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# SOLVING PARAMETER-DEPENDENT LYAPUNOV EQUATIONS USING THE REDUCED BASIS METHOD WITH APPLICATION TO PARAMETRIC MODEL ORDER REDUCTION*

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**Abstract.** Our aim is to numerically solve parameter-dependent Lyapunov equations using the reduced basis method. Such equations arise in parametric model order reduction. We restrict ourselves to the systems that affinely depend on the parameter, as our main strategy is the min-$\theta$ approach. In those cases, we derive various a posteriori error estimates. Based on these estimates, a greedy algorithm for constructing reduced bases is formulated. Thanks to the derived results, a novel so-called parametric balanced truncation model reduction method is developed. Numerical examples are presented.

**Key words.** parameter-dependent Lyapunov equations, reduced basis method, affine dependence, min-$\theta$ approach, error estimates, greedy algorithm, model order reduction, parametric balanced truncation

**AMS subject classifications.** 15A21, 15A24

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## 1. Introduction.

In this paper, we consider the following parametric algebraic Lyapunov equation (PALE):

$$A(\mu)X(\mu)E^T(\mu) + E(\mu)X(\mu)A^T(\mu) = -B(\mu)B^T(\mu), \tag{1}$$

where $A(\mu), E(\mu) \in \mathbb{R}^{N\times N}$ and $B(\mu) \in \mathbb{R}^{N\times m}$ with $m \ll N$ are given. The coefficient matrices and the right-hand side depend on parameter $\mu$ in a compact domain $\mathcal{D} \subset \mathbb{R}^d$. For the rest of this paper, we assume that $E(\mu)$ is nonsingular, and all eigenvalues of a pencil $\lambda E(\mu) - A(\mu)$ have negative real part for all $\mu \in \mathcal{D}$. With these assumptions, (1) has a unique symmetric positive semidefinite solution $X(\mu)$ for all $\mu \in \mathcal{D}$; see, e.g., [23]. Solving Lyapunov equations is of great importance in many control problems including stability analysis, stabilization, model reduction by balanced truncation, and optimal control [2, 10]. The PALE (1) arises naturally in parametric model reduction [4] or in the design of low gain feedback [41].

Because of their role in control theory, various works have been devoted to the numerical solution of Lyapunov equations. First, it is worth mentioning the direct methods by Bartels and Stewart [3] and by Hammarling [16]. To avoid expensive computations in the direct methods, iterative ones such as the sign function method

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[7], the alternating directions implicit method [25, 39], and Krylov subspace methods [21, 30] have been developed. Exploiting the fact that the right-hand sides of Lyapunov equations in most applications have low rank, low-rank versions of the mentioned iterative methods have been formulated. See [9, 24, 28, 31], just to name a few; see also [6, 32] for the recent surveys on the state-of-the-art algorithms.

Although a lot of attention has been paid to Lyapunov equations, only very few publications dedicated to solving the PALEs can be found. To the best of our knowledge, [22] is the only printed work on this subject. Nevertheless, the purpose of the method presented there, computing the solutions for many different parameter values, is not our goal. We would like to compute the solution $X(\mu)$ for any $\mu \in \mathcal{D}$.

For dealing with parameter-dependent problems, the reduced basis method [17, 29] is an effective tool. This method was initially proposed for coercive elliptic partial differential equations and then extended to noncoercive equations [38], Burgers equations [37], and Navier–Stokes equations [36]. Applying this method to PALEs in the present paper is most probably the first time this was ever done, while an extension to parametric Riccati equations can be found in the very recent paper [15].

To use the reduced basis method, first we have to convert the Lyapunov equation to a linear system using the Kronecker product. The size of the resulting linear system is $N^2$, where $N$ is itself already large. This fact results in very expensive computations as well as a huge storage requirement. The key point to avoid these difficulties is to keep all computations with matrices and vectors of dimension $N$. In addition, the norm for error evaluation must be carefully chosen in order to make all computations feasible. Similarly to [17, 29], a posteriori error estimates will be constructed, based on which a greedy algorithm is designed to determine the reduced bases.

To this end, the rest of this paper is organized as follows. In section 2, we repeat how the Lyapunov equation can be converted to a linear equation by application of the Kronecker product. We also provide some formulae that allow us to replace $N^2$-sized operations with $N$-sized ones. Section 3 introduces the reduced basis method for parameter-dependent linear systems. Important components of this method are the greedy algorithm and a posteriori error estimates which will be discussed in sections 3.1 and 3.2, respectively. We also explain how to efficiently compute the residual norm, which is the main strategy for the error estimate in this section. In section 4, we present an extension of the reduced basis framework to parametric Lyapunov equations with symmetric coefficient matrices. The nonsymmetric case is treated in section 5. Based on these results, we develop in section 6 a new approach for parametric model order reduction. Two numerical examples will be presented in section 7. Finally, in section 8, we conclude as well as pose some problems for future work.

Throughout this paper, we will use boldface for matrices and vectors of dimension $N^2$ and the standard font for that of dimension $N$. Given a matrix $A \in \mathbb{R}^{N\times N}$ and a vector $\mathbf{v} \in \mathbb{R}^{N^2}$, $\mathrm{vec}(A)$ will denote the column vector in $\mathbb{R}^{N^2}$ generated from $A$ by stacking all columns of $A$, and $\mathrm{mat}(\mathbf{v})$ is the $N\times N$-matrix such that $\mathrm{vec}(\mathrm{mat}(\mathbf{v})) = \mathbf{v}$. The Kronecker product of the matrices $A$ and $B$ is denoted by $A \otimes B$. The trace of the matrix $A$ is denoted by $\mathrm{trace}(A)$, $\ker(A) = \{v \in \mathbb{R}^N \ : \ Av = 0\}$, $\mathrm{span}(A)$ is the subspace spanned by columns of $A$, $A^T$ stands for the transpose of $A$, and $A > 0$ ($A \geq 0$) means that $A$ is positive definite (semidefinite), i.e., $v^T Av > 0$ ($v^T Av \geq 0$) for all $v \neq 0$. The smallest and largest singular values of $A$ are denoted by $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$, respectively, whereas the smallest and largest eigenvalues of symmetric $A$ are denoted by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. Similarly, $\lambda_{\min}(E, A)$ and $\lambda_{\max}(E, A)$ will denote the smallest and largest eigenvalue of the matrix pencil

$\lambda E - A$, respectively. We denote by $\|v\| = \sqrt{(v^T v)}$ the Euclidean vector norm of $v \in \mathbb{R}^N$, by $\|A\|_2 = \sigma_{\max}(A)$ the spectral norm of $A \in \mathbb{R}^{N\times M}$, and by $\mathrm{cond}(A) = \|A\|_2\|A^{-1}\|_2$ the condition number of the invertible matrix $A$. Furthermore, the inner product of two matrices $A, B \in \mathbb{R}^{N\times M}$ is defined as $\langle A, B\rangle = \mathrm{trace}(B^T A)$, and $\|A\|_F = \sqrt{\langle A, A\rangle}$ is the Frobenius matrix norm.

**2. Lyapunov equations and linear systems.** We begin this section by collecting some useful properties of the Kronecker product, vec- and mat-operators.

LEMMA 2.1. *Let $E, A, X, Y \in \mathbb{R}^{N\times N}$, $\mathbf{x} = \mathrm{vec}(X)$, and $\mathbf{y} = \mathrm{vec}(Y)$. Then*

1. $\mathbf{y}^T\mathbf{x} = \langle X, Y\rangle$, *and*
2. $\mathrm{vec}(AXE^T + EXA^T) = \mathbf{L}\mathbf{x}$ *with* $\mathbf{L} = E \otimes A + A \otimes E$.

*Proof.* While the first equality is straightforward, the second is obtained from [20, Lemma 4.3.1]. ∎

LEMMA 2.2. *Let $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_k] \in \mathbb{R}^{N^2\times k}$, $V_j = \mathrm{mat}(\mathbf{v}_j)$ for $j = 1, \ldots, k$, $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_l] \in \mathbb{R}^{N^2\times l}$, $U_j = \mathrm{mat}(\mathbf{u}_j)$ for $j = 1, \ldots, l$, and $\mathbf{L} = E \otimes A + A \otimes E$.*

1. $\mathbf{L}\mathbf{V} = [\,\mathrm{vec}(AV_1E^T + EV_1A^T), \ldots, \mathrm{vec}(AV_kE^T + EV_kA^T)\,]$.
2. *The entries of $\mathbf{U}^T\mathbf{L}\mathbf{V} \in \mathbb{R}^{l\times k}$ are given by $(\mathbf{U}^T\mathbf{L}\mathbf{V})_{ij} = \langle AV_jE^T + EV_jA^T, U_i\rangle$.*
3. *Let $\mathbf{L}_s = E_s \otimes A_s + A_s \otimes E_s$, $s = 1, 2$. Then the entries of $\mathbf{U}^T\mathbf{L}_1^T\mathbf{L}_2\mathbf{V} \in \mathbb{R}^{l\times k}$ are given by $(\mathbf{U}^T\mathbf{L}_1^T\mathbf{L}_2\mathbf{V})_{ij} = \langle A_2V_jE_2^T + E_2V_jA_2^T, A_1U_iE_1^T + E_1U_iA_1^T\rangle$.*

*Proof.* The equalities can be easily verified by calculation. ∎

Let $\mathbb{S}$ denote a space of $N \times N$ symmetric matrices. Consider a Lyapunov operator $\mathcal{L}_\mu : \mathbb{S} \to \mathbb{S}$ given by

$$\mathcal{L}_\mu(X) = -A(\mu)XE^T(\mu) - E(\mu)XA^T(\mu).$$

Then the PALE (1) takes the form $\mathcal{L}_\mu(X(\mu)) = B(\mu)B^T(\mu)$. Using Lemma 2.1, this equation can also be written as a linear system

$$\mathbf{L}(\mu)\mathbf{x}(\mu) = \mathbf{b}(\mu), \tag{2}$$

where $\mathbf{x}(\mu) = \mathrm{vec}(X(\mu))$, $\mathbf{b}(\mu) = \mathrm{vec}(B(\mu)B^T(\mu))$, and

$$\mathbf{L}(\mu) = -E(\mu) \otimes A(\mu) - A(\mu) \otimes E(\mu) \tag{3}$$

is the matrix representation of the linear Lyapunov operator $\mathcal{L}_\mu$. The following theorem establishes some properties of $\mathcal{L}_\mu$ and $\mathbf{L}(\mu)$.

THEOREM 2.3. *Let $-A(\mu)$ and $E(\mu)$ be symmetric, positive definite for all $\mu \in \mathcal{D}$.*

1. *The matrix $\mathbf{L}(\mu)$ in (3) is symmetric and positive definite for all $\mu \in \mathcal{D}$ and its smallest and largest eigenvalues are bounded as*

$$\lambda_{\min}(\mathbf{L}(\mu)) \geq 2\,\lambda_{\min}(-A(\mu))\,\lambda_{\min}(E(\mu)), \tag{4}$$

$$\lambda_{\max}(\mathbf{L}(\mu)) \leq 2\,\lambda_{\max}(-A(\mu))\,\lambda_{\max}(E(\mu)) \tag{5}$$

   *for all $\mu \in \mathcal{D}$.*

2. *The Lyapunov operator $\mathcal{L}_\mu$ is uniformly coercive, i.e., it holds*

$$\alpha(\mu) := \inf_{V\in\mathbb{R}^{N\times N}\setminus\{0\}} \frac{\langle \mathcal{L}_\mu(V), V\rangle}{\|V\|_F^2} > 0 \tag{6}$$

   *for all $\mu \in \mathcal{D}$.*

3. *The Lyapunov operator $\mathcal{L}_\mu$ is uniformly continuous, i.e., it holds*

$$\gamma(\mu) := \sup_{W,V\in\mathbb{R}^{N\times N}\setminus\{0\}} \frac{\langle \mathcal{L}_\mu(V), W\rangle}{\|W\|_F\|V\|_F} < \infty \tag{7}$$

*for all $\mu \in \mathcal{D}$.*

*Proof.* 1. Since $-A(\mu)$ and $E(\mu)$ are both symmetric, the matrix $\mathbf{L}(\mu)$ in (3) is also symmetric for all $\mu \in \mathcal{D}$. Using Weyl's theorem [19, Theorem 4.3.1] and the multiplicativity property of eigenvalues of the Kronecker product [20, Theorem 4.2.12], we have

$$\begin{aligned}\lambda_{\min}(\mathbf{L}(\mu)) &\geq \lambda_{\min}(-E(\mu)\otimes A(\mu)) + \lambda_{\min}(-A(\mu)\otimes E(\mu))\\ &= 2\lambda_{\min}(-A(\mu))\lambda_{\min}(E(\mu)) > 0,\end{aligned}$$

whereby the last inequality follows from the positive definiteness of $-A(\mu)$ and $E(\mu)$. Thus, the bound (4) holds and $\mathbf{L}(\mu)$ is positive definite. The bound (5) can be proved analogously.

2. We obtain from Lemma 2.1 and the Courant–Fischer theorem [19, Theorem 4.2.6] that

$$\alpha(\mu) = \inf_{V\in\mathbb{R}^{N\times N}\setminus\{0\}} \frac{\langle \mathcal{L}_\mu(V), V\rangle}{\|V\|_F^2} = \inf_{\mathbf{v}\in\mathbb{R}^{N^2}\setminus\{0\}} \frac{\mathbf{v}^T\mathbf{L}(\mu)\mathbf{v}}{\|\mathbf{v}\|^2} = \lambda_{\min}(\mathbf{L}(\mu)) > 0$$

for all $\mu \in \mathcal{D}$, and hence, $\mathcal{L}_\mu$ is uniformly coercive.

