of the deflating subspaces associated with specified eigenvalues of a regular matrix pencil and condition estimations for this problem have been studied extensively in recent years, e.g., [1, 10, 28]. Unfortunately, this problem may be ill-conditioned, since arbitrary small perturbations may change the structure of subspaces and even their dimension. In this section we present an error and perturbation analysis for the spectral projections onto the right and left finite deflating subspaces of the pencil $\lambda A - B$ of index at most one computed by the method described in Section 5.

The computation of the projections P and Π requires as a first step the decision about the numerical rank of A . The usual procedure is to compute the singular value decomposition of A and to set all singular values satisfying $\sigma_j < \epsilon c \|A\|$ to zero, where c is a constant and ϵ is the machine precision. If A and B are perturbed then the same procedure is performed. Due to the perturbation the numerical rank of A may change and, hence, also the spectral projections P and Π may change to \tilde{P} and $\tilde{\Pi}$, respectively. Even if we assume that the rank decision yields the same result r in both cases, then the accuracy of P and Π depends on the gap between σ_r and σ_{r+1} which is defined as

$$d_r = \frac{\|A\|}{\sigma_r(A) - \sigma_{r+1}(A)}. \tag{6.1}$$

Consider the perturbed matrices $\tilde{A} = A + \Delta A$, $\tilde{B} = B + \Delta B$, where $\|\Delta A\| \leq \epsilon \|A\|$ and $\|\Delta B\| \leq \epsilon \|B\|$. Let r be the numerical rank of A and let P^\perp and \tilde{P}^\perp be the orthogonal projections onto the spans of the right singular vectors of A and \tilde{A} , respectively, corresponding to their largest r singular values. Set $A_r = AP^\perp$ and $\tilde{A}_r = \tilde{A}\tilde{P}^\perp$. Then $Q^\perp = I - P^\perp$ and $\tilde{Q}^\perp = I - \tilde{P}^\perp$ are the orthogonal projections onto $\ker A_r$ and $\ker \tilde{A}_r$ with the same r . We will show that if the matrix pencil $\lambda A_r - B$ is regular of index one, then for sufficiently small ϵ the pencil $\lambda \tilde{A}_r - \tilde{B}$ is regular of index one as well.

Lemma 6.1. *Let d_r be as in (6.1). If the matrix $(A_r + BQ^\perp)$ is nonsingular and*

$$\epsilon_r \|(A_r + BQ^\perp)^{-1}\| (\|A\| + \|B\|) < 1,$$

where $\epsilon_r = \epsilon(1 + 2d_r)$, then the matrix $(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)$ is also nonsingular and

$$\|(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1} - (A_r + BQ^\perp)^{-1}\| \leq \frac{\epsilon_r \|(A_r + BQ^\perp)^{-1}\|^2 (\|A\| + \|B\|)}{1 - \epsilon_r \|(A_r + BQ^\perp)^{-1}\| (\|A\| + \|B\|)}. \tag{6.2}$$

Proof: From the relation

$$(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1} = (A_r + BQ^\perp)^{-1} - (\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1} (\tilde{A}_r - A_r + \tilde{B}\tilde{Q}^\perp - BQ^\perp) (A_r + BQ^\perp)^{-1} \tag{6.3}$$

we obtain the estimate

$$\|(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1}\| \leq \frac{\|(A_r + BQ^\perp)^{-1}\|}{1 - \|\tilde{A}_r - A_r + \tilde{B}\tilde{Q}^\perp - BQ^\perp\| \|(A_r + BQ^\perp)^{-1}\|}. \tag{6.4}$$

For $2\varepsilon d_r < 1$ one has the bound

$$\|\tilde{P}^\perp - P^\perp\| = \|\tilde{Q}^\perp - Q^\perp\| \leq \frac{\varepsilon d_r}{1 - \varepsilon d_r}, \tag{6.5}$$

see [17]. Then using the identities $A_r = AP^\perp$, $\tilde{A}_r = \tilde{A}\tilde{P}^\perp$ and $\|\tilde{P}^\perp\| = \|\tilde{Q}^\perp\| = 1$ we have

$$\begin{aligned} \|\tilde{A}_r - A_r + \tilde{B}\tilde{Q}^\perp - BQ^\perp\| &\leq \|\tilde{A} - A\| + \|A\|\|\tilde{P}^\perp - P^\perp\| + \|\tilde{B} - B\| + \|B\|\|\tilde{Q}^\perp - Q^\perp\| \leq \\ &\leq \varepsilon(1 + \frac{d_r}{1 - \varepsilon d_r})(\|A\| + \|B\|) \leq \varepsilon_r(\|A\| + \|B\|). \end{aligned} \tag{6.6}$$

