Model order reduction of electrical circuits with nonlinear elements

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1 Introduction

The efficient and robust numerical simulation of electrical circuits plays a major role in computer aided design of electronic devices. While the structural size of such devices is decreasing, the complexity of the electrical circuits is increasing. This usually leads to a system of model equations in form of differential-algebraic equations (DAE) with a huge number of unknowns. Simulation of such models is unacceptably time and storage consuming. Model order reduction presents a way out of this dilemma. A general idea of model reduction is to replace a large-scale system by a much smaller model which approximates the input-output relation of the original system within a required accuracy. While a large variety of model reduction techniques exists for linear networks, e.g., [1, 3, 4], model reduction of nonlinear circuits is only in its infancy.
In [2], model reduction of nonlinear circuit with only nonlinear resistors was considered. In this paper, we extend these results to more general circuits that may contain other nonlinear elements like nonlinear capacitors or inductors.

2 Circuit Equations

A commonly used tool for modeling electrical circuits is the Modified Nodal Analysis (MNA). An electrical circuit can be modeled as a directed graph whose nodes correspond to the nodes of the circuit and whose branches correspond to the circuit elements. Using Kirchhoff's laws as well as the branch constitutive relations, the dynamics of an electrical circuit can be described by a DAE system of the form

$$\mathscr{E}(x)\tfrac{\mathrm{d}}{\mathrm{d}t}x = \mathscr{A}\,x + f(x) + \mathscr{B}\,u, \qquad y = \mathscr{B}^T x, \tag{1a}$$

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with

$$\mathscr{E}(x)=\begin{bmatrix} A_{\mathcal{C}}\,\mathcal{C}(A_{\mathcal{C}}^T\eta)A_{\mathcal{C}}^T & 0 & 0\\ 0 & \mathcal{L}(\iota_{\mathcal{L}}) & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathscr{A}=\begin{bmatrix} 0 & -A_{\mathcal{L}} & -A_{\mathcal{V}}\\ A_{\mathcal{L}}^T & 0 & 0\\ A_{\mathcal{V}}^T & 0 & 0\end{bmatrix},\quad x=\begin{bmatrix}\eta\\ \iota_{\mathcal{L}}\\ \iota_{\mathcal{V}}\end{bmatrix}, \tag{1b}$$

$$f(x)=\begin{bmatrix} -A_{\mathcal{R}}\,g(A_{\mathcal{R}}^T\eta)\\ 0\\ 0\end{bmatrix},\quad \mathscr{B}=\begin{bmatrix} -A_{\mathcal{J}} & 0\\ 0 & 0\\ 0 & -I\end{bmatrix},\quad u=\begin{bmatrix}\iota_{\mathcal{J}}\\ u_{\mathcal{V}}\end{bmatrix},\quad y=-\begin{bmatrix}u_{\mathcal{J}}\\ \iota_{\mathcal{V}}\end{bmatrix}. \tag{1c}$$