3. Using again Lemma 2.1 and the Courant–Fischer theorem, we have

$$\gamma(\mu) = \sup_{W,V\in\mathbb{R}^{N\times N}\setminus\{0\}} \frac{\langle \mathcal{L}_\mu(V), W\rangle}{\|W\|_F\|V\|_F} = \sup_{\mathbf{w},\mathbf{v}\in\mathbb{R}^{N^2}\setminus\{0\}} \frac{\mathbf{w}^T\mathbf{L}(\mu)\mathbf{v}}{\|\mathbf{w}\|\|\mathbf{v}\|} = \lambda_{\max}(\mathbf{L}(\mu)) < \infty$$

for all $\mu \in \mathcal{D}$. Thus, $\mathcal{L}_\mu$ is uniformly continuous. ∎

The parameter-dependent quantities $\alpha(\mu)$ and $\gamma(\mu)$ are called *coercivity constant* and *continuity constant* of the Lyapunov operator $\mathcal{L}_\mu$, respectively. From the proof of Theorem 2.3 we obtain the important relations

$$\alpha(\mu) = \lambda_{\min}(\mathbf{L}(\mu)), \qquad \gamma(\mu) = \lambda_{\max}(\mathbf{L}(\mu)), \tag{8}$$

which together with the estimates (4) and (5) will be very useful in the following. Note that the coercivity constant $\alpha(\mu)$ coincides with the separation

$$\operatorname{Sep}\big(E(\mu), A(\mu)\big) := \inf_{\|X\|_F=1} \|A(\mu)XE^T(\mu) + E(\mu)XA^T(\mu)\|_F,$$

which measures the separation of the spectrum of the pencil $\lambda E(\mu) - A(\mu)$ from that of $\lambda E(\mu) + A(\mu)$ and is frequently used in the sensitivity analysis of the Lyapunov equations [11, 18].

**3. Reduced basis method.** In this section, we consider the application of the reduced basis method to the linear system (2) with $\mathbf{L}(\mu)$ as in (3), where $-A(\mu)$ and $E(\mu)$ are assumed to be symmetric and positive definite. This method consists of the following steps. For selected parameters $\mu_1, \ldots, \mu_k \in \mathcal{D}$, we construct first a reduced basis matrix

$$\mathbf{V}_k = [\,\mathbf{x}(\mu_1), \ldots, \mathbf{x}(\mu_k)\,], \tag{9}$$

where $\mathbf{x}(\mu_j)$ is the solution of (2) at $\mu = \mu_j$ for $j = 1, \ldots, k$. Then, for any $\mu \in \mathcal{D}$, an approximate solution can be computed by Galerkin projection $\mathbf{x}(\mu) \approx \mathbf{V}_k \hat{\mathbf{x}}(\mu)$, where $\hat{\mathbf{x}}(\mu)$ solves the reduced linear system

$$\hat{\mathbf{L}}(\mu)\hat{\mathbf{x}}(\mu) = \hat{\mathbf{b}}(\mu) \tag{10}$$

with $\hat{\mathbf{L}}(\mu) = \mathbf{V}_k^T \mathbf{L}(\mu)\mathbf{V}_k$ and $\hat{\mathbf{b}}(\mu) = \mathbf{V}_k^T \mathbf{b}(\mu)$. This seemingly simple procedure raises several issues: optimal choice of the parameter sample $\mu_1, \ldots, \mu_k$, providing a good reduced basis subspace that guarantees a rapid convergence of the reduced basis approximation $\mathbf{V}_k \hat{\mathbf{x}}(\mu)$ to $\mathbf{x}(\mu)$ over the entire parameter domain $\mathcal{D}$, rigorous error estimates for the approximate solution, and efficient computations.

**3.1. Greedy algorithm.** The key to the success of the reduced basis method is the construction of an appropriate basis. It should be done in such a way that the error of the approximation is smaller than a given tolerance while the dimension of the reduced basis is kept as small as possible. One way to do this is to employ a greedy algorithm, e.g., [17, section 3.2.2], which successively determines snapshots depending on the error magnitude.

Suppose that we have already computed a basis matrix $\mathbf{V}_k = [\,\mathbf{x}(\mu_1), \ldots, \mathbf{x}(\mu_k)\,]$. Then the error and the residual of the approximate solution $\mathbf{V}_k \hat{\mathbf{x}}(\mu)$ are given by

$$\begin{aligned} \mathbf{e}_k(\mu) &= \mathbf{x}(\mu) - \mathbf{V}_k \hat{\mathbf{x}}(\mu), \\ \mathbf{r}_k(\mu) &= \mathbf{b}(\mu) - \mathbf{L}(\mu)\mathbf{V}_k \hat{\mathbf{x}}(\mu), \end{aligned}$$

respectively. They satisfy the equation

$$\mathbf{L}(\mu)\mathbf{e}_k(\mu) = \mathbf{r}_k(\mu), \tag{11}$$

which immediately implies the error estimate

$$\|\mathbf{e}_k(\mu)\| = \|\mathbf{L}^{-1}(\mu)\mathbf{r}_k(\mu)\| \le \frac{\|\mathbf{r}_k(\mu)\|}{\alpha(\mu)} \le \frac{\|\mathbf{r}_k(\mu)\|}{\alpha_{\mathrm{LB}}(\mu)} =: \Delta_k(\mu). \tag{12}$$

Here, $\alpha_{\mathrm{LB}}(\mu)$ is a positive lower bound for the coercivity constant $\alpha(\mu)$, and $\Delta_k(\mu)$ is the resulting error estimator. To reduce the approximation error, which most probably implies an enlargement of the basis, we find the next value $\mu_{k+1}$ such that $\Delta_k(\mu_{k+1})$ is the largest in $\mathcal{D}$. Of course, we cannot pursue the search on $\mathcal{D}$, which is a continuous set. Instead, one usually does it on a discrete subset of $\mathcal{D}$. To ensure that no good candidates are missed, this training set, denoted by $\mathcal{D}_{\mathrm{train}}$, should be rather dense and, therefore, large. In practice, we choose $\mathcal{D}_{\mathrm{train}}$ first and pick $\mu_1$ arbitrarily in $\mathcal{D}_{\mathrm{train}}$. We also need to specify a tolerance $tol_{\mathrm{rb}}$ for the approximation. The greedy algorithm is then given in the sections that follow.

For the success of this algorithm, an efficient, sharp, and rigorous error estimate is required. This issue will be addressed in the next subsection.

**3.2. Error estimation.** First, we impose some further restrictions on the problems treated in this paper. To wit, we assume that the matrices $A(\mu)$, $E(\mu)$, and $B(\mu)$ are *affine* in the parameter $\mu$, i.e.,

$$\text{(A1)} \qquad A(\mu) = \sum_{j=1}^{n_A} \theta_j^A(\mu) A_j, \quad E(\mu) = \sum_{j=1}^{n_E} \theta_j^E(\mu) E_j, \quad B(\mu) = \sum_{j=1}^{n_B} \theta_j^B(\mu) B_j,$$

where $A_j$, $E_j$, and $B_j$ are independent of $\mu$, $n_A$, $n_E$, and $n_B$ are very small compared to $N$, $\theta_j^A(\mu)$, $\theta_j^E(\mu)$, and $\theta_j^B(\mu)$ are continuous in $\mathcal{D}$ and their evaluations at

**Algorithm 1** Greedy algorithm for linear systems.

**Input:** tolerance $tol_{\text{rb}}$, training set $\mathcal{D}_{\text{train}}$, initial parameter $\mu_1 \in \mathcal{D}_{\text{train}}$.
**Output:** a basis matrix $\mathbf{V}_k$.

Solve $\mathbf{L}(\mu_1)\mathbf{x}(\mu_1) = \mathbf{b}(\mu_1)$.
Set $\Delta_1^{\max} > tol_{\text{rb}}$, $\mathcal{M}_1 = \{\mu_1\}$, $\mathbf{V}_1 = \mathbf{x}(\mu_1)$, and $k = 2$.
**while** $\Delta_{k-1}^{\max} \geq tol_{\text{rb}}$ **do**
  $\mu_k = \underset{\mu \in \mathcal{D}_{\text{train}} \setminus \mathcal{M}_{k-1}}{\arg\max} \Delta_{k-1}(\mu)$
  $\Delta_k^{\max} = \Delta_{k-1}(\mu_k)$
  $\mathcal{M}_k = \mathcal{M}_{k-1} \cup \{\mu_k\}$
  solve $\mathbf{L}(\mu_k)\mathbf{x}(\mu_k) = \mathbf{b}(\mu_k)$
  $\mathbf{V}_k = [\, \mathbf{V}_{k-1},\ \mathbf{x}(\mu_k)\,]$
  $k \leftarrow k + 1$
**end while**

each $\mu \in \mathcal{D}$ are inexpensive. This assumption permits us to decompose the computation of the solution of the linear system (2) into an offline stage (computationally expensive), in which the reduced basis matrix $\mathbf{V}_k$ is constructed and all parameter-independent matrices are computed and stored, and an online stage (computationally inexpensive), in which the reduced system (10) is solved for any $\mu \in \mathcal{D}$ to get the approximate solution $\mathbf{x}(\mu) \approx \mathbf{V}_k \hat{\mathbf{x}}(\mu)$.

Furthermore, we require that the matrices $-A(\mu)$ and $E(\mu)$ are *parametrically coercive*, i.e.,

(A2) $\quad E_j = E_j^T \geq 0 \quad$ and $\quad \theta_j^E(\mu) > 0$ for all $\mu \in \mathcal{D}$ and $j = 1, \ldots, n_E$,

(A3) $\quad -A_j = -A_j^T \geq 0 \quad$ and $\quad \theta_j^A(\mu) > 0$ for all $\mu \in \mathcal{D}$ and $j = 1, \ldots, n_A$.

The following lemma shows that under assumptions (A1)–(A3) the linear system (2) maintains the affine dependence and parametric coercivity.

LEMMA 3.1. *Let $A(\mu)$, $E(\mu)$, and $B(\mu)$ satisfy* (A1)–(A3). *Then the matrix $\mathbf{L}(\mu)$ and the vector $\mathbf{b}(\mu)$ in the linear system* (2) *are affine in the parameter $\mu$. Moreover, $\mathbf{L}(\mu)$ is parametrically coercive.*

*Proof.* The proof of the affine dependence is straightforward and based on the properties of the Kronecker product [20]. Nevertheless, since we will need the explicit form of the affine dependence later on, it is briefly presented here. One can easily verify that the coefficient matrix and the right-hand side in (2) take the form

$$\mathbf{L}(\mu) = \sum_{i=1}^{n_E} \sum_{j=1}^{n_A} \theta_{ij}^{\mathbf{L}}(\mu) \mathbf{L}_{ij}, \qquad \mathbf{b}(\mu) = \sum_{i=1}^{n_B} \sum_{j=1}^{n_B} \theta_{ij}^{\mathbf{b}}(\mu) \mathbf{b}_{ij}, \tag{13}$$

where

$$\begin{aligned} \theta_{ij}^{\mathbf{L}}(\mu) &= \theta_i^E(\mu)\theta_j^A(\mu) > 0, & \mathbf{L}_{ij} &= -E_i \otimes A_j - A_j \otimes E_i, \\ \theta_{ij}^{\mathbf{b}}(\mu) &= \theta_i^B(\mu)\theta_j^B(\mu), & \mathbf{b}_{ij} &= \operatorname{vec}(B_i B_j^T). \end{aligned} \tag{14}$$

Obviously, $\mathbf{L}_{ij}$ is symmetric and positive semidefinite. This fact together with the positiveness of $\theta_{ij}^{\mathbf{L}}$ completes the proof. ∎

*Remark* 3.2. In general, the parametric coercivity of $-A(\mu)$ and $E(\mu)$ does not imply the coercivity of $\mathbf{L}(\mu)$, i.e., the positivity of $\alpha(\mu) = \lambda_{\min}(\mathbf{L}(\mu))$. If, however, in addition to the parametric coercivity, we assume that

(A4) there exist at least one pair $(A_j, E_i)$ such that $-A_j > 0$ and $E_i > 0$,

then the coercivity is satisfied. Assumption (A4) can also be replaced by a condition that

$$\bigcap_{j=1,\dots,n_A} \ker(A_j) = \emptyset \quad \text{and} \quad \bigcap_{j=1,\dots,n_E} \ker(E_j) = \emptyset.$$

To derive a posteriori error estimates, we first have to find positive lower bounds for the coercivity constant $\alpha(\mu)$. We will mainly employ the min-$\theta$ approach [17, section 4.3] which strongly relies on the affine decomposition (A1) and the parametric coercivity (A2) and (A3). To this end, for a fixed value $\bar{\mu} \in \mathcal{D}$, we define the following functions:

$$\theta^{\mathbf{L},\bar{\mu}}_{\min}(\mu) = \min_{\substack{i=1,\dots,n_E\\ j=1,\dots,n_A}} \frac{\theta^{\mathbf{L}}_{ij}(\mu)}{\theta^{\mathbf{L}}_{ij}(\bar{\mu})}, \qquad \theta^{\mathbf{L},\bar{\mu}}_{\max}(\mu) = \max_{\substack{i=1,\dots,n_E\\ j=1,\dots,n_A}} \frac{\theta^{\mathbf{L}}_{ij}(\mu)}{\theta^{\mathbf{L}}_{ij}(\bar{\mu})}, \qquad \theta^{\mathbf{L},\bar{\mu}}(\mu) = \frac{\theta^{\mathbf{L},\bar{\mu}}_{\max}(\mu)}{\theta^{\mathbf{L},\bar{\mu}}_{\min}(\mu)}.$$

The min-$\theta$ approach applied to $\mathbf{L}(\mu)$ is based on lower bounding $\lambda_{\min}(\mathbf{L}(\mu))$ by the min-$\theta$-function multiplied with $\lambda_{\min}(\mathbf{L}(\bar{\mu}))$ computed for a single parameter $\bar{\mu} \in \mathcal{D}$. For convenience, an upper bound for the continuity constant $\gamma(\mu)$ is also included in the following lemma.

