To the Theory of Degenerate Systems of Ordinary Differential Equations

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Abstract

In this paper we propose a quasi-inverse matrix. The application of it to solve the initial value problem for system of ordinary differential equations $A\dot{x}(t) = x(t) + f(t)$ with a singular constant matrix of coefficients is discussed. We give also the notion of a m-quasi-inverse matrix of A, prove the stability of this matrix to perturbations in A and describe algorithms. Numerical examples are given.

A necessity for constructing reliable algorithms for integration of systems of ordinary differential equations

$$A(t)\dot{x}(t) = B(t)x(t) + f(t) \tag{1}$$

with arbitrary coefficient matrices evokes a permanent interest in such problems for several years [**?**] - [**?**]. A recent approach to solving linear-algebraic problems which acquired the name "guaranteed accuracy" [4] offers a new view of the statements of the problems connected with degenerate systems of ordinary differential equations.

In the present article, we consider the case of a constant matrix A and $B = I$. Instead of the Drazin classical inverse matrix that is usually used for solving such systems of differential equations $[1, 2]$, we introduce a quasi-inverse matrix, using which we derive solvability conditions in the form of algebraic relations and prove a uniqueness theorem for a solution. The quasi-inverse matrix is defined as a unique solution to some matrix system of equations and can be calculated by means of the Schur orthogonal decomposition.

In studying the system of differential equations (1) with a degenerate coefficient matrix A, the problem arises of numeric determination of the dimension (and structure) of the invariant subspace of A corresponding to the zero eigenvalue. In view of limited accuracy of calculations, we have to replace the exact spectrum of A with its ε-spectrum constituted by the points λ such that $\|(A - \lambda I)^{-1}\| \geq 1/\varepsilon\|A\|$ (see, for instance, [5]). In the articles [5,], some algorithms were proposed for splitting the spectrum of a matrix into parts inside and outside the unit circle (circular dichotomy). The numeric characteristic of remoteness of the points of the spectrum of the matrix from the unit circle was taken to be the parameter ω of circular dichotomy.

These algorithms are applicable to the spectrum dichotomy problem for an arbitrary circle $|\lambda| = \rho$. If the "almost zero" eigenvalues of the matrix A are separated by some circle of a small radius ρ from the other part of the spectrum, i.e., if the value of the parameter $\omega(\rho)$ is not too large, then we can calculate the projections to the invariant subspaces corresponding to these two groups of eigenvalues with guaranteed accuracy.

Thus, we may assume the original matrix to be preliminarily processed by means of the dichotomy algorithm and to be normalized so that its spectrum lies inside the circle of radius 1 centered at the origin.

In the present article, we also consider an m-quasi-inverse matrix (whose introduction and name are evoked by analogy with an r-pseudo-inverse matrix that is used in solving systems of linear equations [5]) and prove its stability under perturbations of the matrix. To calculate the m-quasi-inverse matrix, we propose an algorithm for stable determination of the subspace corresponding to the "almost zero" eigenvalues of the matrix. This problem has been an objective of the series of articles [7–10]. The proposed algorithm is essentially a modification of the well-known algorithm by V. N. Kublanovskaya [7]. In the latter we use the construction of singular vectors of a matrix which is based on exhaustion of singular values [4]. This provides guaranteed accuracy for results and better estimates. It is in this aspect that the proposed algorithm differs from its prototype. Also, we present examples of numeric tests.

On the one hand, the present article may be regarded as devoted to the study of degenerate systems of ordinary differential equations; on the other hand, it continues research into spectral problems for asymmetric matrices.

1 A quasi-inverse matrix

A quasi-inverse matrix is defined only for a square matrix.

Definition 1 . *A quasi-inverse matrix of a matrix A is defined to be a matrix $A^{\#}$ satisfying the system of the equations*

$$\begin{array}{ll} 1. & A^{\#}AA^{\#} = A^{\#}, \\ 2. & (AA^{\#})^{*} = AA^{\#}, \\ 3. & A^{\#}A^{l+1} = A^{l}, \\ 4. & A^{l+1}(A^{\#})^{l+1} = AA^{\#}, \end{array} \tag{2}$$

where l is the index of A, i.e. the least nonnegative integer such that rank A^{l+1} = rank A^{l}.