Combining (6.4) and (6.6) we obtain

$$\|(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1}\| \leq \frac{\|(A_r + BQ^\perp)^{-1}\|}{1 - \varepsilon_r\|(A_r + BQ^\perp)^{-1}\|(\|A\| + \|B\|)}$$

under the condition that $\varepsilon_r\|(A_r + BQ^\perp)^{-1}\|(\|A\| + \|B\|) < 1$. Hence $(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)$ is nonsingular if $(A_r + BQ^\perp)$ is nonsingular. Bound (6.2) immediately follows from (6.3) and (6.6). \square

As a consequence of Lemma 6.1 we have the following theorem.

Theorem 6.2. *Let r be the numerical rank of the matrix A and let P^\perp be the orthogonal projection onto the span of the right singular vectors of A corresponding to its largest r singular values. Assume that the pencil $\lambda A_r - B$ is of index one, where $A_r = AP^\perp$. Then for $\varepsilon_r \text{cond}(A_r + BQ^\perp)(\|A\| + \|B\|) < \|A_r + BQ^\perp\|$, the perturbed pencil $\lambda \tilde{A}_r - \tilde{B}$ is of index one. Moreover, for the spectral projections P and \tilde{P} onto the right finite deflating subspaces of $\lambda A_r - B$ and $\lambda \tilde{A}_r - \tilde{B}$, respectively, one has the bound*

$$\|\tilde{P} - P\| \leq \frac{3\varepsilon_r \text{cond}^2(A_r + BQ^\perp)(\|A\| + \|B\|)\|B\|}{\|A_r + BQ^\perp\|(\|A_r + BQ^\perp\| - \varepsilon_r \text{cond}(A_r + BQ^\perp)(\|A\| + \|B\|))}. \tag{6.7}$$

Proof: If the pencil $\lambda A_r - B$ has index one, then $(A_r + BQ^\perp)$ is nonsingular and the spectral projection onto the right finite deflating subspace of $\lambda A_r - B$ can be computed as $P = I - Q^\perp(A_r + BQ^\perp)^{-1}B$ [21]. Then by Lemma 6.1 the matrix $(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)$ is nonsingular and, hence, the pencil $\lambda \tilde{A}_r - \tilde{B}$ is of index one and the spectral projection \tilde{P} onto the right finite deflating subspace of $\lambda \tilde{A}_r - \tilde{B}$ has the form $\tilde{P} = I - \tilde{Q}^\perp(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1}\tilde{B}$. Then by adding and subtracting equal terms we obtain

$$\begin{aligned} \|\tilde{P} - P\| &= \|\tilde{Q}^\perp(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1}\tilde{B} - Q^\perp(A_r + BQ^\perp)^{-1}B\| \leq \\ &\leq \|(\tilde{A}_r + \tilde{B}\tilde{Q}^\perp)^{-1} - (A_r + BQ^\perp)^{-1}\|\|\tilde{B}\| + \|(A_r + BQ^\perp)^{-1}\|(\|\tilde{B} - B\| + \|\tilde{Q}^\perp - Q^\perp\|\|B\|). \end{aligned}$$

Using bounds (6.2), (6.5) and $1 \leq \text{cond}(A_r + BQ^\perp) \leq \|(A_r + BQ^\perp)^{-1}\|(\|A\| + \|B\|)$, we obtain that

$$\begin{aligned} \|\tilde{P} - P\| &\leq \frac{\varepsilon_r(1 + \varepsilon)\|(A_r + BQ^\perp)^{-1}\|^2(\|A\| + \|B\|)\|B\|}{1 - \varepsilon_r\|(A_r + BQ^\perp)^{-1}\|(\|A\| + \|B\|)} + \varepsilon_r\|(A_r + BQ^\perp)^{-1}\|\|B\| \leq \\ &\leq \frac{3\varepsilon_r \text{cond}^2(A_r + BQ^\perp)(\|A\| + \|B\|)\|B\|}{\|A_r + BQ^\perp\|(\|A_r + BQ^\perp\| - \varepsilon_r \text{cond}(A_r + BQ^\perp)(\|A\| + \|B\|))}. \end{aligned} \quad \square$$

Bound (6.7) implies that if the gap between the singular values σ_r and σ_{r+1} of the matrix A is not small, i.e., the value d_r is not large, and if the condition number of $A_r + BQ^\perp$ is not large, then the error of the projection P is small for enough small ε . Large values of d_r and $\text{cond}(A_r + BQ^\perp)$ indicate that either the deflating subspace of the matrix pencil $\lambda A_r - B$ corresponding to the finite eigenvalues is ill-conditioned or $\lambda A_r - B$ is near to a pencil with index greater than one.