LEMMA 3.3. *Let $E(\mu)$ and $A(\mu)$ satisfy* (A1)–(A4), *and let $\bar{\mu}, \bar{\mu}_1, \bar{\mu}_2 \in \mathcal{D}$.*

1. *For all $\mu \in \mathcal{D}$, the coercivity constant $\alpha(\mu)$ in* (6) *is bounded from below as*

$$\alpha(\mu) \geq \alpha_{\mathrm{LB}}(\mu) := \max\Big(\alpha^{\mathbf{L},\bar{\mu}}_{\mathrm{LB}}(\mu),\ \alpha^{A,\bar{\mu}_1;E,\bar{\mu}_2}_{\mathrm{LB}}(\mu),\ \alpha^{\mathbf{L}}_{\mathrm{LB}}(\mu)\Big) > 0, \tag{15}$$

*where*

$$\begin{aligned}
\alpha^{\mathbf{L},\bar{\mu}}_{\mathrm{LB}}(\mu) &= 2\,\theta^{\mathbf{L},\bar{\mu}}_{\min}(\mu)\,\lambda_{\min}\big(-A(\bar{\mu})\big)\,\lambda_{\min}\big(E(\bar{\mu})\big),\\
\alpha^{A,\bar{\mu}_1;E,\bar{\mu}_2}_{\mathrm{LB}}(\mu) &= 2\,\theta^{A,\bar{\mu}_1}_{\min}(\mu)\,\theta^{E,\bar{\mu}_2}_{\min}(\mu)\,\lambda_{\min}\big(-A(\bar{\mu}_1)\big)\,\lambda_{\min}\big(E(\bar{\mu}_2)\big),\\
\alpha^{\mathbf{L}}_{\mathrm{LB}}(\mu) &= 2\sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta^{\mathbf{L}}_{ij}(\mu)\,\lambda_{\min}(-A_j)\,\lambda_{\min}(E_i).
\end{aligned}$$

2. *For all $\mu \in \mathcal{D}$, the continuity constant $\gamma(\mu)$ in* (7) *is bounded from above as*

$$\gamma(\mu) \leq \gamma_{\mathrm{UB}}(\mu) := \min\Big(\gamma^{\mathbf{L},\bar{\mu}}_{\mathrm{UB}}(\mu),\ \gamma^{A,\bar{\mu}_1;E,\bar{\mu}_2}_{\mathrm{UB}}(\mu),\ \gamma^{\mathbf{L}}_{\mathrm{UB}}(\mu)\Big), \tag{16}$$

*where*

$$\begin{aligned}
\gamma^{\mathbf{L},\bar{\mu}}_{\mathrm{UB}}(\mu) &= 2\,\theta^{\mathbf{L},\bar{\mu}}_{\max}(\mu)\,\lambda_{\max}\big(-A(\bar{\mu})\big)\,\lambda_{\max}\big(E(\bar{\mu})\big),\\
\gamma^{A,\bar{\mu}_1;E,\bar{\mu}_2}_{\mathrm{UB}}(\mu) &= 2\,\theta^{A,\bar{\mu}_1}_{\max}(\mu)\,\theta^{E,\bar{\mu}_2}_{\max}(\mu)\,\lambda_{\max}\big(-A(\bar{\mu}_1)\big)\,\lambda_{\max}\big(E(\bar{\mu}_2)\big),\\
\gamma^{\mathbf{L}}_{\mathrm{UB}}(\mu) &= 2\sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta^{\mathbf{L}}_{ij}(\mu)\,\lambda_{\max}(-A_j)\,\lambda_{\max}(E_i).
\end{aligned}$$

*Proof.* 1. Using the Courant–Fischer theorem and the min-$\theta$ approach we obtain that

$$\begin{aligned}
\lambda_{\min}\big(\mathbf{L}(\mu)\big) &= \min_{\|v\|=1} \mathbf{v}^T\mathbf{L}(\mu)\mathbf{v} = \min_{\|v\|=1} \sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta^{\mathbf{L}}_{ij}(\mu)\,\mathbf{v}^T\mathbf{L}_{ij}\mathbf{v}\\
&= \min_{\|v\|=1} \sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\frac{\theta^{\mathbf{L}}_{ij}(\mu)}{\theta^{\mathbf{L}}_{ij}(\bar{\mu})}\,\theta^{\mathbf{L}}_{ij}(\bar{\mu})\,\mathbf{v}^T\mathbf{L}_{ij}\mathbf{v} \geq \theta^{\mathbf{L},\bar{\mu}}_{\min}(\mu)\,\lambda_{\min}\big(\mathbf{L}(\bar{\mu})\big).
\end{aligned}$$

Then (4) implies the lower bound

$$\lambda_{\min}\big(\mathbf{L}(\mu)\big) \geq 2\,\theta_{\min}^{\mathbf{L},\bar{\mu}}(\mu)\,\lambda_{\min}\big(-A(\bar{\mu})\big)\,\lambda_{\min}\big(E(\bar{\mu})\big).$$

On the other hand, using (4) again and applying the min-$\theta$ approach to the matrices $A(\mu)$ and $E(\mu)$, we have

$$\begin{aligned}\lambda_{\min}\big(\mathbf{L}(\mu)\big) &\geq 2\,\lambda_{\min}\big(-A(\mu)\big)\,\lambda_{\min}\big(E(\mu)\big)\\ &\geq 2\,\theta_{\min}^{A,\bar{\mu}_1}(\mu)\,\theta_{\min}^{E,\bar{\mu}_2}(\mu)\lambda_{\min}\big(-A(\bar{\mu}_1)\big)\,\lambda_{\min}\big(E(\bar{\mu}_2)\big).\end{aligned}$$

Finally, it follows from Weyl's theorem [19, Theorem 4.3.1] that

$$\lambda_{\min}\big(\mathbf{L}(\mu)\big) \geq \sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta_{ij}^{\mathbf{L}}(\mu)\,\lambda_{\min}(\mathbf{L}_{ij}) \geq 2\sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta_{ij}^{\mathbf{L}}(\mu)\,\lambda_{\min}(-A_j)\,\lambda_{\min}(E_i).$$

Thus, the bound (15) holds.

2. The bound (16) can be proved similarly using (5). ∎

*Remark* 3.4. One can observe that if $E(\mu) \equiv E$ is constant and $\bar{\mu} = \bar{\mu}_1 = \bar{\mu}_2$, then $\alpha_{\mathrm{LB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu)$ and $\gamma_{\mathrm{UB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu)$ coincide with $\alpha_{\mathrm{LB}}^{\mathbf{L},\bar{\mu}}(\mu)$ and $\gamma_{\mathrm{UB}}^{\mathbf{L},\bar{\mu}}(\mu)$, respectively. Note also that the quantities $\alpha_{\mathrm{LB}}^{\mathbf{L}}(\mu)$ and $\gamma_{\mathrm{UB}}^{\mathbf{L}}(\mu)$ in (15) and (16), respectively, have no relation to the min-$\theta$ approach. We have presented them in order to possibly tighten the bounds.

*Remark* 3.5. A slightly improved version of the min-$\theta$ approach, called the multi-parameter min-$\theta$ approach, was presented in [17, section 4.3.2]. Roughly speaking, it is the same as the former but applied several times with several reference parameters $\bar{\mu}_i$, $i = 1, \ldots, l$, and the new lower bound is the maximum of the lower bounds corresponding to $\bar{\mu}_i$, $i = 1, \ldots, l$.

**3.2.1. Euclidean norm error estimate.** The following theorem provides an a posteriori error estimate for the reduced basis solution $\mathbf{V}_k\hat{\mathbf{x}}(\mu)$.

THEOREM 3.6. *Let assumptions* (A1)–(A4) *be fulfilled, and let* $\alpha_{\mathrm{LB}}(\mu)$ *and* $\gamma_{\mathrm{UB}}(\mu)$ *be as in* (15) *and* (16)*, respectively. Then the error* $\mathbf{e}_k(\mu) = \mathbf{x}(\mu) - \mathbf{V}_k\hat{\mathbf{x}}(\mu)$ *satisfies the bounds*

$$\|\mathbf{e}_k(\mu)\| \leq \Delta_k(\mu) \leq \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}\,\|\mathbf{e}_k(\mu)\|, \tag{17}$$

*where the error estimator* $\Delta_k(\mu)$ *is given by*

$$\Delta_k(\mu) = \frac{\|\mathbf{r}_k(\mu)\|}{\alpha_{\mathrm{LB}}(\mu)}. \tag{18}$$

*Proof.* The error estimate $\|\mathbf{e}_k(\mu)\| \leq \Delta_k(\mu)$ immediately follows from (12) and (15). Furthermore, using (8), (11), and (16), we have

$$\Delta_k(\mu) = \frac{\|\mathbf{r}_k(\mu)\|}{\alpha_{\mathrm{LB}}(\mu)} \leq \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}\,\|\mathbf{e}_k(\mu)\|.$$

This completes the proof. ∎

The effectivity of the error estimator $\Delta_k(\mu)$ is measured by the quantity

$$\eta_k(\mu) = \frac{\Delta_k(\mu)}{\|\mathbf{e}_k(\mu)\|}.$$

The error estimate is tight if $\eta_k(\mu)$ is very close to 1. It follows from (17) that

$$1 \le \eta_k(\mu) \le \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)}. \tag{19}$$

Therefore, to sharpen the error estimate, one could choose the parameter $\bar{\mu} \in \mathcal{D}$ such that the quotient $\gamma_{\mathrm{UB}}(\mu)/\alpha_{\mathrm{LB}}(\mu)$ is as small as possible. The solution to this optimization problem, however, goes beyond the purpose of this paper.

Recall that in the greedy algorithm, finding the maximizer of the error estimator $\Delta_k(\mu)$ on a large discrete set $\mathcal{D}_{\mathrm{train}}$ is required. This involves the repeated computation of the residual norm $\|\mathbf{r}_k(\mu)\|$ in the vector space of huge dimension $N^2$ for all $\mu \in \mathcal{D}_{\mathrm{train}}$. These unfeasibly expensive computations can be avoided thanks to the choice of the norm, the affine dependence in $\mathbf{L}(\mu)$ and $\mathbf{b}(\mu)$, and an appropriate arrangement of various computation steps.

To simplify the notation, we reindex the expressions for the coefficient matrix and the right-hand side in (13) by replacing the two-index system by a one-index system, say

$$\mathbf{L}(\mu) = \sum_{p=1}^{n_A n_E} \theta_p^{\mathbf{L}}(\mu)\mathbf{L}_p, \qquad \mathbf{b}(\mu) = \sum_{p=1}^{n_B^2} \theta_p^{\mathbf{b}}(\mu)\mathbf{b}_p, \tag{20}$$

where

$$\begin{aligned} &\mathbf{L}_p = \mathbf{L}_{ij}, \qquad \theta_p^{\mathbf{L}}(\mu) = \theta_{ij}^{\mathbf{L}}(\mu) \quad \text{for} \quad p = (i-1)n_A + j, \\ &\mathbf{b}_p = \mathbf{b}_{ij}, \qquad \theta_p^{\mathbf{b}}(\mu) = \theta_{ij}^{\mathbf{b}}(\mu) \quad \text{for} \quad p = (i-1)n_B + j. \end{aligned} \tag{21}$$

Then the residual norm can be represented as

$$\begin{aligned} \|\mathbf{r}_k(\mu)\|^2 &= \big(\mathbf{b}(\mu) - \mathbf{L}(\mu)\mathbf{V}_k\hat{\mathbf{x}}(\mu)\big)^T\big(\mathbf{b}(\mu) - \mathbf{L}(\mu)\mathbf{V}_k\hat{\mathbf{x}}(\mu)\big) \\ &= \sum_{p,q=1}^{n_B^2} \theta_p^{\mathbf{b}}(\mu)\theta_q^{\mathbf{b}}(\mu)\mathbf{b}_p^T\mathbf{b}_q - 2\sum_{p=1}^{n_B^2}\sum_{q=1}^{n_A n_E} \theta_p^{\mathbf{b}}(\mu)\theta_q^{\mathbf{L}}(\mu)\mathbf{b}_p^T\mathbf{L}_q\mathbf{V}_k\hat{\mathbf{x}}(\mu) \\ &\quad + \sum_{p,q=1}^{n_A n_E} \theta_p^{\mathbf{L}}(\mu)\theta_q^{\mathbf{L}}(\mu)\hat{\mathbf{x}}^T(\mu)\mathbf{V}_k^T\mathbf{L}_p^T\mathbf{L}_q\mathbf{V}_k\hat{\mathbf{x}}(\mu), \end{aligned} \tag{22}$$

where $\hat{\mathbf{x}}(\mu)$ is the solution of the reduced linear system (10). Note that the parameter-dependent coefficient matrix and the right-hand side in (10) also admit the affine decompositions

$$\hat{\mathbf{L}}(\mu) = \sum_{p=1}^{n_A n_E} \theta_p^{\mathbf{L}}(\mu)\mathbf{V}_k^T\mathbf{L}_p\mathbf{V}_k, \qquad \hat{\mathbf{b}}(\mu) = \sum_{p=1}^{n_B^2} \theta_p^{\mathbf{b}}(\mu)\mathbf{V}_k^T\mathbf{b}_p. \tag{23}$$

One can easily realize that several matrix-matrix and matrix-vector products in (22) and (23), namely $\mathbf{b}_p^T\mathbf{b}_q$, $\mathbf{b}_p^T\mathbf{L}_q\mathbf{V}_k$, $\mathbf{V}_k^T\mathbf{L}_p^T\mathbf{L}_q\mathbf{V}_k$, $\mathbf{V}_k^T\mathbf{L}_p\mathbf{V}_k$, and $\mathbf{V}_k^T\mathbf{b}_p$, are independent

of $\mu$. They are expensive to compute but not to store. Another worthwhile advantage is that all of these parameter-independent quantities can be computed and stored hierarchically with respect to the step $k$ of the greedy algorithm. In other words, at each greedy step only one entry of the vectors, one row, and one column of the matrices must be computed and added to their previous versions. Once all parameter-independent quantities are available, for each $\mu \in \mathcal{D}_{\text{train}}$, one first computes (23), then solves (10) for $\hat{\mathbf{x}}(\mu)$ and, finally, computes (22). These three steps are cheap since their computational complexity depends only on $k$, $n_E$, $n_A$, and $n_B$, which are very small. Therefore, the search on $\mathcal{D}_{\text{train}}$ is quite fast, which makes the greedy algorithm feasible and efficient even for large-scale problems and the large training set $\mathcal{D}_{\text{train}}$.