It is easy to prove that equalities (2) are independent, i.e., none of them is derivable from the others. To this end, it suffices to indicate two matrices that satisfy any three of the equalities but not the fourth.

Note that for $l = 0$ the matrix A is nondegenerate. In this case a quasi-inverse matrix coincides with the conventional inverse matrix: $A^{\#} = A^{-1}$.

The correctness of the definition of quasi-inverse matrix is guaranteed by the following theorem:

Theorem 1 . *Every matrix A has a quasi-inverse matrix which is moreover unique.*

Proof. Represent the matrix A in Schur block triangular form:

$$A = U \begin{pmatrix} C & D \\ 0 & B \end{pmatrix} U^{*} \tag{3}$$

where U is an orthogonal transformation, C is a nondegenerate matrix and B is an upper triangular matrix with each diagonal entry zero. It is easy to verify that the matrix

$$A^{\#} = U \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^{*} \tag{4}$$

satisfies all equalities in (2). Thereby the existence of a quasi-inverse matrix is proven.

Prove that the matrix A cannot have two different quasi-inverse matrices $A_1^{\#}$ and $A_2^{\#}$. Let $R = A_1^{\#} - A_2^{\#}$. The third equality in (**??**) yields

$$RA^{l+1} = 0, \tag{5}$$

and the second and fourth yield $(AR)^{*} = AR, \quad RAR = 0$. Consequently,

$$(AR)^{*}AR = ARAR = 0.$$

Hence, we obtain $AR = 0$ or, alternatively,

$$AA_1^{\#} = AA_2^{\#}. \qquad (\ )$$

Now, we make use of the first equality in (2). Involving (5), ( ), and the fourth equality in (2), we infer

$$\begin{aligned} A_1^{\#} &= A_1^{\#}AA_1^{\#} = A_1^{\#}AA_2^{\#} = A_1^{\#}A^{l+1}(A_2^{\#})^{l+1} = \\ &= A_2^{\#}A^{l+1}(A_2^{\#})^{l+1} = A_2^{\#}AA_2^{\#} = A_2^{\#}. \end{aligned}$$

The theorem is proven.

As a consequence of the theorem, we obtain representability of the quasi-inverse matrix $A^{\#}$ in the form (4).

Note that the quasi-inverse matrix belongs to none of the familiar classes of generalized inverse matrices (see, for instance, [2]). In particular, the matrix $A^{\#}$ is not the Drazin inverse matrix. Indeed, the Drazin inverse matrix A^D of A is defined as the only solution to the matrix system of the equations

$$AX = XA, \qquad XAX = X, \qquad XA^{l+1} = A^{l}, \tag{7}$$

where l is the index of A [**?**]. It is easy to verify that $A^{\#}$ fails to satisfy the first relation in (7).

Demonstrate that the quasi-inverse matrix can be defined by means of an orthogonal projection. To this end, consider the system of the equations

$$1. AX = P, \qquad 2. P^{*} = P, \qquad 3. XP = X, \qquad 4. XP = PX, \qquad 5. APX = XAP. \tag{8}$$

Theorem 2 . *The system of matrix equations (8) is solvable for P and X. Moreover, for a fixed matrix P, the matrix X is determined uniquely.*

Proof. The first and third equalities in (8) yield

$$P^2 = AXP = AX = P,$$

i.e. P is a projection. The second equality in (8) implies that P is an orthogonal projection. Then there exists an orthogonal matrix V such that

$$P = V \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

where $N_1 = \text{rank } P$, I_{N_1} is the identity matrix of order N_1. Denote

$$V^*AV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \text{and} \qquad V^*XV = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Show that $X_{12} = X_{21} = X_{22} = 0$. To this end, it suffices to use the third and fourth identities in (8) together with the equalities

$$V^*XVV^*PV = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix},$$

$$V^*PVV^*XV = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix}.$$

It is easy to demonstrate that the first equality in (8) yields $X_{11} = A_{11}^{-1}$ and $A_{21} = 0$. Indeed,

$$V^*AVV^*XV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11}X_{11} & 0 \\ A_{21}X_{11} & 0 \end{pmatrix} = V^*PV = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $A_{21}X_{11} = 0$ and $A_{11}X_{11} = I$. The last equality implies that the matrices A_{11} and X_{11} are nondegenerate.