Here, x, u and y are state, input and output, respectively, η is the vector of node potentials, $\iota_{\mathcal{L}}$, $\iota_{\mathcal{V}}$ and $\iota_{\mathcal{J}}$ are the vectors of currents through inductors, voltage sources and current sources, respectively, $u_{\mathcal{V}}$ and $u_{\mathcal{J}}$ are the vectors of voltages of voltage sources and current sources, respectively. We will distinguish between linear circuit elements (denoted by a bar) that are characterized by linear current-voltage relations and nonlinear circuit components (denoted by a tilde) that are characterized by nonlinear current-voltage relations. Without loss of generality, we assume that the circuit elements are ordered such that the incidence matrices describing the circuit topology have the form $A_{\mathcal{C}}=[A_{\bar{\mathcal{C}}}\ A_{\widetilde{\mathcal{C}}}]\in\mathbb{R}^{n_\eta,n_{\bar{\mathcal{C}}}+n_{\widetilde{\mathcal{C}}}}$, $A_{\mathcal{L}}=[A_{\bar{\mathcal{L}}}\ A_{\widetilde{\mathcal{L}}}]\in\mathbb{R}^{n_\eta,n_{\bar{\mathcal{L}}}+n_{\widetilde{\mathcal{L}}}}$, $A_{\mathcal{R}}=[A_{\bar{\mathcal{R}}}\ A_{\widetilde{\mathcal{R}}}]\in\mathbb{R}^{n_\eta,n_{\bar{\mathcal{R}}}+n_{\widetilde{\mathcal{R}}}}$, $A_{\mathcal{V}}\in\mathbb{R}^{n_\eta,n_{\mathcal{V}}}$ and $A_{\mathcal{J}}\in\mathbb{R}^{n_\eta,n_{\mathcal{J}}}$, where the incidence matrices $A_{\bar{\mathcal{C}}}$, $A_{\bar{\mathcal{L}}}$ and $A_{\bar{\mathcal{R}}}$ correspond to the linear circuit components, and $A_{\widetilde{\mathcal{C}}}$, $A_{\widetilde{\mathcal{L}}}$ and $A_{\widetilde{\mathcal{R}}}$ correspond to the nonlinear circuit components. Furthermore, the conductance matrix-valued function $\mathcal{C}:\mathbb{R}^{n_{\mathcal{C}}}\to\mathbb{R}^{n_{\mathcal{C}},n_{\mathcal{C}}}$, the inductance matrix-valued function $\mathcal{L}:\mathbb{R}^{n_{\mathcal{L}}}\to\mathbb{R}^{n_{\mathcal{L}},n_{\mathcal{L}}}$ and the resistor relation $g:\mathbb{R}^{n_{\mathcal{R}}}\to\mathbb{R}^{n_{\mathcal{R}}}$ given by

$$\mathcal{C}(A_{\mathcal{C}}^T\eta)=\begin{bmatrix}\bar{\mathcal{C}} & 0\\ 0 & \widetilde{\mathcal{C}}(A_{\widetilde{\mathcal{C}}}^T\eta)\end{bmatrix},\quad \mathcal{L}(\iota_{\mathcal{L}})=\begin{bmatrix}\bar{\mathcal{L}} & 0\\ 0 & \widetilde{\mathcal{L}}(\iota_{\widetilde{\mathcal{L}}})\end{bmatrix},\quad g(A_{\mathcal{R}}^T\eta)=\begin{bmatrix}\bar{\mathcal{G}}A_{\bar{\mathcal{R}}}^T\eta\\ \widetilde{g}(A_{\widetilde{\mathcal{R}}}^T\eta)\end{bmatrix} \tag{1d}$$

characterize the physical properties of the capacitors, inductors and resistors, respectively. Here $\iota_{\widetilde{\mathcal{L}}}\in\mathbb{R}^{n_{\widetilde{\mathcal{L}}}}$ is the vector of currents through the nonlinear inductors. We will assume that the matrices $A_{\mathcal{V}}$ and $[A_{\mathcal{C}}\ A_{\mathcal{L}}\ A_{\mathcal{R}}\ A_{\mathcal{V}}]$ have full rank, the matrices $\mathcal{C}(A_{\mathcal{C}}^T\eta)$ and $\mathcal{L}(\iota_{\mathcal{L}})$ are symmetric, positive definite and the function $g(A_{\mathcal{R}}^T\eta)$ is monotonically increasing for all admissible η and $\iota_{\mathcal{L}}$. This assumptions imply that the circuit elements do not generate energy, i.e., the circuit is passive.

3 Model Reduction for Nonlinear Circuits

In this section, we present a model reduction technique for nonlinear circuits with a small number of nonlinear elements. This technique is based on decoupling of the linear and nonlinear subcircuits in a suitable way, reduction of the linear part using the PABTEC method [4] followed by an adequate recoupling of the unchanged nonlinear part and the reduced linear part to obtain a nonlinear reduced-order model.