**3.2.2. Energy norm error estimate.** Instead of the Euclidean vector norm, one can also quantify the approximation error in the energy norm, which is frequently used in the reduced basis method [17, 29]. In our setting, the energy vector norm is defined by $\|\mathbf{x}\|_\mu = \sqrt{\mathbf{x}^T \mathbf{L}(\mu)\mathbf{x}}$. It is easy to verify that

$$\sqrt{\alpha(\mu)}\|\mathbf{x}\| \le \|\mathbf{x}\|_\mu \le \sqrt{\gamma(\mu)}\|\mathbf{x}\|,$$

with $\alpha(\mu)$ and $\gamma(\mu)$ as in (6) and (7), respectively. From (11) and (15) we obtain the energy norm error estimate

$$\|\mathbf{e}_k(\mu)\|_\mu = \sqrt{\mathbf{r}_k^T(\mu)\mathbf{L}^{-1}(\mu)\mathbf{r}_k(\mu)} \le \frac{\|\mathbf{r}_k(\mu)\|}{\sqrt{\alpha_{\text{LB}}(\mu)}} =: \Delta_k^{en}(\mu). \tag{24}$$

Then the corresponding effectivity constant satisfies

$$1 \le \eta_k^{en}(\mu) := \frac{\Delta_k^{en}(\mu)}{\|\mathbf{e}_k(\mu)\|_\mu} \le \sqrt{\frac{\gamma_{\text{UB}}(\mu)}{\alpha_{\text{LB}}(\mu)}}.$$

We see that the upper bound for $\eta_k^{en}(\mu)$ is smaller than that for $\eta_k(\mu)$ in (19). This implies that the estimate (24) is sharper than (17).

**4. Low-rank reduced basis method for Lyapunov equation.** A major drawback of the reduced basis method described above is that it operates with matrices and vectors of huge dimension $N^2$ and, as a consequence, suffers from high computational complexity and large storage requirements. Fortunately, thanks to Lemma 2.2, the reduced basis method can directly be applied to the PALE (1) in which all operations are still with $N \times N$-matrices and $N$-vectors. Moreover, assuming that the solution of the PALE (1) with the low-rank right-hand side $B(\mu)B^T(\mu)$ is well approximated by a low-rank matrix $X(\mu) \approx Z(\mu)Z^T(\mu)$, we can further reduce computational cost and memory requirements both in offline and online stages.

**4.1. Offline phase.** In the offline phase, instead of solving the linear system (2), we compute the solutions of the PALE (1) for selected parameters $\mu_1, \ldots, \mu_k$. Let $Z_j \in \mathbb{R}^{N \times n_j}$ be the low-rank Cholesky factor of the solution $X(\mu_j) \approx Z_j Z_j^T$ of (1) at $\mu = \mu_j$, $j = 1, \ldots, k$. These factors can efficiently be computed by the low-rank alternating directions implicit (LR-ADI) method [6, 24, 28], Krylov subspace method [9, 31], or the Riemannian method [35]. Then for any $\mu \in \mathcal{D}$, we determine an approximate solution of (1) as

$$X(\mu) \approx \text{mat}(\mathbf{V}_k \hat{\mathbf{x}}(\mu)) =: X_{\text{RB}}(\mu),$$

where

$$\mathbf{V}_k = \big[\, \mathrm{vec}(Z_1 Z_1^T), \ldots, \mathrm{vec}(Z_k Z_k^T) \,\big] \tag{25}$$

and $\hat{\mathbf{x}}(\mu)$ solves the reduced linear system (10). Note that, for simplicity, we denote the new reduced basis matrix (25) again by $\mathbf{V}_k$ although it differs from that in (9) since it is constructed from the low-rank approximate solutions $X(\mu_j) \approx Z_j Z_j^T$ instead of the exact solutions $X(\mu_j) = \mathrm{mat}\big(\mathbf{x}(\mu_j)\big)$ used in (9). Clearly, this results in a different reduced basis approximation.

The solution $X_{\mathrm{RB}}(\mu)$ can also be written as

$$X_{\mathrm{RB}}(\mu) = \sum_{j=1}^{k} \hat{x}_j(\mu) Z_j Z_j^T = V_k \begin{bmatrix} \hat{x}_1(\mu) I_{n_1} & & \\ & \ddots & \\ & & \hat{x}_k(\mu) I_{n_k} \end{bmatrix} V_k^T, \tag{26}$$

where $[\hat{x}_1(\mu), \ldots, \hat{x}_1(\mu)]^T = \hat{\mathbf{x}}(\mu)$ and

$$V_k = [\, Z_1, \ldots, Z_k \,]. \tag{27}$$

Note that we never form the matrix $\mathbf{V}_k$ explicitly to construct the reduced linear system (10). Instead, we exploit the affine decomposition (23) of the coefficient matrix $\hat{\mathbf{L}}(\mu)$ and the right-hand side $\hat{\mathbf{b}}(\mu)$ and compute the entries of the parameter-independent matrices $\mathbf{V}_k^T \mathbf{L}_p \mathbf{V}_k$ for $p = (i-1)n_A + j$ with $i = 1, \ldots, n_E$, $j = 1, \ldots, n_A$, and the vectors $\mathbf{V}_k^T \mathbf{b}_p$ for $p = (i-1)n_B + j$ with $i, j = 1, \ldots, n_B$, using Lemmas 2.1 and 2.2 and relations (14) and (21) as follows:

$$\begin{aligned} (\mathbf{V}_k^T \mathbf{L}_p \mathbf{V}_k)_{rl} &= \langle -E_i Z_l Z_l^T A_j^T - A_j Z_l Z_l^T E_i^T, Z_r Z_r^T \rangle \\ &= -2\mathrm{trace}\big(Z_r^T (E_i Z_l)(A_j Z_l)^T Z_r\big), \\ (\mathbf{V}_k^T \mathbf{b}_p)_r &= \langle B_i B_j^T, Z_r Z_r^T \rangle = \mathrm{trace}\big((B_j^T Z_r)(Z_r^T B_i)\big) \end{aligned}$$

for $r, l = 1, \ldots, k$. Thus, taking advantage of the structure of $\mathbf{L}_p$, $\mathbf{b}_p$, and $\mathbf{V}_k$ reduces the computational cost, for example, of $\mathbf{V}_k^T \mathbf{L}_p \mathbf{V}_k$ from $\mathcal{O}(N^4 k)$ to $\mathcal{O}(N^2 n)$ with $n = n_1 + \cdots + n_k$. In counting, we did not exploit the sparsity of $E_i$ and $A_j$.

*Remark* 4.1. Note that even $E_i$ and $A_j$ are assumed to be symmetric, and we always write $E_i^T$ and $A_j^T$ if the transpose matrices are needed. This will simplify the extension of the reduced basis method to nonsymmetric problems; see section 5.

For the approximate solution $X_{\mathrm{RB}}(\mu)$, we obtain from Theorem 3.6 the error estimate

$$\|X(\mu) - X_{\mathrm{RB}}(\mu)\|_F = \|\mathbf{e}_k(\mu)\| \le \Delta_k(\mu) \tag{28}$$

with $\Delta_k(\mu)$ as in (18). This error estimator can now be utilized in the greedy parameter sampling procedure presented in Algorithm 2.

For an efficient computation of the residual norm $\|\mathbf{r}_k(\mu)\|$, we again make use of the affine representation (22), where computing the parameter-independent quantities demands a more detailed discussion. Consider first $\mathbf{b}_p^T \mathbf{b}_q$ for $p = (i-1)n_B + j$, $q = (f-1)n_B + g$, and $i, j, f, g = 1, \ldots, n_B$. Using (14) and (21) we obtain

$$\mathbf{b}_p^T \mathbf{b}_q = \langle B_f B_g^T, B_i B_j^T \rangle = \mathrm{trace}\big((B_i^T B_f)(B_g^T B_j)\big).$$

**Algorithm 2** Greedy algorithm for Lyapunov equations.

**Input:** tolerance $tol_{\mathrm{rb}}$, training set $\mathcal{D}_{\mathrm{train}}$, initial parameter $\mu_1 \in \mathcal{D}_{\mathrm{train}}$.
**Output:** a basis matrix $V_k$.

Solve the PALE (1) at $\mu = \mu_1$ for $X(\mu_1) \approx Z_1 Z_1^T$.
Set $\Delta_1^{\max} > tol_{\mathrm{rb}}$, $\mathcal{M}_1 = \{\mu_1\}$, $V_1 = Z_1$, and $k = 2$.
**while** $\Delta_{k-1}^{\max} \geq tol_{\mathrm{rb}}$ **do**
  $\mu_k = \operatorname*{arg\,max}_{\mu \in \mathcal{D}_{\mathrm{train}} \setminus \mathcal{M}_{k-1}} \Delta_{k-1}(\mu)$
  $\Delta_k^{\max} = \Delta_{k-1}(\mu_k)$
  $\mathcal{M}_k = \mathcal{M}_{k-1} \cup \{\mu_k\}$
  solve the PALE (1) at $\mu = \mu_k$ for $X(\mu_k) \approx Z_k Z_k^T$
  $V_k = [\, V_{k-1},\ Z_k \,]$
  $k \leftarrow k + 1$
**end while**

The components of the vector $\mathbf{b}_p^T \mathbf{L}_q \mathbf{V}_k$ for $p = (i-1)n_B + j$, $q = (f-1)n_A + g$, $i, j = 1, \ldots, n_B$, $f = 1, \ldots, n_E$, and $g = 1, \ldots, n_A$ can be expressed as

$$\begin{aligned}(\mathbf{b}_p^T \mathbf{L}_q \mathbf{V}_k)_l &= \langle -A_g Z_l Z_l^T E_f^T - E_f Z_l Z_l^T A_g^T, B_i B_j^T \rangle \\ &= -\mathrm{trace}\big(B_i^T (E_f Z_l)(A_g Z_l)^T B_j + B_i^T (A_g Z_l)(E_f Z_l)^T B_j\big), \quad l = 1, \ldots, k.\end{aligned}$$

Finally, the matrix $\mathbf{V}_k^T \mathbf{L}_p^T \mathbf{L}_q \mathbf{V}_k$ for $p = (i-1)n_A + j$, $q = (f-1)n_A + g$, and $i, f = 1, \ldots, n_E$, $j, g = 1, \ldots, n_A$, can be determined elementwise:

$$\begin{aligned}(\mathbf{V}_k^T \mathbf{L}_p^T \mathbf{L}_q \mathbf{V}_k)_{rl} &= \langle -A_g Z_l Z_l^T E_f^T - E_f Z_l Z_l^T A_g^T, -A_j Z_r Z_r^T E_i^T - E_i Z_r Z_r^T A_j^T \rangle \\ &= 2\,\mathrm{trace}\big((E_i Z_r)^T (E_f Z_l)(A_g Z_l)^T (A_j Z_r) + (E_i Z_r)^T (A_g Z_l)(E_f Z_l)^T (A_j Z_r)\big)\end{aligned}$$

for $r, l = 1, \ldots, k$. Here, we used Lemma 2.2 and the relations (14) and (21).

**4.2. Online phase.** Once the reduced basis matrix $V_k$ is constructed such that the error estimator does not exceed a given tolerance, the solution of the PALE (1) at any $\mu \in \mathcal{D}$ can be obtained in the online phase as in (26). Nevertheless, a serious disadvantage of this approach is that the resulting approximate solution $X_{\mathrm{RB}}(\mu)$ is not necessarily positive semidefinite since the solution $\hat{\mathbf{x}}(\mu)$ of (10) may have negative entries. This difficulty can be circumvented by computing the approximate solution in the form $X(\mu) \approx V_k \hat{X}(\mu) V_k^T =: \hat{X}_{\mathrm{RB}}(\mu)$, where $V_k$ is as in (27) and $\hat{X}(\mu)$ solves the reduced Lyapunov equation

$$\hat{A}(\mu)\hat{X}(\mu)\hat{E}^T(\mu) + \hat{E}(\mu)\hat{X}(\mu)\hat{A}^T(\mu) = -\hat{B}(\mu)\hat{B}^T(\mu) \tag{29}$$

with $\hat{E}(\mu) = V_k^T E(\mu) V_k$, $\hat{A}(\mu) = V_k^T A(\mu) V_k$, and $\hat{B}(\mu) = V_k^T B(\mu)$. Since $-A(\mu)$ and $E(\mu)$ are symmetric and positive definite, this equation has a unique symmetric positive semidefinite solution $\hat{X}(\mu) = \hat{Z}(\mu)\hat{Z}^T(\mu)$. Then $\hat{X}_{\mathrm{RB}}(\mu)$ can be written in the factorized form $\hat{X}_{\mathrm{RB}}(\mu) = Z_{\mathrm{RB}}(\mu) Z_{\mathrm{RB}}^T(\mu)$ with $Z_{\mathrm{RB}}(\mu) = V_k \hat{Z}(\mu)$.