For the projection $\Pi = I - BQ^\perp(A_r + BQ^\perp)^{-1}$, the same perturbation estimate holds.

7 Sensitivity analysis for the generalized Lyapunov equation

In this section we present a bound on the sensitivity of the solution Z of the generalized Lyapunov equation (4.8). The perturbation analysis for the standard Lyapunov equation was the topic of numerous papers [18, 23, 24, 26]. The sensitivity of the generalized Lyapunov equation with nonsingular A is studied in [29]. The analysis of the general problem with a singular matrix A is very complicated and still not completely known. The difficulty is that small perturbations in the stable pencil $\lambda A - B$ may alter strongly its eigenstructure. This may lead to the change of the dimension of the finite deflating subspace, loss of the regularity or jumping of eigenvalues to the closed right half-plane [6].

In the sequel we consider only perturbations which exclude the case when the dimension of the finite deflating subspace of the pencil is changed. In many practical applications this is justified. Consider, for example, semi-explicit differential-algebraic equations

$$A_{11}\dot{x}_1(t) = B_{11}x_1(t) + B_{12}x_2(t), \tag{7.1}$$

$$0 = B_{21}x_1(t) + B_{22}x_2(t), \tag{7.2}$$

with a nonsingular matrix A_{11} [4, 32]. Equation (7.1) describes the dynamic behavior of the system, while equation (7.2) gives algebraic constraints on the states. Obviously, it is unreasonable to consider perturbations which cause the algebraic constraints to become differential.

Note that in the study of the asymptotic stability of the differential-algebraic equation (2.3) it is allowed for the index of the matrix pencil $\lambda A - B$ to be changed by perturbations. It is important only that finite eigenvalues stay finite and infinite eigenvalues must stay infinite. However, the perturbation analysis in this case is very complicated. We will deal only with perturbations which preserve the nilpotency structure of the pencil $\lambda A - B$, i.e., the right and left infinite deflating subspaces of $\lambda A - B$ are not changed. In this case

$$\ker P = \ker \tilde{P}, \quad \ker \Pi = \ker \tilde{\Pi}, \quad (7.3)$$

where P and \tilde{P} (Π and $\tilde{\Pi}$) are the spectral projections onto the right (left) finite deflating subspaces of the pencil $\lambda A - B$ and the perturbed pencil $\lambda \tilde{A} - \tilde{B}$, respectively. It follows from (7.3) that

$$\begin{aligned} \tilde{P}P &= \tilde{P} & \text{and} & & P\tilde{P} &= P, \\ \tilde{\Pi}\Pi &= \tilde{\Pi} & \text{and} & & \Pi\tilde{\Pi} &= \Pi. \end{aligned} \quad (7.4)$$

Moreover, we will assume that for the allowable perturbations ΔA and ΔB of the matrix pencil $\lambda A - B$ such that $\|\Delta A\| \leq \varepsilon\|A\|$ and $\|\Delta B\| \leq \varepsilon\|B\|$, we have an error bound $\|\tilde{P} - P\| \leq \varepsilon K$ with some constant K . This estimate implies that the right finite deflating subspace of the perturbed pencil $\lambda \tilde{A} - \tilde{B} = \lambda(A + \Delta A) - (B + \Delta B)$ is close to the right finite deflating subspace of $\lambda A - B$. For example, in the case of a matrix pencil $\lambda A - B$ of index at most one bound (6.7) implies that

$$K = \frac{3(1 + 2d_r) \text{cond}^2(A + BQ^\perp)(\|A\| + \|B\|)\|B\|}{\|A + BQ^\perp\|(\|A + BQ^\perp\| - \varepsilon_r \text{cond}(A + BQ^\perp)(\|A\| + \|B\|))}.$$

Nevertheless, under allowable perturbations the perturbed pencil may have a finite eigenvalue in the closed right half-plane. We will show that if all finite eigenvalues of the regular pencil $\lambda A - B$ lie in the open left half-plane, then for small enough ε , the pencil $\lambda \tilde{A} - \tilde{B}$ is regular and it has no finite eigenvalues with nonnegative real part.