3.1 Decoupling of Linear and Nonlinear Subcircuits

Consider the circuit equations (1). Let $A_{\widetilde{\mathcal{R}}} = A^1_{\widetilde{\mathcal{R}}} + A^2_{\widetilde{\mathcal{R}}}$ be decomposed such that $A^1_{\widetilde{\mathcal{R}}} \in \{0,1\}^{n_\eta, n_{\widetilde{\mathcal{R}}}}$ and $A^2_{\widetilde{\mathcal{R}}} \in \{-1,0\}^{n_\eta, n_{\widetilde{\mathcal{R}}}}$ and let $G_1, G_2 \in \mathbb{R}^{n_{\widetilde{\mathcal{R}}}, n_{\widetilde{\mathcal{R}}}}$ be given such that G_1 and $G_1 + G_2$ are both symmetric, positive definite and G_2 is positive semidefinite. Assume that $u_{\widetilde{\mathcal{C}}} \in \mathbb{R}^{n_{\widetilde{\mathcal{C}}}}$ and $\imath_z \in \mathbb{R}^{n_{\widetilde{\mathcal{R}}}}$ satisfy the relations

$$u_{\widetilde{\mathcal{C}}} = A^T_{\widetilde{\mathcal{C}}}\eta \quad \text{and} \quad \imath_z = (G_1 + G_2)G_1^{-1}\widetilde{g}(A^T_{\widetilde{\mathcal{R}}}\eta) - G_2 A^T_{\widetilde{\mathcal{R}}}\eta. \tag{2}$$

Then system (1) together with the relations

$$\eta_z = (G_1 + G_2)^{-1}\big((G_1(A^1_{\widetilde{\mathcal{R}}})^T - G_2(A^2_{\widetilde{\mathcal{R}}})^T)\eta - \imath_z\big), \tag{3a}$$

$$\imath_{\widetilde{\mathcal{C}}} = \widetilde{\mathcal{C}}(u_{\widetilde{\mathcal{C}}})\tfrac{\mathrm{d}}{\mathrm{d}t}u_{\widetilde{\mathcal{C}}} \tag{3b}$$

for the additional unknowns $\eta_z \in \mathbb{R}^{n_{\widetilde{\mathcal{R}}}}$ and $\imath_{\widetilde{\mathcal{C}}} \in \mathbb{R}^{n_{\widetilde{\mathcal{C}}}}$, is equivalent to the system

$$\widetilde{\mathcal{L}}(\imath_{\widetilde{\mathcal{L}}})\tfrac{\mathrm{d}}{\mathrm{d}t}\imath_{\widetilde{\mathcal{L}}} = A^T_{\widetilde{\mathcal{L}}}\eta, \tag{4}$$

coupled with the linear system

$$E\tfrac{\mathrm{d}}{\mathrm{d}t}x_\ell = Ax_\ell + Bu_\ell, \qquad y_\ell = B^T x_\ell. \tag{5a}$$

Here, the system matrices are in the MNA form

$$E = \begin{bmatrix} A_C C A_C^T & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_R G A_R^T & -A_L & -A_V \\ A_L^T & 0 & 0 \\ A_V^T & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \tag{5b}$$

with incidence and element matrices

$$A_C = \begin{bmatrix} A_{\bar{\mathcal{C}}} \\ 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} A_{\bar{\mathcal{R}}} & A^1_{\widetilde{\mathcal{R}}} & A^2_{\widetilde{\mathcal{R}}} \\ 0 & -I & I \end{bmatrix}, \quad A_L = \begin{bmatrix} A_{\bar{\mathcal{L}}} \\ 0 \end{bmatrix}, \quad A_V = \begin{bmatrix} A_{\mathcal{V}} & A_{\widetilde{\mathcal{C}}} \\ 0 & 0 \end{bmatrix}, \tag{5c}$$

$$A_I = \begin{bmatrix} A_{\mathcal{J}} & A^2_{\widetilde{\mathcal{R}}} & A_{\widetilde{\mathcal{L}}} \\ 0 & I & 0 \end{bmatrix}, \quad C = \bar{\mathcal{C}}, \quad L = \bar{\mathcal{L}}, \quad G = \begin{bmatrix} \bar{\mathcal{G}} & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_2 \end{bmatrix}, \tag{5d}$$