Let

$$\hat{R}_k(\mu) = A(\mu)\hat{X}_{\mathrm{RB}}(\mu)E^T(\mu) + E(\mu)\hat{X}_{\mathrm{RB}}(\mu)A^T(\mu) + B(\mu)B^T(\mu) \tag{30}$$

be the residual associated with the approximate solution $\hat{X}_{\mathrm{RB}}(\mu)$. Then the error $X(\mu) - \hat{X}_{\mathrm{RB}}(\mu)$ can be estimated similarly to the linear system case as

$$\|X(\mu) - \hat{X}_{\mathrm{RB}}(\mu)\|_F \leq \frac{\|\hat{R}_k(\mu)\|_F}{\alpha(\mu)} \leq \frac{\|\hat{R}_k(\mu)\|_F}{\alpha_{\mathrm{LB}}(\mu)} =: \hat{\Delta}_k(\mu) \tag{31}$$

with $\alpha_{\mathrm{LB}}(\mu)$ as in (15). Replacing the approximate solution $\mathbf{V}_k\hat{\mathbf{x}}(\mu)$ in (22) by

$$\mathrm{vec}(\hat{X}_{\mathrm{RB}}(\mu)) = (V_k \otimes V_k)\mathrm{vec}(\hat{X}(\mu)),$$

we obtain the following expression for the residual:

$$\begin{aligned}
\|\hat{R}_k(\mu)\|_F^2 &= \|\mathbf{b}(\mu) - \mathbf{L}(\mu)(V_k \otimes V_k)\mathrm{vec}(\hat{X}(\mu))\|^2 \\
&= \sum_{i,j=1}^{n_B}\sum_{f,g=1}^{n_B} \theta^B_{ijfg}(\mu)\mathrm{trace}\big((B_i^TB_f)(B_g^TB_j)\big) \\
&+ 2\sum_{i,j=1}^{n_B}\sum_{f=1}^{n_E}\sum_{g=1}^{n_A} \theta^{AEB}_{ijfg}(\mu)\mathrm{trace}\big(B_i^T(E_fV_k)\hat{X}(\mu)(A_gV_k)^TB_j\big) \\
&+ 2\sum_{i,j=1}^{n_B}\sum_{f=1}^{n_E}\sum_{g=1}^{n_A} \theta^{AEB}_{ijfg}(\mu)\mathrm{trace}\big(B_i^T(A_gV_k)\hat{X}(\mu)(E_fV_k)^TB_j\big) \\
&+ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A} \theta^{AE}_{ijfg}(\mu)\mathrm{trace}\big((E_fV_k)^T(E_iV_k)\hat{X}(\mu)(A_jV_k)^T(A_gV_k)\hat{X}(\mu)\big) \\
&+ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A} \theta^{AE}_{ijfg}(\mu)\mathrm{trace}\big((E_fV_k)^T(A_jV_k)\hat{X}(\mu)(E_iV_k)^T(A_gV_k)\hat{X}(\mu)\big),
\end{aligned}$$

where $\theta^B_{ijfg}(\mu) = \theta^B_i(\mu)\theta^B_j(\mu)\theta^B_f(\mu)\theta^B_g(\mu)$, $\theta^{AEB}_{ijfg}(\mu) = \theta^B_i(\mu)\theta^B_j(\mu)\theta^E_f(\mu)\theta^A_g(\mu)$, and $\theta^{AE}_{ijfg}(\mu) = \theta^E_i(\mu)\theta^A_j(\mu)\theta^E_f(\mu)\theta^A_g(\mu)$. Again, all parameter-independent matrices can be precomputed and stored in the offline stage. Then the error estimator $\hat{\Delta}_k(\mu)$ can be calculated in the online stage at low computational cost which is independent of the large dimension $N$.

Note that in the greedy algorithm, instead of $\Delta_k(\mu)$ one can also use the estimator $\hat{\Delta}_k(\mu)$. It should, however, be emphasized that the computation of the reduced basis solution $X_{\mathrm{RB}}(\mu)$ in (26) is less expensive than that of $\hat{X}_{\mathrm{RB}}(\mu) = V_k\hat{X}(\mu)V_k^T$ because solving the linear system (10) with $\hat{\mathbf{L}}(\mu) \in \mathbb{R}^{k\times k}$ is cheaper than solving the Lyapunov equation (29) with $\hat{E}(\mu)$, $\hat{A}(\mu) \in \mathbb{R}^{n\times n}$, where $n = n_1 + \cdots + n_k$ may be significantly larger than $k$. The dimension $n$ depends on $m$ (number of columns of $B(\mu)$), the convergence rate of the iterative method used for solving the PALE (1), and the number of greedy steps. To keep the offline computational cost low, we compute the error estimator based on $X_{\mathrm{RB}}(\mu)$, whereas in the online phase, where the positive semidefiniteness of the solution is most essential, we calculate $\hat{X}_{\mathrm{RB}}(\mu)$. Moreover, due to the fact that $\mathrm{span}(\mathbf{V}_k) \subset \mathrm{span}(V_k \otimes V_k)$, the approximate solution $\hat{X}_{\mathrm{RB}}(\mu)$ is predicted to have a smaller error. In order to retain a low-rank structure of $\hat{X}_{\mathrm{RB}}(\mu)$ and, therefore, to reduce the online computational cost, the column compression in $V_k$ should be performed with a prescribed tolerance $tol_{\mathrm{cc}}$. This can be done by computing a rank-revealing QR decomposition or a singular value decomposition (SVD) of $V_k$.

**5. Nonsymmetric case.** The reduced basis method described above can also be applied to the PALE (1), where $A(\mu)$ is nonsymmetric, with some adjustments in formulating the error estimates.

Assume that the pencil $\lambda E(\mu) - A(\mu)$ is *strictly dissipative*, i.e.,

$$E(\mu) = E^T(\mu) > 0, \qquad A(\mu) + A^T(\mu) < 0 \tag{32}$$

for all $\mu \in \mathcal{D}$. These conditions guarantee the solvability of the reduced Lyapunov equation (29) for any projection matrix $V_k$. They are fulfilled if assumptions (A1),

(A2), together with

(A3′) $A_j + A_j^T \le 0$ and $\theta_j^A(\mu) > 0$ for all $\mu \in \mathcal{D}$ and $j = 1, \ldots, n_A$;

(A4′) there exist at least one pair $(A_j, E_i)$ such that $A_j + A_j^T < 0$ and $E_i > 0$,

hold. Taking (11) and (12) into account, we need only find an upper bound for $\|\mathbf{L}(\mu)\|_2 = \sigma_{\max}\big(\mathbf{L}(\mu)\big)$ and a lower bound for $\|\mathbf{L}^{-1}(\mu)\|_2^{-1} = \sigma_{\min}\big(\mathbf{L}(\mu)\big)$. For this purpose, we introduce

$$S(\mu) = \frac{1}{2}\big(A(\mu) + A^T(\mu)\big), \tag{33}$$
$$\mathbf{L}_S(\mu) = \frac{1}{2}\big(\mathbf{L}(\mu) + \mathbf{L}^T(\mu)\big) = -E(\mu) \otimes S(\mu) - S(\mu) \otimes E(\mu).$$

Obviously, these matrices inherit the affine structure. Moreover, $S(\mu) = S^T(\mu) < 0$ and $\mathbf{L}_S(\mu) = \mathbf{L}_S^T(\mu) > 0$ for all $\mu \in \mathcal{D}$.

**5.1. Frobenius norm error estimates.** In this subsection we derive the Frobenius norm error estimates for the approximate solutions $X_{\mathrm{RB}}(\mu)$ and $\hat{X}_{\mathrm{RB}}(\mu)$. The following lemma establishes lower and upper bounds for the smallest and largest singular values of $\mathbf{L}(\mu)$, respectively.

LEMMA 5.1. *Let $E(\mu)$ and $A(\mu)$ satisfy* (A1), (A2), (A3′), *and* (A4′), *and let $\bar{\mu}, \bar{\mu}_1, \bar{\mu}_2 \in \mathcal{D}$.*

1. *For all $\mu \in \mathcal{D}$, the smallest singular value of $\mathbf{L}(\mu)$ is bounded from below as*

$$\sigma_{\min}\big(\mathbf{L}(\mu)\big) \ge \tilde{\alpha}_{\mathrm{LB}}(\mu) := \max\Big(\tilde{\alpha}_{\mathrm{LB}}^{\mathbf{L},\bar{\mu}}(\mu),\ \tilde{\alpha}_{\mathrm{LB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu),\ \tilde{\alpha}_{\mathrm{LB}}^{\mathbf{L}}(\mu)\Big) > 0, \tag{34}$$

*where*

$$\begin{aligned}
\tilde{\alpha}_{\mathrm{LB}}^{\mathbf{L},\bar{\mu}}(\mu) &= 2\,\theta_{\min}^{\mathbf{L},\bar{\mu}}(\mu)\,\lambda_{\min}\big(-S(\bar{\mu})\big)\,\lambda_{\min}\big(E(\bar{\mu})\big),\\
\tilde{\alpha}_{\mathrm{LB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu) &= 2\,\theta_{\min}^{A,\bar{\mu}_1}(\mu)\,\theta_{\min}^{E,\bar{\mu}_2}(\mu)\,\lambda_{\min}\big(-S(\bar{\mu}_1)\big)\,\lambda_{\min}\big(E(\bar{\mu}_2)\big),\\
\tilde{\alpha}_{\mathrm{LB}}^{\mathbf{L}}(\mu) &= 2\sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta_{ij}^{\mathbf{L}}(\mu)\,\lambda_{\min}(-S_j)\,\lambda_{\min}(E_i)
\end{aligned}$$

*with $S(\mu)$ as in* (33) *and $S_j = (A_j + A_j^T)/2$.*

2. *For all $\mu \in \mathcal{D}$, the largest singular value of $\mathbf{L}(\mu)$ is bounded from above as*

$$\sigma_{\max}\big(\mathbf{L}(\mu)\big) \le \tilde{\gamma}_{\mathrm{UB}}(\mu) := \min\Big(\tilde{\gamma}_{\mathrm{UB}}^{\mathbf{L},\bar{\mu}}(\mu),\ \tilde{\gamma}_{\mathrm{UB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu),\ \tilde{\gamma}_{\mathrm{UB}}^{\mathbf{L}}(\mu)\Big), \tag{35}$$

*where*

$$\begin{aligned}
\tilde{\gamma}_{\mathrm{UB}}^{\mathbf{L},\bar{\mu}}(\mu) &= 2\,\theta_{\max}^{\mathbf{L},\bar{\mu}}(\mu)\,\sigma_{\max}\big(A(\bar{\mu})\big)\,\lambda_{\max}\big(E(\bar{\mu})\big),\\
\tilde{\gamma}_{\mathrm{UB}}^{A,\bar{\mu}_1;E,\bar{\mu}_2}(\mu) &= 2\,\theta_{\max}^{A,\bar{\mu}_1}(\mu)\,\theta_{\max}^{E,\bar{\mu}_2}(\mu)\,\sigma_{\max}\big(A(\bar{\mu}_1)\big)\,\lambda_{\max}\big(E(\bar{\mu}_2)\big),\\
\tilde{\gamma}_{\mathrm{UB}}^{\mathbf{L}}(\mu) &= 2\sum_{i=1}^{n_E}\sum_{j=1}^{n_A}\theta_{ij}^{\mathbf{L}}(\mu)\,\sigma_{\max}(A_j)\,\lambda_{\max}(E_i).
\end{aligned}$$

*Proof.* 1. Based on an important inequality between singular values of a matrix and eigenvalues of its symmetric part [20, Corollary 3.1.5], we get

$$\sigma_{\min}\big(\mathbf{L}(\mu)\big) \ge \lambda_{\min}\big(\mathbf{L}_S(\mu)\big).$$

Then the bound (34) immediately follows from Lemma 3.3, part 1.

2. Using the multiplicativity property of the singular values of the Kronecker product [20, Theorem 4.2.15], we obtain that

$$\sigma_{\max}\big(\mathbf{L}(\mu)\big) \le 2\,\sigma_{\max}\big(A(\mu)\big)\,\lambda_{\max}\big(E(\mu)\big).$$

Then the bound (35) can be proved analogously to Lemma 3.3, part 2. ∎

We use now the bounds (34) and (35) to derive the error estimates for the reduced basis solutions of the PALE (1).

THEOREM 5.2. *Let $E(\mu)$ and $A(\mu)$ satisfy* (A1), (A2), (A3´), *and* (A4´), *and let $X_{\rm RB}(\mu)$ and $\hat{X}_{\rm RB}(\mu)$ be the reduced basis approximations to the solution of the PALE* (1). *Then the errors $X(\mu) - X_{\rm RB}(\mu)$ and $X(\mu) - \hat{X}_{\rm RB}(\mu)$ can be estimated as*

$$\|X(\mu) - X_{\rm RB}(\mu)\|_F \le \frac{\|\mathrm{mat}(\mathbf{r}_k(\mu))\|_F}{\tilde{\alpha}_{\rm LB}(\mu)} =: \Delta_k^{ns}(\mu) \le \frac{\tilde{\gamma}_{\rm UB}(\mu)}{\tilde{\alpha}_{\rm LB}(\mu)}\|X(\mu) - X_{\rm RB}(\mu)\|_F,$$
$$\|X(\mu) - \hat{X}_{\rm RB}(\mu)\|_F \le \frac{\|\hat{R}_k(\mu)\|_F}{\tilde{\alpha}_{\rm LB}(\mu)} =: \hat{\Delta}_k^{ns}(\mu) \le \frac{\tilde{\gamma}_{\rm UB}(\mu)}{\tilde{\alpha}_{\rm LB}(\mu)}\|X(\mu) - \hat{X}_{\rm RB}(\mu)\|_F,$$

*where $\tilde{\alpha}_{\rm LB}(\mu)$ and $\tilde{\gamma}_{\rm UB}(\mu)$ are as in* (34) *and* (35), *respectively.*

*Proof.* The result follows from (11), (12), and Lemma 5.1. ∎

**5.2. Logarithmic norm based error estimates.** Alternative error estimates can be derived using a 2-logarithmic norm of the pencil $\lambda E(\mu) - A(\mu)$, defined as

$$\ell\big(E(\mu), A(\mu)\big) = \lambda_{\max}\big(E(\mu), S(\mu)\big).$$

If $E(\mu) \equiv I$, then $\ell\big(I, A(\mu)\big) = \ell\big(A(\mu)\big) = \lambda_{\max}\big(S(\mu)\big)$ is the 2-logarithmic matrix norm, which is frequently used in differential equations and numerical analysis [33]. Conditions (32) imply that $\ell\big(E(\mu), A(\mu)\big) < 0$ for all $\mu \in \mathcal{D}$, so it is not a norm in the usual sense. Define a weighted matrix norm

$$\|X\|_{E(\mu)} = \|G^T(\mu) X G(\mu)\|_F,$$

where $G(\mu)$ is a Cholesky factor of $E(\mu) = G(\mu)G^T(\mu)$. The following theorem establishes an error estimate for the reduced basis solution $\hat{X}_{RB}(\mu) = V_k \hat{X}(\mu) V_k^T$.