Thus,

$$A = V \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} V^*, \quad X = V \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} V^*, \quad P = V \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V^*.$$

It is easy to verify that the so-obtained matrices X and P satisfy the system of equations (8). Thus, the system is solvable with respect to X and P. Show that, given A and some fixed orthogonal projection P, the matrix X in (8) is determined uniquely.

Suppose that there are two different matrices X_1 and X_2 satisfying (8). The first, third, and fourth equalities in (8) yield

$$APX_1 = AX_1P = P = AX_2P = APX_2,$$

$$X_i = X_iP = X_iAX_iP = X_iAPX_i \quad i = 1, 2.$$

Furthermore, using these equalities together with the fifth equality in (8), we obtain

$$X_1 = X_1APX_1 = X_1APX_2 = APX_1X_2 = APX_2X_2 = X_2APX_2 = X_2.$$

The theorem is proven.

Remark 1. If we take as P the orthogonal projection to the maximal invariant subspace corresponding to the nonzero eigenvalues of the matrix A then it is easy to verify that the matrix $A^{\#}$ determined by equality (4) satisfies the system of equations (8). In view of uniqueness of a solution to (8), we can say that the matrix systems of equations (2) and (8) are equivalent and can define the quasi-inverse matrix $A^{\#}$ as a unique solution to the system

$$AA^{\#} = P, \quad A^{\#}P = A^{\#}, \quad A^{\#}P = PA^{\#}, \quad APA^{\#} = A^{\#}AP. \tag{9}$$

2 Application of the quasi-inverse matrix

In this section we show that the quasi-inverse matrix can be used for solution of the initial value problem

$$\begin{cases} A\dot{x}(t) &= x(t) + f(t), \\ x(a) &= x_0, \end{cases} \tag{10}$$

where A is a singular $(N \times N)$-matrix, $f(t) \in C^{l+1}$ (l is the index of A).

Let k be a multiplicity of zero eigenvalue of the matrix A. We present A in the block triangular Schur form (**??**). Denote

$$U^* x(t) = y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

where $y_1(t)$ and $y_2(t)$ are the vectors of dimensions $(N-k)$ and k, respectively. Then the equations of system (**??**) can be rewritten in the form

$$\begin{cases} C\dot{y}_1(t) + D\dot{y}_2(t) &= y_1(t) + g_1(t), \\ B\dot{y}_2(t) &= y_2(t) + g_2(t), \end{cases} \tag{11}$$

where $U^* f(t) = g(t)$. Let $y_2(t)$ be the solution of the second equation of (**??**). Then the following identities hold:

$$\begin{array}{rcl} Ty_2(t) &=& y_2(t) + g_2(t), \\ T^2 y_2(t) &=& Ty_2(t) + Tg_2(t), \\ & \dots & \\ T^s y_2(t) &=& T^{s-1} y_2(t) + T^{s-1} g_2(t), \end{array} \tag{12}$$

where T is the operator whose action on a function $\varphi(t)$ is determined by the formula $T\varphi(t) = B\dot{\varphi}(t)$ and T^s is the sth power of the operator. Taking the sum of equalities (12), we find

$$T^s y_2(t) = y_2(t) + \sum_{i=0}^{s-1} T^i g_2(t).$$

Since $B^s = 0$ for $s \geq l$, the solution $y_2(t)$ to system (11) satisfies the relation

$$y_2(t) = -\sum_{i=0}^{l-1} B^i g_2^{(i)}(t). \tag{13}$$

In view of the nondegeneracy of the matrix C in the first equation of system (11), we can now calculate the vector $y_1(t)$.