and $x_\ell^T = [\,\eta^T \; \eta_z^T \,|\, \imath_{\bar{\mathcal{L}}}^T \,|\, \imath_{\mathcal{V}}^T \; \imath_{\widetilde{\mathcal{C}}}^T\,]$, $u_\ell^T = [\,\imath_{\mathcal{J}}^T \; \imath_z^T \,|\, \imath_{\widetilde{\mathcal{L}}}^T \,|\, u_{\mathcal{V}}^T \; u_{\widetilde{\mathcal{C}}}^T\,]$, $y_\ell^T = [\,y_1^T \; y_2^T \,|\, y_3^T \,|\, y_4^T \; y_5^T\,]$. With (2), equivalency here means that $[x^T \; \eta_z^T \; \imath_{\widetilde{\mathcal{C}}}^T]^T$ solves (1) and (3) if and only if $[x_\ell^T \; \imath_{\widetilde{\mathcal{L}}}^T]^T$ solves (5) and (4). More details and the proof can be found in [5].

3.2 Model-Order Reduction of the Linear Subsystem

We now apply the PABTEC method [4] to the linear system (5) with a transfer function $G(s) = B^T(sE - A)^{-1}B$. The assumptions above on the nonlinear system (1) guarantee that the following projected Lur'e equations

$$\begin{gathered} EX(A - BB^T)^T + (A - BB^T)XE^T + 2P_l BB^T P_l^T = -2K_c K_c^T, \\ EXB - P_l BM_0^T = -K_c J_c^T, \qquad J_c J_c^T = I - M_0 M_0^T, \qquad X = P_r X P_r^T \end{gathered} \tag{6}$$

and

$$\begin{aligned} E^TY(A-BB^T)+(A-BB^T)^TYE+2P_r^TBB^TP_r &= -2K_o^TK_o, \\ -E^TYB+P_r^TBM_0 = -K_o^TJ_o, \qquad J_o^TJ_o = I-M_0^TM_0, & \qquad Y=P_l^TYP_l \end{aligned} \tag{7}$$

are solvable for X, K_c, J_c and Y, K_o, J_o, respectively, Here P_r and P_l are the spectral projectors onto the right and left deflating subspaces of $\lambda E-(A-BB^T)$ corresponding to the finite eigenvalues and $M_0 = I-2\lim_{s\to\infty} B^T(sE-A+BB^T)^{-1}B$. The minimal solutions $X_{\min}$ and $Y_{\min}$ of (6) and (7) that satisfy $0\le X_{\min}\le X$ and $0\le Y_{\min}\le Y$ for all symmetric solutions X and Y of (6) and (7), are called the controllability and observability Gramians of system (5). Using the block structure of the system matrices (5b), we can show that $P_l = S_{\text{int}}P_rS_{\text{int}}$ and $Y_{\min} = S_{\text{int}}X_{\min}S_{\text{int}}$ with $S_{\text{int}} = \text{diag}(I,-I,-I)$.

Model reduction consists in approximating the large-scale DAE system (5) of order $n_\ell = n_\eta + n_{\widetilde{\mathcal{R}}} + n_{\widetilde{\mathcal{L}}} + n_{\mathcal{V}} + n_{\widetilde{\mathcal{C}}}$ by a reduced-order model

$$\hat{E}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{x}_\ell = \hat{A}\hat{x}_\ell + \hat{B}u_\ell, \qquad \hat{y} = \hat{C}\hat{x}_\ell, \tag{8}$$

where $\hat{x}_\ell\in\mathbb{R}^r$ and $r\ll n_\ell$. It is required that the approximate system (8) captures the input-output behavior of (5) to a required accuracy and preserves passivity.

The PABTEC model reduction method is based on transforming system (5) into a balanced form whose controllability and observability Gramians are equal and diagonal. Then a reduced-order model is computed by truncation the states corresponding to the small diagonal elements of the balanced Gramians. We summarize the PABTEC method in Algorithm 1.