THEOREM 5.3. *Let $X(\mu)$ and $\hat{X}_{RB}(\mu)$ be the exact and approximate solutions of the PALE* (1). *Then the error $X(\mu) - \hat{X}_{RB}(\mu)$ can be estimated as*

$$\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)} \le \frac{\|\hat{R}_k(\mu)\|_F}{\alpha_{\rm LB}^{E,A,\bar{\mu}}(\mu)} =: \hat{\Delta}_k^{E,A,\bar{\mu}}(\mu) \le \frac{\gamma_{\rm UB}^{E,A,\bar{\mu}}(\mu)}{\alpha_{\rm LB}^{E,A,\bar{\mu}}(\mu)}\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)},$$

*where*

$$\alpha_{\rm LB}^{E,A,\bar{\mu}}(\mu) = 2\,\frac{\theta_{\min}^{A,\bar{\mu}}(\mu)}{\theta^{E,\bar{\mu}}(\mu)}\lambda_{\min}\big(E(\bar{\mu})\big)\,\lambda_{\min}\big(E(\bar{\mu}), -S(\bar{\mu})\big), \tag{36}$$

$$\gamma_{\rm UB}^{E,A,\bar{\mu}}(\mu) = 2\,\theta_{\max}^{A,\bar{\mu}}(\mu)\sigma_{\max}\big(A(\bar{\mu})\big)\sqrt{\theta^{E,\bar{\mu}}(\mu)\frac{\lambda_{\max}\big(E(\bar{\mu})\big)}{\lambda_{\min}\big(E(\bar{\mu})\big)}}, \tag{37}$$

*and $\hat{R}_k(\mu)$ is the residual given in* (30).

*Proof.* Let $\Xi(\mu) = X(\mu) - \hat{X}_{RB}(\mu)$. It follows from

$$G^{-1}(\mu)\hat{R}_k(\mu)G^{-T}(\mu) = -A_G(\mu)G^T(\mu)\Xi(\mu)G(\mu) - G^T(\mu)\Xi(\mu)G(\mu)A_G^T(\mu)$$

with $A_G(\mu) = G^{-1}(\mu)A(\mu)G^{-T}(\mu)$ that

$$-G^T(\mu)\Xi(\mu)G(\mu) = \int_0^\infty e^{A_G(\mu)t}G^{-1}(\mu)\hat{R}_k(\mu)G^{-T}(\mu)e^{A_G^T(\mu)t}dt.$$

The matrix exponential is estimated as

$$\|e^{A_G(\mu)t}\|_2 \le e^{\ell\left(G^{-1}(\mu)A(\mu)G^{-T}(\mu)\right)t} = e^{\ell\left(E(\mu),A(\mu)\right)t};$$

see [8, 27]. Therefore,

$$\|\Xi(\mu)\|_{E(\mu)} \le \|G^{-1}(\mu)\hat{R}_k(\mu)G^{-T}(\mu)\|_F \int_0^\infty e^{2\ell(E(\mu),A(\mu))t}dt \le \frac{\|\hat{R}_k(\mu)\|_F\|E^{-1}(\mu)\|_2}{-2\,\ell\big(E(\mu),A(\mu)\big)}.$$

It remains to find a lower bound for $-\ell\big(E(\mu),A(\mu)\big)$. We again employ the min-$\theta$ approach. For

$$S(\mu) = \sum_{j=1}^{n_A}\theta_j^A(\mu)S_j$$

with $S_j = (A_j + A_j^T)/2$, we obtain similarly to the proof of Lemma 3.3 that

$$\begin{aligned}
-\ell\big(E(\mu),A(\mu)\big) &= \lambda_{\min}\big(E(\mu),-S(\mu)\big) = \min_{v\in\mathbb{R}^n\setminus\{0\}}\frac{v^T(-S(\mu))v}{v^TE(\mu)v}\\
&= \min_{v\in\mathbb{R}^n\setminus\{0\}}\frac{\sum_{j=1}^{n_A}\theta_j^A(\mu)v^T(-S_j)v}{\sum_{j=1}^{n_E}\theta_j^E(\mu)v^TE_jv} \ge \min_{v\in\mathbb{R}^n\setminus\{0\}}\frac{\theta_{\min}^{A,\bar{\mu}}(\mu)v^T(-S(\bar{\mu}))v}{\theta_{\max}^{E,\bar{\mu}}(\mu)v^TE(\bar{\mu})v}\\
&= \frac{\theta_{\min}^{A,\bar{\mu}}(\mu)}{\theta_{\max}^{E,\bar{\mu}}(\mu)}\lambda_{\min}\big(E(\bar{\mu}),-S(\bar{\mu})\big).
\end{aligned}$$

Then taking into account that

$$\|E^{-1}(\mu)\|_2 = \frac{1}{\lambda_{\min}\big(E(\mu)\big)} \le \frac{1}{\theta_{\min}^{E,\bar{\mu}}(\mu)\lambda_{\min}\big(E(\bar{\mu})\big)}$$

and $\theta^{E,\bar{\mu}}(\mu) = \theta_{\max}^{E,\bar{\mu}}(\mu)/\theta_{\min}^{E,\bar{\mu}}(\mu)$, we get the error estimate

$$\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)} \le \frac{\|\hat{R}_k(\mu)\|_F}{\alpha_{\mathrm{LB}}^{E,A,\bar{\mu}}(\mu)} =: \hat{\Delta}_k^{E,A,\bar{\mu}}(\mu)$$

with $\alpha_{\mathrm{LB}}^{E,A,\bar{\mu}}(\mu)$ as in (36). Furthermore, using

$$\begin{aligned}
\|\hat{R}_k(\mu)\|_F &= \|A(\mu)(\hat{X}_{\mathrm{RB}}(\mu) - X(\mu))E^T(\mu) + E(\mu)(\hat{X}_{\mathrm{RB}}(\mu) - X(\mu))A^T(\mu)\|_F\\
&\le 2\,\|A(\mu)\|_2\sqrt{\|E(\mu)\|_2\|E^{-1}(\mu)\|_2}\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)}\\
&\le 2\,\theta_{\max}^{A,\bar{\mu}}(\mu)\sigma_{\max}\big(A(\bar{\mu})\big)\sqrt{\theta^{E,\bar{\mu}}(\mu)\frac{\lambda_{\max}\big(E(\bar{\mu})\big)}{\lambda_{\min}\big(E(\bar{\mu})\big)}}\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)}\\
&= \gamma_{\mathrm{UB}}^{E,A,\bar{\mu}}(\mu)\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)}
\end{aligned}$$

with $\gamma_{\mathrm{UB}}^{E,A,\bar{\mu}}(\mu)$ as in (37), the effectivity constant can be estimated as

$$1 \le \eta_k^{E,A,\bar{\mu}}(\mu) := \frac{\hat{\Delta}_k^{E,A,\bar{\mu}}(\mu)}{\|X(\mu) - \hat{X}_{RB}(\mu)\|_{E(\mu)}} \le \frac{\gamma_{\mathrm{UB}}^{E,A,\bar{\mu}}(\mu)}{\alpha_{\mathrm{LB}}^{E,A,\bar{\mu}}(\mu)}.$$

This completes the proof. ∎

**6. Application to parametric model order reduction.** Given a parametric linear dynamical system

$$\begin{aligned} E(\mu)\dot{x}(t,\mu) &= A(\mu)x(t,\mu) + B(\mu)u(t,\mu), \\ y(t,\mu) &= C(\mu)x(t,\mu), \end{aligned} \tag{38}$$

where $A(\mu)$, $E(\mu)$, and $B(\mu)$ satisfy (A1), and the output matrix $C(\mu) \in \mathbb{R}^{l\times N}$ with $l \ll N$ also depends affinely on $\mu$, i.e.,

$$C(\mu) = \sum_{j=1}^{n_C} \theta_j^C(\mu) C_j,$$

the main goal of model reduction is to approximate system (38) by a reduced-order model

$$\begin{aligned} \tilde{E}(\mu)\dot{\tilde{x}}(t,\mu) &= \tilde{A}(\mu)\tilde{x}(t,\mu) + \tilde{B}(\mu)u(t,\mu), \\ \tilde{y}(t,\mu) &= \tilde{C}(\mu)\tilde{x}(t,\mu), \end{aligned} \tag{39}$$

where $\tilde{A}(\mu)$, $\tilde{E}(\mu) \in \mathbb{R}^{r\times r}$, $\tilde{B}(\mu) \in \mathbb{R}^{r\times m}$, and $C(\mu) \in \mathbb{R}^{l\times r}$ with $r \ll N$. This model can be computed by balanced truncation, probably the most effective model order reduction method for linear control systems. This method is based on the controllability and observability Gramians $X(\mu)$ and $Y(\mu)$ defined as the solutions of the PALE (1) and the dual PALE

$$A^T(\mu)Y(\mu)E(\mu) + E^T(\mu)Y(\mu)A(\mu) = -C^T(\mu)C(\mu), \tag{40}$$

respectively. For brevity, we do not review the balanced truncation method here, but refer the reader who is not familiar with this method to [2, 34].

Based on the derived results, we develop a so-called parametric balanced truncation method as follows. One can observe that for the parametric system (38), the balanced truncation method admits the offline-online decomposition. In the offline phase, we determine the reduced basis matrices $V_X \in \mathbb{R}^{N\times n_X}$ and $V_Y \in \mathbb{R}^{N\times n_Y}$ by the greedy algorithm applied to the PALEs (1) and (40), respectively. Then in the online phase, for any $\mu \in \mathcal{D}$, we first find the approximate Gramians

$$X(\mu) \approx V_X Z_X(\mu) Z_X^T(\mu) V_X^T, \qquad Y(\mu) \approx V_Y Z_Y(\mu) Z_Y^T(\mu) V_Y^T,$$

where $\hat{X}(\mu) = Z_X(\mu)Z_X^T(\mu)$ and $\breve{Y}(\mu) = Z_Y(\mu)Z_Y^T(\mu)$ solve, respectively, the reduced Lyapunov equations (29) with $\hat{A}(\mu) = V_X^T A(\mu) V_X$, $\hat{E}(\mu) = V_X^T E(\mu) V_X$, and $\hat{B}(\mu) = V_X^T B(\mu)$ and

$$\breve{A}^T(\mu)\breve{Y}(\mu)\breve{E}(\mu) + \breve{E}^T(\mu)\breve{Y}(\mu)\breve{A}(\mu) = -\breve{C}^T(\mu)\breve{C}(\mu) \tag{41}$$

with $\breve{A}(\mu) = V_Y^T A(\mu) V_Y$, $\breve{E}(\mu) = V_Y^T E(\mu) V_Y$, and $\breve{C}(\mu) = C(\mu)V_Y$. Computing the SVD

$$\begin{aligned} Z_Y^T(\mu)V_Y^T E(\mu) V_X Z_X(\mu) &= \sum_{j=1}^{n_E} \theta_j^E(\mu) Z_Y^T(\mu) V_Y^T E_j V_X Z_X(\mu) \\ &= [U_1(\mu),\ U_2(\mu)] \begin{bmatrix} \Sigma_1(\mu) & 0 \\ 0 & \Sigma_2(\mu) \end{bmatrix} [V_1(\mu),\ V_2(\mu)]^T \end{aligned} \tag{42}$$

with $\Sigma_1(\mu) \in \mathbb{R}^{r\times r}$, we obtain the projection matrices

$$W(\mu) = V_Y Z_Y(\mu) U_1(\mu) \Sigma_1^{-1/2}(\mu), \quad T(\mu) = V_X Z_X(\mu) V_1(\mu) \Sigma_1^{-1/2}(\mu).$$

Then the reduced-order system matrices in (39) take the forms

$$\begin{aligned} \tilde{E}(\mu) &= W^T(\mu) E(\mu) T(\mu), & \tilde{A}(\mu) &= W^T(\mu) A(\mu) T(\mu), \\ \tilde{B}(\mu) &= W^T(\mu) B(\mu), & \tilde{C}(\mu) &= C(\mu) T(\mu). \end{aligned}$$

Exploiting the affine structure and the parametric formulations of the Gramians, we get

$$\begin{aligned} \tilde{E}(\mu) &= \sum_{j=1}^{n_E} \theta_j^E(\mu) \tilde{W}^T(\mu) V_Y^T E_j V_X \tilde{T}(\mu), & \tilde{B}(\mu) &= \sum_{j=1}^{n_B} \theta_j^B(\mu) \tilde{W}^T(\mu) V_Y^T B_j, \\ \tilde{A}(\mu) &= \sum_{j=1}^{n_A} \theta_j^A(\mu) \tilde{W}^T(\mu) V_Y^T A_j V_X \tilde{T}(\mu), & \tilde{C}(\mu) &= \sum_{j=1}^{n_C} \theta_j^C(\mu) C_j V_X \tilde{T}(\mu) \end{aligned} \tag{43}$$

with $\tilde{W}(\mu) = Z_Y(\mu) U_1(\mu) \Sigma_1^{-1/2}(\mu)$ and $\tilde{T}(\mu) = Z_X(\mu) V_1(\mu) \Sigma_1^{-1/2}(\mu)$. All parameter-independent terms should be computed and stored before running the online stage. We summarize the parametric balanced truncation model reduction method as follows.

**Offline:** Given the parametric system (38):

- Compute the reduced basis matrices $V_X$ and $V_Y$.
- Compute and store all parameter-independent matrices:
  $V_X^T E_j V_X$, $V_Y^T E_j V_Y$, and $V_Y^T E_j V_X$ for $j = 1, \ldots, n_E$;
  $V_X^T A_j V_X$, $V_Y^T A_j V_Y$, and $V_Y^T A_j V_X$ for $j = 1, \ldots, n_A$;
  $V_X^T B_j$ and $V_Y^T B_j$ for $j = 1, \ldots, n_B$; $C_j V_X$ and $C_j V_Y$ for $j = 1, \ldots, n_C$.