Denote

$$\varphi(t) = (AA^{\#} - I)\sum_{i=0}^{l} A^i f^{(i)}(t). \tag{14}$$

It is easy to demonstrate that a solution to system (10) satisfies the algebraic equality

$$(I - AA^{\#})x(t) = \varphi(t). \tag{15}$$

To this end, it suffices to make use of representations (3) and (4) and equality (13). Afterwards, solving the Cauchy problem (10), from (15) we, by necessity, obtain the following condition on the initial data:

$$(I - AA^{\#})x_0 = \varphi(a). \qquad (1\ \ )$$

Remark 2. In equality (14) the index l of the matrix A can be replaced with an arbitrary integer $N \geq s > l$, since the index of the matrix is, as a rule, unknown. In this event, the smoothness requirement on the function $f(t)$ becomes more stringent.

Remark 3. When equality (1) is satisfied, the solutions y_1 and y_2 to system (11) with the corresponding initial data are determined uniquely. This allows us to formulate the following theorem.

Theorem 3 . *If condition (16) is satisfied then there exists a unique solution to Cauchy problem (10).*

Demonstrate that solving problem (10) can be reduced to solving the system that is solved with respect to the derivatives,

$$\begin{cases} \dot{x}(t) &= A^{\#}x(t) + (I - A^{\#}A)\dot{\varphi}(t) + A^{\#}f(t), \\ x(a) &= x_0. \end{cases} \tag{17}$$

Theorem 4 . *If condition (16) is satisfied then a solution to Cauchy problem (17) is a solution to Cauchy problem (10).*

Proof. As a preliminary, we observe the easy equalities

$$A^{\#}A(I - A^{\#}A) = 0, \tag{18}$$

$$(I - AA^{\#})(I - A^{\#}A) = I - AA^{\#}, \tag{19}$$

$$(I - AA^{\#})^2 = I - AA^{\#}, \tag{20}$$

$$(I - A^{\#}A)(I - AA^{\#}) = I - A^{\#}A. \tag{21}$$

Now, show that if condition (1) is satisfied then a solution to Cauchy problem (17) satisfies algebraic equation (15). Indeed, multiplying (17) by the matrix $A^{\#}A$ and using (18) and the first equality in (2), we obtain

$$A^{\#}A\dot{x}(t) = A^{\#}x(t) + A^{\#}f(t). \tag{22}$$

Afterwards, we subtract (22) from (17) and multiply the result by the matrix $(I - AA^{\#})$. As a result of using (19) and (20), we arrive at the equality

$$(I - AA^{\#})\dot{x}(t) = \dot{\varphi}(t).$$

Denoting $u(t) = (I - AA^{\#})x(t) - \varphi(t)$, we obtain $\dot{u}(t) = 0$. By condition (1), we have $u(t) \equiv 0$. Consequently, equality (15) is satisfied.

Now, substitute the derivative $\dot{x}(t)$ given by (17) into equation (10):

$$A\dot{x}(t) = AA^{\#}x(t) + A(I - A^{\#}A)\dot{\varphi}(t) + AA^{\#}f(t). \tag{23}$$

Using the third property of the matrix $A^{\#}$ in (2) and equalities (14), (20), and (21), we infer

$$\begin{aligned} A(I - A^{\#}A)\dot{\varphi}(t) &= -(I - AA^{\#})\sum_{i=0}^{l} A^{i+1}f^{(i+1)}(t) = \\ &= -(I - AA^{\#})(\sum_{i=0}^{l} A^{i}f^{(i)}(t) + A^{l+1}f^{(l+1)}(t) - f(t)) = \\ &= \varphi(t) + (I - AA^{\#})f(t). \end{aligned}$$

Taking (15) into account, we continue equality (23) as follows:

$$A\dot{x}(t) = AA^{\#}x(t) + \varphi(t) + (I - AA^{\#})f(t) + AA^{\#}f(t) = x(t) + f(t).$$

The theorem is proven.

Thus, to solve system (**??**), with a degenerate matrix a coefficient of the derivatives, it suffices to know the quasi-inverse matrix.

3 An Algorithm for Reducing a Matrix to Schur Block Triangular Form

At the first step the calculation of the quasi-inverse matrix involves the algorithm for reducing the original matrix to block triangular form.