Consider now the perturbed equations

$$\tilde{A}^* \tilde{Z} \tilde{B} + \tilde{B}^* \tilde{Z} \tilde{A} = -\tilde{P}^* \tilde{P}, \quad \tilde{Z} = \tilde{Z} \tilde{\Pi}. \quad (7.5)$$

The following theorem gives an error bound for the solution of (4.8).

Theorem 7.1. *Let $\lambda A - B$ be stable and let Z be a solution of the generalized Lyapunov equation (4.8) together with $Z = Z\Pi$. Assume that for the spectral projections \tilde{P} and $\tilde{\Pi}$ onto the right and left finite deflating subspaces of the perturbed pencil $\lambda \tilde{A} - \tilde{B} = \lambda(A + \Delta A) - (B + \Delta B)$ with $\|\Delta A\| \leq \varepsilon\|A\|$ and $\|\Delta B\| \leq \varepsilon\|B\|$, relations (7.4) are satisfied and a bound $\|\tilde{P} - P\| \leq \varepsilon K < 1$ holds with some constant K . If $3\varepsilon\kappa(A, B) < 1$, then the pencil $\lambda \tilde{A} - \tilde{B}$ is stable, the perturbed equations (7.5) have a unique solution \tilde{Z} and*

$$\frac{\|\tilde{Z} - Z\|}{\|\tilde{Z}\|} \leq \frac{3\varepsilon(K\|P\| + \kappa(A, B))}{1 - 3\varepsilon\kappa(A, B)}. \quad (7.6)$$

Proof: The perturbed generalized Lyapunov equation in (7.5) can be rewritten as

$$A^* \tilde{Z} B + B^* \tilde{Z} A = -\left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z})\right),$$

where $\mathcal{D}(\tilde{Z}) = (\Delta A)^* \tilde{Z} \tilde{B} + A^* \tilde{Z} \Delta B + (\Delta B)^* \tilde{Z} A + \tilde{B}^* \tilde{Z} \Delta A$. Using (2.1) we can verify that $\Pi A = \Pi A P = A P$ and $\Pi B = \Pi B P = B P$. Analogous relations hold for the perturbed pencil $\lambda \tilde{A} - \tilde{B}$. Then by (7.4) we obtain that $\tilde{Z} = \tilde{Z} \Pi = \tilde{Z} \Pi \tilde{\Pi} = \tilde{Z} \tilde{\Pi}$ and

$$\tilde{Z} A = \tilde{Z} \Pi A = \tilde{Z} A P = \tilde{Z} \Pi A P \tilde{P} = \tilde{Z} A \tilde{P}, \quad \tilde{Z} \tilde{A} = \tilde{Z} \tilde{\Pi} \tilde{A} = \tilde{Z} \tilde{A} \tilde{P} = \tilde{Z} \tilde{\Pi} \tilde{A} \tilde{P} = \tilde{Z} \tilde{A} \tilde{P}.$$

These relationships remain valid if we replace A by B and \tilde{A} by \tilde{B} . In this case we obtain

$$\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z}) = P^* \left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z})\right) P = \tilde{P}^* \left(I + \mathcal{D}(\tilde{Z})\right) \tilde{P}. \quad (7.7)$$

Then the perturbed equations (7.5) are equivalent to

$$A^* \tilde{Z} B + B^* \tilde{Z} A = -P^* \left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z})\right) P, \quad \tilde{Z} = \tilde{Z} \Pi.$$

Since the pencil $\lambda A - B$ is stable, then by Theorem 4.1 these equations have a unique solution \tilde{Z} that has the form

$$\tilde{Z} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* \left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z})\right) P (i\xi A - B)^{-1} d\xi. \quad (7.8)$$

Thus, we have the integral equation $\tilde{Z} = \mathcal{I}(\tilde{Z})$ with respect to unknown matrix \tilde{Z} , where

$$\mathcal{I}(\tilde{Z}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* \left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z}) \right) P (i\xi A - B)^{-1} d\xi.$$