**Online:** Given $\mu \in \mathcal{D}$:

- Compute $\hat{A}(\mu) = V_X^T A(\mu) V_X$, $\hat{E}(\mu) = V_X^T E(\mu) V_X$, $\hat{B}(\mu) = V_X^T B(\mu)$, and $\breve{A}(\mu) = V_Y^T A(\mu) V_Y$, $\breve{E}(\mu) = V_Y^T E(\mu) V_Y$, $\breve{C}(\mu) = C(\mu) V_Y$ using precomputed parameter-independent terms and the affine structure.
- Solve (29) and (41) for the Cholesky factors $Z_X(\mu)$ and $Z_Y(\mu)$, respectively.
- Compute the SVD (42).
- Compute the reduced-order model (39), (43).

Ignoring the small numbers $n_A, n_E, n_B$, and $n_C$, one can verify that the computational complexity of the online stage does not exceed $\mathcal{O}(n_X^3 + n_Y^3)$.

The parametric reduced-order system (39) can also be determined by applying the reduced basis (RB) method directly to (38); see [14]. Next, we sketch a comparison between this method and the reduced basis balanced truncation (RBBT) method presented here. The latter method has a control-theoretic background and gains all benefits of balanced truncation once accurate approximations to the Gramians are available. It should, however, be noticed that while the RB method provably preserves the stability for systems with a strictly dissipative pencil $\lambda E(\mu) - A(\mu)$, the RBBT method provides stable reduced-order models under stronger assumptions that $E(\mu)$ and $-A(\mu)$ are symmetric, positive definite, and $B(\mu) = C^T(\mu)$. Since in the RBBT method only (low-rank) approximations to the Gramians are computed, in general, one can no longer guarantee the stability of the reduced-order models. Though this issue has been addressed in [12, 40], to the best of our knowledge, there is still no rigorous analysis on the impact of the inexactness in the Gramians on the preservation of stability in balanced truncation even for nonparametric systems. Note that in our experiments, we have never observed instability which can be explained by small errors

in the approximated Gramians. Furthermore, in contrast to the RB method, the RBBT method does not rely on the state snapshots, and the projection subspaces are independent of the training input. Comparing the computational complexity of both approaches, we conclude that the offline phase in the RBBT method is less expensive than that in the RB method. Indeed, at the $k$th greedy step, the computation of the error estimates requires in our method solving $K$ linear systems of dimension $k$, where $K$ is the number of training parameters in $\mathcal{D}_{\text{train}}$, whereas in the RB method, one has to solve $K$ differential equations whose dimension is equal to $k$ or larger than $k$ if at at least one greedy step the basis is enriched by more than one vector. As mentioned in [5] and also observed in our numerical experiments, adding more than one basis vector usually results in a faster convergence of the greedy iteration. On the other hand, the online phase in the RBBT method is more expensive than that in the RB method, since for every new parameter, we additionally have to solve two reduced Lyapunov equations, compute the SVD, and then construct the reduced-order model by projection. Finally, the RBBT method provides an a priori error bound which can be computed by summing up the truncated Hankel singular values, whereas the RB method comes with a posteriori error estimates based on residual computations.

**7. Numerical examples.** In this section, we present some results of numerical experiments to demonstrate the properties of the RB method applied to the PALEs as well as the parametric balanced truncation model reduction. For solving the Lyapunov equations for fixed parameter values, we use the LR-ADI method as described in [6]. All computations are performed with MATLAB R2014a on a laptop using 64-bit OS Windows 8.1, equipped with 2.40 GHz 8 GB Intel Core i7-4500U CPU.

**7.1. A heat equation.** The first model is taken from [22]. Consider the heat equation

$$\begin{aligned} \frac{\partial \vartheta}{\partial t} - \nabla(\sigma(\xi)\nabla\vartheta) &= f \quad \text{in} \quad \Omega \times (0,T), \\ \vartheta &= 0 \quad \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{44}$$

with the heat conductivity coefficient

$$\sigma(\xi) = \begin{cases} 1+\mu_i & \text{for} \quad \xi \in D_i,\ i = 1,\dots,4, \\ 1 & \text{for} \quad \xi \in \Omega \backslash (\cup_{i=1}^4 D_i), \end{cases} \tag{45}$$

where $D_i \subset \Omega = (0,4)^2$, $i = 1,\dots,4$, are four discs of radius 0.5 centered at $(1,1)$, $(3,1)$, $(1,3)$, and $(3,3)$, respectively, and the parameter $\mu = [\mu_1,\mu_2,\mu_3,\mu_4]^T$ varies in $\mathcal{D} = [0.1,10]^4$. Equation (44) with the source term $f \equiv 1$ is discretized using the finite element method with piecewise linear basis functions resulting in a system (38) of dimension $N = 1580$ with the symmetric positive definite mass matrix $E(\mu) \equiv E$ and the stiffness matrix

$$A(\mu) = \mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 + \mu_4 A_4 + A_5, \tag{46}$$

where $A_i$, $i = 1,\dots,4$, are symmetric negative semidefinite, and $A_5$ is symmetric negative definite. The input matrix $B(\mu) \equiv B \in \mathbb{R}^N$ originates from the source function $f$, and the output matrix $C(\mu) \equiv C = 1/N[1,\dots,1] \in \mathbb{R}^{1\times N}$. The data were thankfully provided by the authors of [22] and can be downloaded from http://anchp.epfl.ch/htucker.

For solving the controllability and observability PALEs (1) and (40), we employ the RB method with the same setting for both equations. The training set $\mathcal{D}_{\text{train}}$,

which consists of 10000 random parameters, is obtained successively, starting with the random number generator "twister" and the seed 0. The reference parameter $\bar{\mu}$ required in the min-$\theta$ approach is simply chosen to be the 5000th point in the training set. The tolerance for stopping the greedy algorithm and the maximal number of greedy iterations are given by $tol_{\mathrm{rb}} = 10^{-4}$ and $k_{\max} = 40$, respectively. The test set $\mathcal{D}_{\mathrm{test}}$ containing 50 random parameters is obtained with the generator "twister" and the seed 1 on which the online phase is tested. Finally, the tolerance for column compression in the online phase is taken as $tol_{\mathrm{cc}} = 10^{-6}$. For simulation in the time domain $[0, 10]$, we use the implicit Euler method with 200 equidistant time steps.

In Figure 1(a), we plot the coercivity constant $\alpha(\mu)$ and its lower bounds as in Lemma 3.3 on the test parameter set $\mathcal{D}_{\mathrm{test}}$. The horizontal axis shows the index of the parameters from $\mathcal{D}_{\mathrm{test}}$. Note that for the heat model, we have $\alpha_{\mathrm{LB}}^{\mathbf{L},\bar{\mu}}(\mu) = \alpha_{\mathrm{LB}}^{A,\bar{\mu};E,\bar{\mu}}(\mu)$. For computing $\alpha(\mu) = \lambda_{\min}\big(\mathbf{L}(\mu)\big)$, we use the Lanczos method applied to $\mathbf{L}^{-1}(\mu)$, where the matrix-vector products $\mathbf{L}^{-1}(\mu)\mathbf{v}$ are determined by solving the Lyapunov equations

$$-A(\mu)XE^T(\mu) - E(\mu)XA^T(\mu) = \mathrm{mat}(\mathbf{v}).$$

It is worth mentioning that the right-hand side in these equations is not necessarily of low rank, which leads to high computational effort.

In Figure 1(b), we present the convergence history of the maximal error estimate $\Delta_k^{\max}$ in the greedy algorithm for both PALEs. One can observe that the convergence of the greedy algorithm is not very satisfactory: the iteration indeed stops before the tolerance is reached. The true errors are, however, acceptable, and the computed reduced bases provide reasonably good approximations to the solutions of the PALEs on the test set $D_{\mathrm{test}}$. Figure 2 shows the relative errors

$$\frac{\|X(\mu) - X_{\mathrm{RB}}(\mu)\|_F}{\|X_{\mathrm{RB}}(\mu)\|_F} \qquad \text{and} \qquad \frac{\|X(\mu) - \hat{X}_{\mathrm{RB}}(\mu)\|_F}{\|\hat{X}_{\mathrm{RB}}(\mu)\|_F},$$

the relative error estimates $\Delta_{40}(\mu)/\|X_{\mathrm{RB}}(\mu)\|_F$ and $\hat{\Delta}_{40}(\mu)/\|\hat{X}_{\mathrm{RB}}(\mu)\|_F$, and the error efficiencies

$$\frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)} \frac{\|X(\mu) - X_{\mathrm{RB}}(\mu)\|_F}{\|X_{\mathrm{RB}}(\mu)\|_F} \qquad \text{and} \qquad \frac{\gamma_{\mathrm{UB}}(\mu)}{\alpha_{\mathrm{LB}}(\mu)} \frac{\|X(\mu) - \hat{X}_{\mathrm{RB}}(\mu)\|_F}{\|\hat{X}_{\mathrm{RB}}(\mu)\|_F}$$

for the controllability PALE (1) on the test set $\mathcal{D}_{\mathrm{test}}$. One can see that the corresponding errors and error estimates are different from each other by a multiplicative factor of about $10^3$. Moreover, the error for $\hat{X}_{RB}(\mu)$ is smaller than that for $X_{RB}(\mu)$, which supports our observation in section 4.2 and encourages the use of the method by the fact that the error in the online phase is even smaller than that in the offline phase.

Finally, we compare the presented RBBT method with the interpolation based balanced truncation approach [4] and the RB method from [14]. For the first comparison, we use both multivariate Lagrange polynomial interpolation and linear interpolation in the frequency domain on the uniformly spaced parameter grid with 256 nodes. Due to the local properties of linear interpolation, the resulting reduced-order models have dimension about 125 which is much smaller than that in the Lagrange interpolation, but still larger than the average reduced dimension by about 7 in our method (the reduced dimensions may vary for different parameters). In Figure 3(a), we plot the approximate $\mathcal{H}_\infty$-norm of the absolute errors in the frequency response

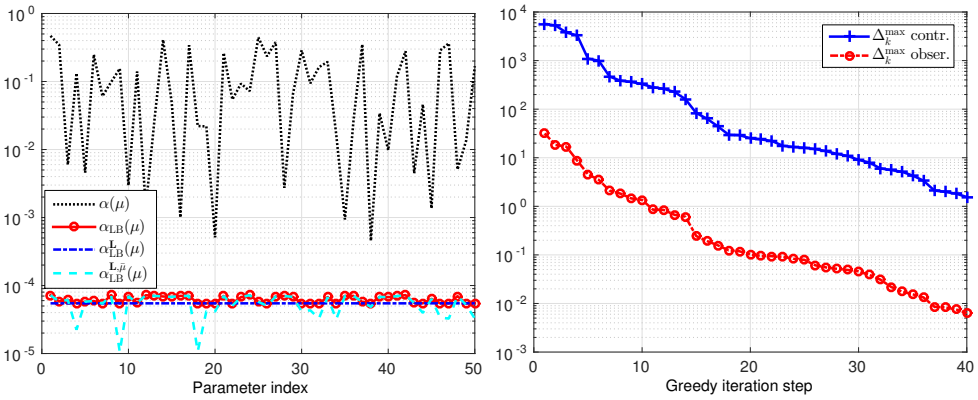


FIG. 1. *Heat equation:* (a) *the coercivity constant* $\alpha(\mu)$ *and its lower bounds for* $\mu \in \mathcal{D}_{\text{test}}$; (b) *the maximal error estimate during the greedy iteration.*

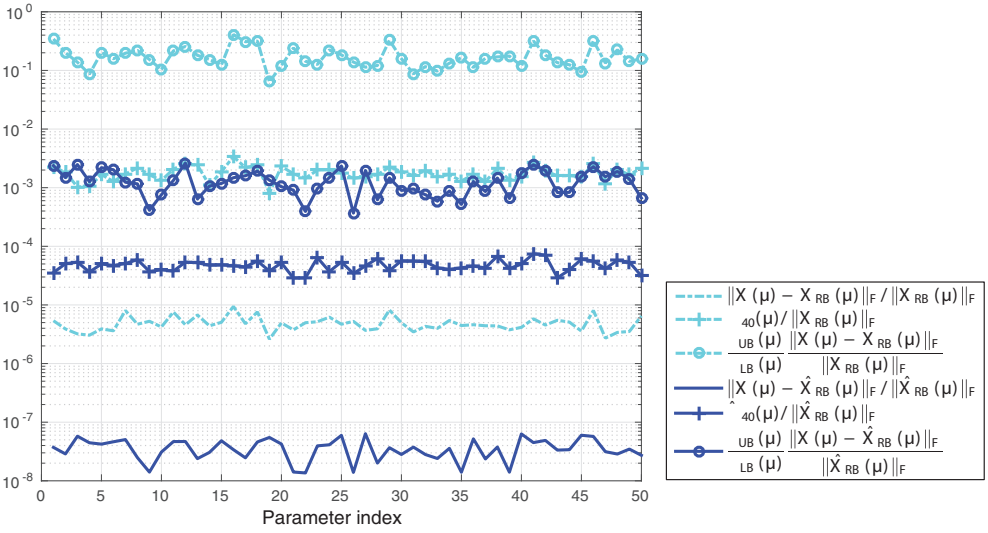


FIG. 2. *Heat equation: relative errors, error estimates, and error efficiencies measured in two ways for the controllability PALE* (1).

defined as

$$\begin{aligned} \|H(\cdot,\mu)-\tilde{H}(\cdot,\mu)\|_{\mathcal{H}_\infty} &= \sup_{\omega\in\mathbb{R}} \|H(i\omega,\mu)-\tilde{H}(i\omega,\mu)\|_2 \\ &\approx \sup_{\omega_j\in[\omega_{\min},\omega_{\max}]} \|H(i\omega_j,\mu)-\tilde{H}(i\omega_j,\mu)\|_2, \end{aligned} \tag{47}$$

where $H(s,\mu) = C\big(sE - A(\mu)\big)^{-1}B$ and $\tilde{H}(s,\mu) = \tilde{C}\big(s\tilde{E} - \tilde{A}(\mu)\big)^{-1}\tilde{B}$ are the transfer functions of the original and the reduced-order systems and $\mu \in \mathcal{D}_{\text{test}}$. It shows that our method delivers much smaller errors than the others. We also compare

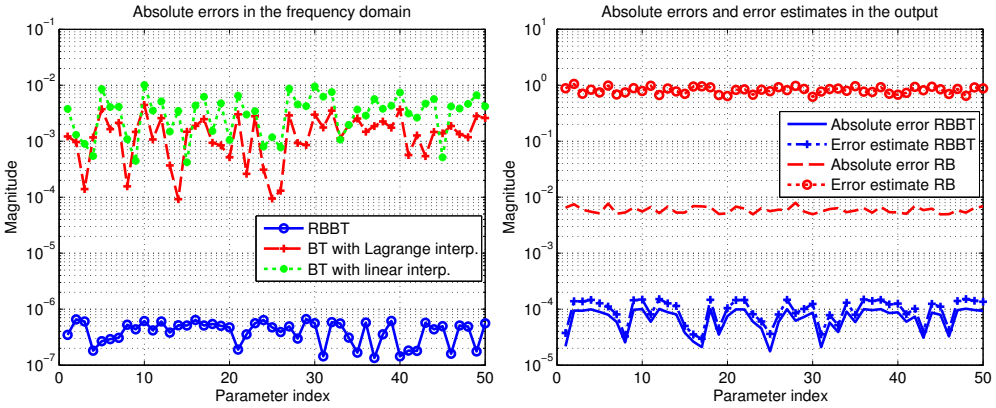


FIG. 3. *Heat equation: a comparison of the parametric balanced truncation with* (a) *the interpolation based balanced truncation and* (b) *the RB method in the time domain.*

the computation time. As anticipated, the offline phase of our method takes longer, 315 sec, while the interpolation based methods require 171 sec. In the online phase, counting from the moment a new parameter value $\mu$ is given until the corresponding reduced-order model is constructed and the frequency response is computed, our method takes 0.06 sec, the linear interpolation based balanced truncation method runs 0.08 sec, while the Lagrange polynomial interpolation based method takes 0.11 sec.