In practice, the matrix A is often given with errors. Therefore, it may occur that the matrix has no zero eigenvalues but is ill-conditioned. The problem arises of determining a perturbation matrix E with possibly least norm and such that

$$A + E = U \begin{pmatrix} C & D \\ 0 & B \end{pmatrix} U^{*}.$$

The algorithm for reducing an ill-conditioned matrix to block triangular form by means of additional perturbations can be represented as follows:

Fix a small nonnegative number σ^{*}, this number playing the role of "significance level." Split the singular spectrum of the matrix A into the subset of "zero" singular values (i.e., either exactly equal to zero or positive but small in magnitude) and the subset of the remaining singular values. The meaning of σ^* is that the singular values of the matrix satisfying the condition $\sigma_j < \sigma^*$ are considered as "zero."

Let m_0 denote the number of "zero" singular values of A. At the first step, we calculate the left singular vectors that correspond to the singular values $\sigma_1(A), \ldots, \sigma_{m_0}(A)$. Without loss of generality we may assume that they are normalized. Complement this orthonormal system of vectors to an orthogonal matrix $U_1 = [V_1, V_2]$, where the columns of the $(N \times m_0)$-matrix V_2 are the calculated left singular vectors.

It is easy to understand that the entries of the last m_0 rows of the matrix $F_1 = U_1^* A U_1$ have the form

$$f_{\;-m_0+i,j} = \sigma_i(A) u_i^* v_j, \quad 1 \le i \le m_0,\ 1 \le j \le N,$$

where u_i is the right singular vector of A corresponding to the singular value $\sigma_i(A)$ and v_j is the jth column of the matrix U_1. The elements are small in magnitude, because they satisfy the inequality

$$|\, f_{\;-m_0+i,j} \,| < \sigma^*, \qquad 1 \le i \le m_0, \quad 1 \le j \le N.$$

Introduce a perturbation by replacing the elements with zeros. As a result of this, we obtain a matrix whose block structure can be represented as

$$F_1^0 = \begin{pmatrix} A_1 & D_1 \\ 0 & 0 \end{pmatrix},$$

where A_1 is an $(N - m_0) \times (N - m_0)$-matrix.

At the next step, denote the number of "zero" singular values of the matrix A_1 by $m_1 - m_0$. Given the left singular vectors corresponding to the "zero" singular values of A_1, construct an orthogonal transformation $U^{(2)}$ as above. Define

$$U_2 = \begin{pmatrix} U^{(2)} & 0 \\ 0 & I_{m_0} \end{pmatrix}.$$

Nullify the first $N - m_0$ elements of the ith row of $F_2 = U_2^* F_1^0 U_2$, where i ranges from $N - m_1 + 1$ to $N - m_0$. We thus obtain

$$F_2^0 = \begin{pmatrix} A_2 & D_2 \\ 0 & B_2 \end{pmatrix},$$

where A_2 is an $(N-m_1)\times(N-m_1)$-matrix and B_2 is an upper triangular $(m_1 \times m_1)$-matrix with zero diagonal entries.

Continue the process, each time considering matrices of decreasing orders, until at some step h we obtain a matrix

$$F_h^0 = \begin{pmatrix} A_h & D_h \\ 0 & B_h \end{pmatrix},$$

in which the matrix A_h has no "zero" singular values. The so-obtained matrix B_h is obviously nilpotent.

The result of the above transformations has the form

$$\begin{pmatrix} C & D \\ 0 & B \end{pmatrix} = U^*(A+E)U,$$

where $C = A_h$, $D = D_h$, $B = B_h$, $U = U_1 \cdot \ldots \cdot U_h$ is an orthogonal transformation, and E is the matrix of additional perturbations caused by nullifying some elements.

Observe that in the case when the matrix A has the exactly zero eigenvalue the above-described algorithm presents a constructive proof of Schur's theorem [11] restated in terms of singular vectors.