Algorithm 1. Passivity-preserving balanced truncation for electrical circuits (PABTEC).

1. Compute the Cholesky factor R of the minimal solution $X_{\min} = RR^T$ of (6).
2. Compute the eigenvalue decompositions $R^TS_{\text{int}}ER = [U_1\ \ U_2]\text{diag}(\Lambda_1,\Lambda_2)[V_1\ \ V_2]^T$ and $(I-M_0)S_{\text{ext}} = U_0\Lambda_0U_0^T$, where $[U_1\ U_2]$, $[V_1\ V_2]$ and U_0 are orthogonal, $\Lambda_1 = \text{diag}(\lambda_1,\ldots,\lambda_{r_f})$, $\Lambda_2 = \text{diag}(\lambda_{r_f+1},\ldots,\lambda_q)$ and $\Lambda_0 = \text{diag}(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)$.
3. Compute the reduced-order system (8) with
$$\hat{E} = \begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},\quad \hat{A} = \frac{1}{2}\begin{bmatrix} 2W^TAT & \sqrt{2}W^TBC_\infty \\ -\sqrt{2}B_\infty B^TT & 2I-B_\infty C_\infty\end{bmatrix},\quad \hat{B} = \begin{bmatrix} W^TB \\ -B_\infty/\sqrt{2}\end{bmatrix},\quad \hat{C}^T = \begin{bmatrix} T^TB \\ C_\infty^T/\sqrt{2}\end{bmatrix},$$
where $B_\infty = S_0|\Lambda_0|^{1/2}U_0^TS_{\text{ext}}$, $C_\infty = U_0|\Lambda_0|^{1/2}$, $S_{\text{ext}} = \text{diag}(I,-I)$,
$W = S_{\text{int}}RU_1|\Lambda_1|^{-1/2}$, $S_0 = \text{diag}(\text{sign}(\hat{\lambda}_1),\ldots,\text{sign}(\hat{\lambda}_m))$, $|\Lambda_0| = \text{diag}(|\hat{\lambda}_1|,\ldots,|\hat{\lambda}_m|)$,
$T = RU_1S_1|\Lambda_1|^{-1/2}$, $S_1 = \text{diag}(\text{sign}(\lambda_1),\ldots,\text{sign}(\lambda_{r_f}))$, $|\Lambda_1| = \text{diag}(|\lambda_1|,\ldots,|\lambda_{r_f}|)$.

One can show that the reduced model computed by the PABTEC method preserves passivity and we have the error bound $\|\widetilde{G}-G\|_{\mathbb{H}_\infty} \le 2\|I+G\|^2_{\mathbb{H}_\infty}(\lambda_{r_f+1}+\ldots+\lambda_q)$, provided $2\|I+G\|_{\mathbb{H}_\infty}(\lambda_{r_f+1}+\ldots+\lambda_q)<1$, see [4] for details.

If $D_c = I-M_0M_0^T$ is nonsingular, then $D_o = I-M_0^TM_0$ is also nonsingular and the projected Lur'e equation (6) is equivalent to the projected Riccati equation

$$P_lHP_l^T + FXF^T + EXE^T + EXQXE^T = 0, \qquad X = P_rXP_r^T \tag{9}$$

with $F = A-BB^T-2P_lBD_o^{-1}M_0^TB^TP_r$, $H = 2BD_o^{-1}B^T$ and $Q = 2BD_c^{-1}B^T$. Such an equation can be solved via Newton's method [4]. Note that the matrix M_0 and the projectors P_l and P_r required in (9) can be constructed in explicit form exploiting the topological structure of the circuit equations (5), see [4]. For large-scale problems, the numerical solution of projected Lur'e equations is currently under investigation.