In the RB method, we construct the projection matrix $V$ from the solution snapshots of system (38) with the initial condition $x(0) = 0$ and the impulsive input $u(t) = \delta(t)$. For this purpose, we employ the POD-Greedy approach as described in [13], with only the difference that at every greedy step, we enrich the basis matrix by one up to three columns depending on the magnitude of the singular values in the POD computations. To verify the dependence of the approximation quality on the training input in the RB method, we simulate the reduced-order models with the input $u_{\text{test}}(t) = 3 - 100\cos(t)$ on the test parameter set $\mathcal{D}_{\text{test}}$. Taking into account that all eigenvalues of $\lambda E - A(\mu)$ have negative real part, the output error in the RB method can be estimated as $\|y(\cdot,\mu) - \tilde{y}(\cdot,\mu)\|_{\mathcal{L}_2} \leq \|\Delta_y(\cdot,\mu)\|_{\mathcal{L}_2}$, where

$$\Delta_y(t,\mu) = \|C\|_2 \int_0^t \|EV\dot{\tilde{x}}(\tau,\mu) - A(\mu)V\tilde{x}(\tau,\mu) - Bu_{\text{test}}(\tau)\|d\tau; \tag{48}$$

see [14] for details. Note that since for the RBBT method in the time-domain-only error bound in $\mathcal{L}_2$-norm is available, we use this norm also in the RB method to make the reduction results comparable. Besides, the error estimate (48) is only based on the primal problem. Using other norms and other error estimation techniques relying on both primal and dual problems, e.g., [14, 17, 29], may improve the performance of the RB method.

Figure 3(b) shows the absolute errors $\|y(\cdot,\mu) - \tilde{y}(\cdot,\mu)\|_{\mathcal{L}_2}$ for the RB and RBBT methods. In the latter method, the output error is estimated as

$$\|y(\cdot,\mu) - \tilde{y}(\cdot,\mu)\|_{\mathcal{L}_2} \leq 2\,\text{trace}\big(\Sigma_2(\mu)\big)\|u_{\text{test}}\|_{\mathcal{L}_2},$$

where $\Sigma_2(\mu)$ contains truncated Hankel singular values [2]. Both error estimates are also presented in Figure 3(b). One can see that the errors in the RBBT method are

smaller than those in the RB method, although the average reduced dimension is 7 compared to 63 in the RB method. Note that for the training input $u(t) \equiv 1$, we get the errors of about same order of magnitude in both reduction methods. The offline phase in the RBBT and RB methods takes 315 sec and 3811 sec, respectively, while in the online phase the average time in the RBBT method (0.055 sec) is higher than in the RB method (0.014 sec).

**7.2. An anemometer model.** We consider now an anemometer model describing a thermal based flow sensor; see [26] and references therein. Simulation of this device requires solving a convection-diffusion partial differential equation of the form

$$\rho c \frac{\partial \vartheta}{\partial t} = \nabla(\kappa \nabla \vartheta) - \rho c v \nabla \vartheta + \dot{q}, \tag{49}$$

where $\rho$ denotes the mass density, $c \in [0, 1]$ is the specific heat, $\kappa \in [1, 2]$ is the thermal conductivity, $v \in [0.1, 2]$ is the fluid velocity, $\vartheta$ is the temperature, and $\dot{q}$ is the heat flow into the system caused by the heater. The considered model is restricted to the case $\rho = 1$. The finite element discretization of (49) leads to system (38) of order $N = 29008$ with the mass matrix $E(\mu) = E_1 + cE_2$, where $E_1 = E_2$ are symmetric positive definite matrices and the stiffness matrix $A(\mu) = A_1 + kA_2 + cvA_3$, where $A_1$ is symmetric negative definite, $A_2$ is nonsymmetric but negative semidefinite, $A_3$ is symmetric negative semidefinite, and $\mu = [c, k, v]^T$. The input matrix $B \in \mathbb{R}^N$ and the output matrix $C \in \mathbb{R}^{1\times N}$ are parameter-independent. The data can be downloaded from [1].

In this example, we want to test the error for nonsymmetric systems and the reliability of the method when applied to really large systems. The time interval is $[0, 2]$; the number of time steps and the sets $\mathcal{D}_{\text{train}}$, $\mathcal{D}_{\text{ref}}$, $\mathcal{D}_{\text{test}}$ are chosen with the same settings as in the previous example. Note, however, that each set in this example is different from its counterpart due to the difference in the dimension of the parameter domain. We run the greedy algorithm for 20 steps. The convergence histories of the greedy iteration for both (1) and (40) are shown in Figure 4(a). The relative errors and the nonsymmetric error estimates as in Theorems 5.2 and 5.3 are presented in Figure 5. The situation is quite similar to the previous example, except for the fact that the error in the reduced basis solutions of the observability PALE (40), which is not presented here, is rather large compared to that of the controllability PALE (1). The reason is most probably that the corresponding greedy iteration stagnates after the first five steps; see Figure 4(a). Together with the numerical results in section 7.1, we believe that the reduced basis approximation is better when the maximal error in the greedy search (almost) monotonously decreases. Nevertheless, the resulting errors in the frequency response as defined in (47), as shown in Figure 4(b), are small.

Now we turn our attention to computation time shown in Figure 6(a). One can see that most of the time (73%) used in the offline phase is spent on solving the Lyapunov equation at twenty different points sought by the greedy algorithm. Thanks to suitable arrangement of parameter-dependent and parameter-independent terms, the computation time of the seemingly expensive search on the training set can be almost ignored (7%). The time for computing the parameter independent terms is quite remarkable (18%). Note, however, that these terms will be stored and used in the online phase, which helps to reduce the cost. To wit, in the online phase, for the true error, we have to solve the original Lyapunov equation and the reduced Lyapunov equation at 50 points. The first task takes 849.9 sec, while the second takes only 8.48 sec, which accelerates the computation by a factor of 100.

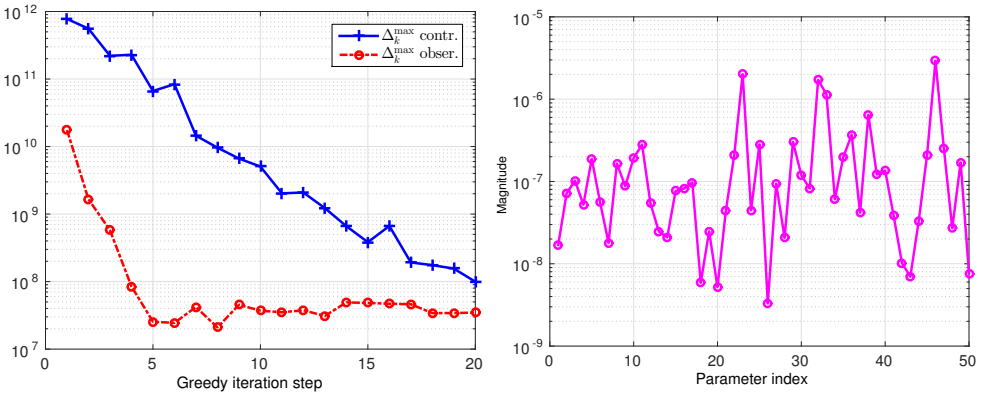


FIG. 4. *Anemometer model:* (a) *the maximal error estimate during the greedy iteration;* (b) *the absolute error in the frequency response for parametric model order reduction.*

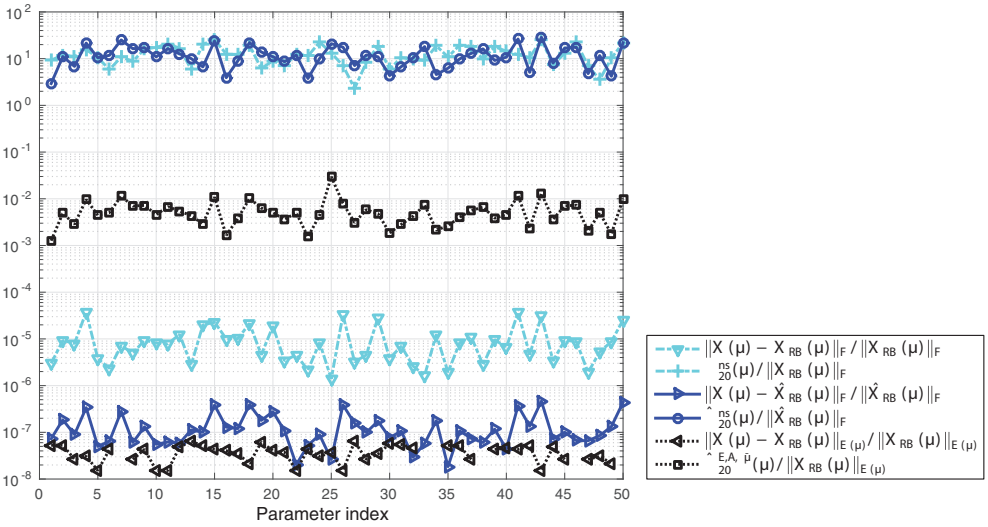


FIG. 5. *Anemometer model: relative errors and error estimates measured in three ways for the controllability PALE* (1).

Finally, in Figure 6(b), we compare the errors and the error estimates of the RBBT method and the RB method with the training input $u(t) \equiv 1$. The test input is again $u_{\text{test}}(t) = 3 - 100\cos(t)$ for both methods. A situation similar to the previous example can be observed. The offline phase takes 1506 sec in the RBBT method and 2065 sec in the RB method. The order of the reduced systems obtained by the RBBT method and the RB method is about 46 and 56, respectively, and the duration of the online phase is 0.37 sec for the RBBT method and 0.02 sec for the RB method. The overestimation in the RB method could be explained by using the primal problem based output bounds, whereas the RBBT error estimates are based on both primal and adjoint problems.

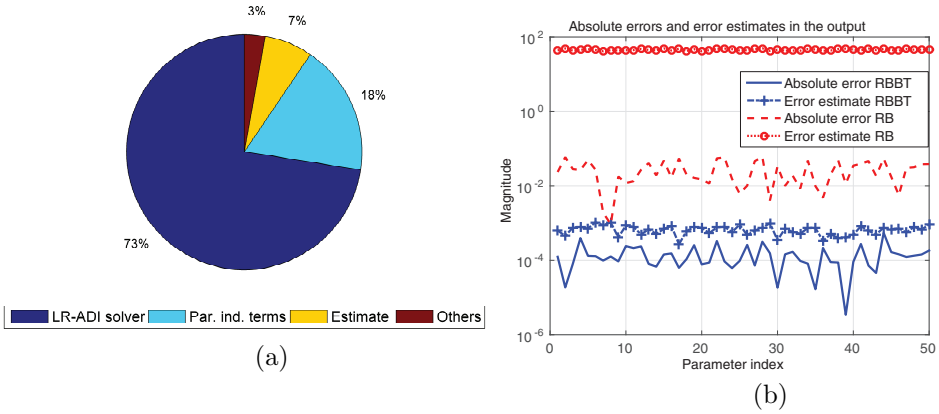


FIG. 6. *Anemometer model:* (a) *time spent for different tasks and* (b) *a comparison of the parametric balanced truncation with the reduced basis in the time domain.*

By considering two examples, we observed that practically the proposed RB method works well for both symmetric and nonsymmetric large-scale PALEs. The combination of this method with balanced truncation provides a competitive alternative to other parametric model reduction approaches such as the direct use of the RB method for dynamical systems and the interpolation based balanced truncation.

**8. Conclusion.** In this paper, we presented the RB method for solving large-scale PALEs. For deriving the error estimates for approximate solutions to PALEs with symmetric and nonsymmetric matrix coefficients depending affinely on parameters, we used the min-$\theta$ approach. The RB method was then used to extend the standard balanced truncation model reduction approach to the parametric systems which, unlike [4], does not require interpolation. Numerical examples show that, on the one hand, the proposed method worked well for large problems, but on the other hand, the error estimate can be poor, especially in the nonsymmetric case. Tightening the error estimates by using other matrix norms and/or other techniques such as the natural norm approach, the successive constraint method or their combination remains for the future work.

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