It is easily seen that the matrix V_1 obtained at the first step of the algorithm is a basis for the subspace spanned by the left singular vectors of A corresponding to the "nonzero" singular values. Using the definition of the orthogonal projection to the corresponding subspace of the matrix A [4], we can write down $P^{(0)} = V_1 V_1^*$. Straightforward verification easily yields validity of the following chain of equalities:

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \left[\frac{V_1^*}{0} \right] A \, [\, V_1 \mid 0 \,] = U_1^* P^{(0)} A P^{(0)} U_1.$$

Arguing in a similar way, we can represent the described algorithm in terms of projections by the formulas

$$\begin{aligned} A^{(0)} &:= A, \\ A^{(i+1)} &:= P^{(i)} A^{(i)} P^{(i)}, \quad i = 0, 1, \ldots, h-1, \end{aligned} \tag{24}$$

where

$$A^{(i)} = U_1 \ldots U_{i-1} U_i \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} U_i^* U_{i-1}^* \ldots U_1^*, \tag{25}$$

and $P^{(i)}$ is the orthogonal projection to the linear span of the left singular vectors corresponding to the "nonzero" singular values of $A^{(i)}$.

Thus, we have isolated a nilpotent block of the perturbed matrix $A+E$. In this event we say that the matrix A possesses a quasinilpotent block.

To study stability of the algorithm, we introduce the following

Definition 2 . *Parameters δ and α correctly determine a quasinilpotent block of A of order k if the inequalities*

$$\sigma_{m_i}(A^{(i)}) \le \delta < \alpha \le \sigma_{m_i+1}(A^{(i)}), \quad i = 0, 1, \ldots, h \tag{2 }$$

hold for some collection $m = (m_0, m_1, \ldots, m_h)$.

Consider the sequence of natural numbers

$$m_0 < m_1 < m_2 < \ldots < m_{h-1} = m_h = k,$$

with m_i standing for the number of the "zero" singular values of the matrix $A^{(i)}$ with the first m_{i-1} values being exactly equal to zero. Put

$$\delta = \max_{0 \le i \le h-1} \sigma_{m_i}(A^{(i)}), \qquad \alpha = \min_{0 \le i \le h} \sigma_{m_i+1}(A^{(i)}).$$

It is easy to see that these parameters correctly determine a quasinilpotent block of A.

Remark 4. For the block A_h of the matrix orthogonally similar to $A^{(h)}$ (see (25)), we have $\sigma_1(A_h) = \sigma_{m_h+1}(A^{(h)}) \geq \alpha$. Therefore, while choosing the collection m that governs the parameters δ and α when we talk about the correct determination of a quasinilpotent block, it is natural to require that the block A_h be not too ill-conditioned.

We prove the following continuity theorem.

Theorem 5 *Suppose that parameters δ and α correctly determine a quasinilpotent block of order k of a matrix A. Then for every sufficiently small ε there are parameters $\widetilde{\delta} = \widetilde{\delta}(\varepsilon)$ and $\widetilde{\alpha} = \widetilde{\alpha}(\varepsilon)$ that correctly determine a quasinilpotent block of order k for an arbitrary matrix $\widetilde{A}$ with $\|\widetilde{A} - A\| \leq \varepsilon \|A\|$.*

Proof. Take a small positive number ε. Suppose that matrices A and $\widetilde{A}$ are close so that

$$\|\widetilde{A} - A\| \leq \varepsilon \|A\|. \tag{27}$$

Using the continuity estimate

$$|\, \sigma_i(\widetilde{A}) - \sigma_i(A)| \leq \|\widetilde{A} - A\|$$

for the singular values [4] and inequalities (2), we obtain

$$\sigma_{m_i}(\widetilde{A}^{(i)}) \leq \sigma_{m_i}(A^{(i)}) + \|\widetilde{A}^{(i)} - A^{(i)}\| \leq \delta + \|\widetilde{A}^{(i)} - A^{(i)}\|,$$

$$\sigma_{m_i+1}(\widetilde{A}^{(i)}) \geq \sigma_{m_i+1}(A^{(i)}) - \|\widetilde{A}^{(i)} - A^{(i)}\| \geq \alpha - \|\widetilde{A}^{(i)} - A^{(i)}\|.$$

Estimate the norm $\|\widetilde{A}^{(i)} - A^{(i)}\|$. It is easy to verify that

$$\begin{aligned} \widetilde{A}^{(i)} - A^{(i)} &= \widetilde{P}^{(i-1)}(\widetilde{A}^{(i-1)} - A^{(i-1)})\widetilde{P}^{(i-1)} + \widetilde{P}^{(i-1)}A^{(i-1)}(\widetilde{P}^{(i-1)} - P^{(i-1)}) + \\ &+ (\widetilde{P}^{(i-1)} - P^{(i-1)})A^{(i-1)}P^{(i-1)}. \end{aligned}$$