From $\|\mathcal{D}(\tilde{Z})\| \leq 2(\|\Delta A\| \|\tilde{B}\| + \|\Delta B\| \|A\|) \|\tilde{Z}\| \leq 6\|A\| \|B\| \|\tilde{Z}\|$ we obtain that

$$\begin{aligned} \|\mathcal{I}(Z_1) - \mathcal{I}(Z_2)\| &= \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* \mathcal{D}(Z_1 - Z_2) P (i\xi A - B)^{-1} d\xi \right\| \leq \\ &\leq \|\mathcal{D}(Z_1 - Z_2)\| \|Z\| \leq 3\varepsilon \kappa(A, B) \|Z_1 - Z_2\| \end{aligned}$$

for any matrices Z_1 and Z_2 . Since $3\varepsilon \kappa(A, B) < 1$, the operator $\mathcal{I}(\tilde{Z})$ is contracting. Then by the fixed point theorem [27] the equation $\tilde{Z} = \mathcal{I}(\tilde{Z})$ has a unique solution \tilde{Z} and we can estimate the error

$$\begin{aligned} \|\tilde{Z} - Z\| &= \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi A - B)^{-*} P^* \left(\tilde{P}^* \tilde{P} + \mathcal{D}(\tilde{Z}) - P^* P \right) P (i\xi A - B)^{-1} d\xi \right\| \leq \\ &\leq \left(\|\tilde{P}^* \tilde{P} - P^* P\| + \|\mathcal{D}(\tilde{Z})\| \right) \|Z\|. \end{aligned}$$

Taking into account that

$$\|\tilde{P}^* \tilde{P} - P^* P\| \leq \|\tilde{P} - P\| (\|\tilde{P} - P\| + 2\|P\|) \leq 3\varepsilon K \|P\|$$

and $\|\mathcal{D}(\tilde{Z})\| \leq 6\varepsilon \|A\| \|B\| (\|Z\| + \|\tilde{Z} - Z\|)$ we obtain the relative perturbation bound (7.6). □

Bound (7.6) shows that for not large $\kappa(A, B)$ and $K\|P\|$, the solution of the perturbed equations (7.5) is a small perturbation of the solution of (4.8) together with $Z = Z\Pi$. The parameter $\kappa(A, B)$ may be used as a condition number for the generalized Lyapunov equation (4.8).

8 Numerical experiments

In this section we present results of numerical experiments of computing the projection P and the parameters $\varkappa(A, B)$, $\kappa(A, B)$. Computations were performed in MATLAB 5.2 on HP-UX 10.20 workstation using double precision arithmetic with machine precision $\epsilon \approx 2.2 \cdot 10^{-16}$. In the rank decision problem we set the computed singular value $\sigma_j(A)$ to zero if $\sigma_j(A) \leq \epsilon n \|A\|$. The number of remaining nonzero singular values is taken to be the numerical rank of the matrix. To solve the generalized Lyapunov equation (5.9) we use the generalized Bartels-Stewart method from [34]. The normalized residual

$$\Delta = \frac{\|A^* Z B + B^* Z A + P^* P\|}{2\|A\| \|B\| \|Z\|}$$

is a measure of the quality of the computed solution of the generalized Lyapunov equation (4.8).

Example 3. [9, Example 1-3.1] Consider the system

$$A\dot{x}(t) = Bx(t) + Fv_s(t) \tag{8.1}$$

with the measured output $y(t) = Gx(t)$, where

$$A = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \tag{8.2}$$

$$G = (0 \ 0 \ 1 \ 0), \quad x(t) = (I(t) \ v_L(t) \ v_C(t) \ v_R(t))^T.$$

Equation (8.1) with (8.2) describes a simple RLC electrical circuit. The voltage source $v_s(t)$ is the control input, $R = 2$, $L = 1.1$ and $C = 10^{-4}$ are the resistance, inductance and capacitance, respectively, $v_R(t)$, $v_L(t)$ and $v_C(t)$ are the corresponding voltage drops and $I(t)$ is the current. For the proportional output feedback control $v_s(t) = Ky(t) = KGx(t)$ we have the closed loop system $A\dot{x}(t) = (B + FKG)x(t)$. The finite eigenvalues of the matrix pencil $\lambda A - B_K$ with $B_K = B + FKG$ are given by

$$\lambda_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{K-1}{CL}}.$$

It is easy to see that if $K \geq 1$, then the pencil $\lambda A - B_K$ has one eigenvalue in the closed right half-plane, and both its finite eigenvalues have negative real part, otherwise.