3.3 Recoupling of the Reduced Linear Subsystem and the Nonlinear Subsystem

Let $\hat{B} = \begin{bmatrix} \hat{B}_1 & \hat{B}_2 & \hat{B}_3 & \hat{B}_4 & \hat{B}_5 \end{bmatrix}$ and $\hat{C}^T = \begin{bmatrix} \hat{C}_1^T & \hat{C}_2^T & \hat{C}_3^T & \hat{C}_4^T & \hat{C}_5^T \end{bmatrix}$ in (8) be partitioned in blocks according to u_ℓ and y_ℓ, respectively. Since the vector $\hat{y}_\ell = \hat{C}\hat{x}_\ell$ is an approximation to the output vector y_ℓ of system (5), we have

$$-(A_{\widetilde{\mathcal{R}}}^2)^T\eta - \eta_z \approx \hat{C}_2\hat{x}_\ell, \qquad -A_{\widetilde{\mathcal{L}}}^T\eta \approx \hat{C}_3\hat{x}_\ell, \qquad -\iota_{\widetilde{\mathcal{C}}} \approx \hat{C}_5\hat{x}_\ell. \tag{10}$$

Then equations (4) and (3b) are approximated by

$$\widetilde{\mathcal{L}}(\hat{\iota}_{\widetilde{\mathcal{L}}})\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\iota}_{\widetilde{\mathcal{L}}} = -\hat{C}_3\hat{x}_\ell, \qquad \widetilde{\mathcal{C}}(\hat{u}_{\widetilde{\mathcal{C}}})\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{\widetilde{\mathcal{C}}} = -\hat{C}_5\hat{x}_\ell, \tag{11}$$

respectively, where $\hat{\iota}_{\widetilde{\mathcal{L}}}$ and $\hat{u}_{\widetilde{\mathcal{C}}}$ are approximations to $\iota_{\widetilde{\mathcal{L}}}$ and $u_{\widetilde{\mathcal{C}}}$, respectively. Furthermore, for ι_z defined in (2), η_z defined in (3a) and $u_{\widetilde{\mathcal{R}}} = A_{\widetilde{\mathcal{R}}}^T\eta \in \mathbb{R}^{n_{\widetilde{\mathcal{R}}}}$, we have

$$\iota_z = (G_1 + G_2)G_1^{-1}\widetilde{g}(u_{\widetilde{\mathcal{R}}}) - G_2 u_{\widetilde{\mathcal{R}}},$$

$$-(A_{\widetilde{\mathcal{R}}}^2)^T\eta - \eta_z = -A_{\widetilde{\mathcal{R}}}^T\eta + G_1^{-1}\widetilde{g}(A_{\widetilde{\mathcal{R}}}^T\eta) = -u_{\widetilde{\mathcal{R}}} + G_1^{-1}\widetilde{g}(u_{\widetilde{\mathcal{R}}}). \tag{12}$$

Then the first equation in (10) together with (12) imply the relation

$$0 = -G_1\hat{C}_2\hat{x}_\ell - G_1\hat{u}_{\widetilde{\mathcal{R}}} + \widetilde{g}(\hat{u}_{\widetilde{\mathcal{R}}}), \tag{13}$$

where $\hat{u}_{\widetilde{\mathcal{R}}}$ approximates $u_{\widetilde{\mathcal{R}}}$. Combining (8), (11) and (13) and adding to $\hat{x}_\ell$ also the approximations $\hat{\iota}_{\widetilde{\mathcal{L}}}$, $\hat{u}_{\widetilde{\mathcal{C}}}$, and $\hat{u}_{\widetilde{\mathcal{R}}}$ as state variables, we get the reduced-order nonlinear DAE system

$$\hat{\mathscr{E}}(\hat{x})\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{x} = \hat{\mathscr{A}}\hat{x} + \hat{f}(\hat{x}) + \hat{\mathscr{B}}u, \qquad \hat{y} = \hat{\mathscr{C}}\hat{x}$$