Table 1 gives the numerical results for different values of K . For all K the gap is $d_2 = 1.1$. We see that as K approaches to 1, the values of $\|A^* Z A\|$ and, respectively, $\varkappa(A, B_K)$, $\kappa(A, B_K)$ increase. For $K = 1$, the Lyapunov equation (4.8) is not solvable.

Table 1: Example 3

K	$\text{cond}(A_2 + B_K Q^\perp)$	Δ	$\ A^* Z A\ $	$\varkappa(A, B_K)$	$\kappa(A, B_K)$
0	3.9022	$3.57 \cdot 10^{-20}$	$6.0525 \cdot 10^3$	$1.7114 \cdot 10^8$	$3.3018 \cdot 10^8$
$1 - 10^{-2}$	3.9022	$2.36 \cdot 10^{-20}$	$3.0253 \cdot 10^5$	$6.0509 \cdot 10^9$	$1.6502 \cdot 10^{10}$
$1 - 10^{-4}$	3.9022	$2.53 \cdot 10^{-20}$	$3.0250 \cdot 10^7$	$6.0500 \cdot 10^{11}$	$1.6500 \cdot 10^{12}$
$1 - 10^{-6}$	3.9022	$1.73 \cdot 10^{-20}$	$3.0250 \cdot 10^9$	$6.0500 \cdot 10^{13}$	$1.6500 \cdot 10^{14}$
1	3.9022	–	∞	∞	∞

Example 4. [22] The following example is a model for the transistor amplifier. The equation has the form

$$A \frac{dy}{dt} = f(y), \tag{8.3}$$

where

$$A = \begin{bmatrix} -C_1 & C_1 & 0 & 0 & 0 \\ C_1 & -C_1 & 0 & 0 & 0 \\ 0 & 0 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & -C_3 & C_3 \\ 0 & 0 & 0 & C_3 & -C_3 \end{bmatrix}, \quad f(y) = \begin{bmatrix} -\frac{U_e(t)}{R_0} + \frac{y_1}{R_0} \\ -\frac{6}{R_2} + y_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + 0.01g(y_2 - y_3) \\ -g(y_2 - y_3) + \frac{y_3}{R_3} \\ -\frac{6}{R_4} + \frac{y_4}{R_4} + 0.99g(y_2 - y_3) \\ \frac{y_5}{R_5} \end{bmatrix}$$

with

$$g(x) = 10^{-6} \left(e^{x/0.026} - 1 \right); \quad U_e(t) = 0.1 \sin(200\pi t); \quad R_0 = 1000;$$

$$C_k = k \cdot 10^{-6}, \quad k = 1, 2, 3; \quad R_k = 9000, \quad k = 1, \dots, 5.$$

Asymptotic stability of the stationary solution y_* of (8.3) is equivalent to asymptotic stability of the trivial solution of the linearized system $A\dot{x}(t) = Bx(t)$ with $B = f'(y_*)$ [30].

The stationary solution of (8.3) is given by $y_* = (0, 2.98582, 2.83616, 3.19220, 0)^T$. The following computed parameters

$$d_3 = 3, \quad \text{cond}(A_3 + BQ^\perp) = 7.9915 \cdot 10^4, \quad \|P\| = 1.0656 \cdot 10^2,$$

$$\Delta = 1.0661 \cdot 10^{-18}, \quad \varkappa(A, B) = 1.2077 \cdot 10^6, \quad \kappa(A, B) = 4.7521 \cdot 10^6.$$

show that the pencil $\lambda A - B$ is of index 1 and has no finite eigenvalues in the closed right half-plane, i.e., the stationary solution y_* of (8.3) is asymptotically stable.

Conclusion

We have derived parameters that can be used to investigate the asymptotic stability of the trivial solution of linear DAE without computing the eigenvalues of the corresponding matrix pencil explicitly. To determine numerically these parameters it is necessary to compute the spectral projections onto the right and left deflating subspaces of the pencil corresponding to the finite eigenvalues and to solve a generalized Lyapunov equation. We have described a method for computing such projections and for solving the generalized Lyapunov equation for the matrix pencil of index at most one. This method is based on the singular value decomposition and admits error analysis for the computed projections. The sensitivity of the generalized Lyapunov equation under allowable perturbations which preserve the nilpotency structure of the pencil has been discussed. The computation of the projection onto the finite deflating subspace and the solution of the generalized Lyapunov equation for a pencil of higher index together with a complete perturbation analysis are still open problems and currently under investigation.

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