with

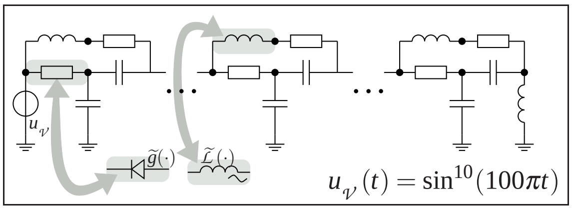
$$\hat{\mathscr{E}}(\hat{x}) = \begin{bmatrix} \hat{E} & 0 & 0 & 0 \\ 0 & \widetilde{\mathcal{L}}(\hat{\iota}_{\widetilde{\mathcal{L}}}) & 0 & 0 \\ 0 & 0 & \widetilde{\mathcal{C}}(\hat{u}_{\widetilde{\mathcal{C}}}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\mathscr{A}} = \begin{bmatrix} \hat{A} + \hat{B}_2(G_1 + G_2)\hat{C}_2 & \hat{B}_3 & \hat{B}_5 & \hat{B}_2G_1 \\ -\hat{C}_3 & 0 & 0 & 0 \\ -\hat{C}_5 & 0 & 0 & 0 \\ -G_1\hat{C}_2 & 0 & 0 & -G_1 \end{bmatrix},$$

$$\hat{f}(\hat{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \widetilde{g}(\hat{u}_{\widetilde{\mathcal{R}}}) \end{bmatrix}, \quad \hat{\mathscr{B}} = \begin{bmatrix} \hat{B}_1 & \hat{B}_4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_r^T = \begin{bmatrix} \hat{C}_1^T & \hat{C}_4^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} \hat{x}_\ell \\ \hat{\iota}_{\widetilde{\mathcal{L}}} \\ \hat{u}_{\widetilde{\mathcal{C}}} \\ \hat{u}_{\widetilde{\mathcal{R}}} \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_4 \end{bmatrix}$$

that approximates the original nonlinear system (1). This reduced model can now be used instead of (1) in the analysis of the dynamical behavior of the circuit.

4 Numerical Experiments

Consider an electrical circuit constructed by 1000 repetitions of subcircuits containing one inductor, two capacitors, and two resistors, as shown in the figure on the right. In the 1st, 101st, 201st, etc., subcircuits, a linear resistor is replaced by a diode. Furthermore, in the 100th, 200th, 300th, etc., subcircuits, the linear inductor is replaced by a nonlinear inductor. For more details, we refer to [5].



original system: 1 voltage source, 1990 linear resistors, 10 diodes, 991 linear inductors, 10 nonlinear inductors, 2000 linear capacitors	
dimension of the original system	4003
simulation time for the original system (t_o)	4557s
tolerance for model reduction	1e-05
time for model reduction	822s
dimension of the reduced system	203
simulation time for the reduced system (t_r)	67s
absolute error in the output	4.4e-06
speedup (t_o/t_r)	68.5

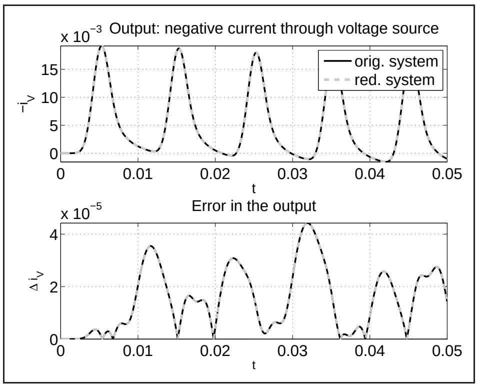


Fig. 1 (Left) numerical results; (right) simulation results for the original and the reduced systems.

The numerical simulation is done for $t \in [0, 0.05]$ seconds using the BDF method of order 2 with fixed stepsize of length $5 \cdot 10^{-5}$. The numerical results for the prescribed tolerance is given in Figures 1. In the upper plot, the output $y(t) = -\iota_{\mathcal{V}}(t)$ of the original system and the output $\hat{y}(t) = -\hat{\iota}_{\mathcal{V}}(t)$ of the reduced system are presented. In the lower plot, the error $\Delta \iota_{\mathcal{V}}(t) = |\hat{y}(t) - y(t)|$ is shown.

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