Using the properties of a norm and the fact that the norm of an orthogonal projection equals 1, we obtain

$$\|A^{(i)}\| \leq \|A\|,$$

$$\|\widetilde{A}^{(i)} - A^{(i)}\| \leq \|\widetilde{A}^{(i-1)} - A^{(i-1)}\| + 2\|A\|\|\widetilde{P}^{(i-1)} - P^{(i-1)}\|. \tag{28}$$

By (**??**) and (**??**)

$$\|\widetilde{A}^{(0)} - A^{(0)}\| \leq \varepsilon\|A\| = \varepsilon_0\|A\|.$$

Furthermore, applying inequality (28), we obtain

$$\|\widetilde{A}^{(1)} - A^{(1)}\| \leq \varepsilon_0\|A\| + 2\|A\|\|\widetilde{P}^{(0)} - P^{(0)}\|. \tag{29}$$

For the projections $\widetilde{P}^{(0)}$ and $P^{(0)}$ to the linear spans of the left singular vectors of the matrices $\widetilde{A}^{(0)}$ and $A^{(0)}$ corresponding to the last $N - m_0$ singular values we now use the estimate [4]

$$\|\widetilde{P}^{(0)} - P^{(0)}\| \leq \frac{\varepsilon_0 d_0}{1 - \varepsilon_0 d_0},$$

where $d_0 = \|A\|/(\sigma_{m_0+1}(A^{(0)}) - \sigma_{m_0}(A^{(0)}))$ is the "clearance" between the singular values $\sigma_{m_0}(A^{(0)})$ and $\sigma_{m_0+1}(A^{(0)})$. The estimate holds under the condition

$$2\varepsilon_0 d_0 < 1, \tag{30}$$

which prohibits the singular values of $\widetilde{A}^{(0)}$ from "sticking together."

If we denote $d = \|A\|/(\alpha - \delta)$ then (2) obviously yields the estimate $d_0 \leq d$. Now, we rewrite condition (30) as $2\varepsilon_0 d < 1$. Continuing inequality (29), we obtain

$$\|\widetilde{A}^{(1)} - A^{(1)}\| \leq \varepsilon_0 \left(1 + \frac{2d}{1 - \varepsilon_0 d}\right) \|A\| = \varepsilon_1 \|A\|,$$

where $\varepsilon_1 = \varepsilon_0(1 + 2d/(1 - \varepsilon_0 d))$. In the general case, for $2\varepsilon_{i-1} d < 1$ we similarly obtain

$$\|\widetilde{A}^{(i)} - A^{(i)}\| \leq \varepsilon_i \|A\|,$$

where $\varepsilon_i = \varepsilon_{i-1}(1 + 2d/(1 - \varepsilon_{i-1} d))$.

It is immediate from the condition $2\varepsilon_{i-1} d < 1$ that $\varepsilon_i < \varepsilon_{i-1}(1 + 4d)$. Consequently, $\varepsilon_i < \varepsilon_0(1 + 4d)^i = \varepsilon(1 + 4d)^i$. Thus, we arrive at the estimate

$$\|\widetilde{A}^{(i)} - A^{(i)}\| < \varepsilon(1 + 4d)^i \|A\|, \tag{31}$$

which holds for $2\varepsilon(1 + 4d)^{i-1} d < 1$.

Returning now to the beginning of the proof, we conclude that

$$\sigma_{m_i}(\widetilde{A}^{(i)}) < \delta + \varepsilon(1 + 4d)^i \|A\|, \qquad \sigma_{m_i+1}(\widetilde{A}^{(i)}) > \alpha - \varepsilon(1 + 4d)^i \|A\|.$$

Denote

$$\widetilde{\delta} = \delta + \max_{0 \leq i \leq h-1} \varepsilon(1 + 4d)^i \|A\| = \delta + \varepsilon(1 + 4d)^{h-1} \|A\|, \tag{32}$$

$$\widetilde{\alpha} = \alpha - \max_{0 \leq i \leq h} \varepsilon(1 + 4d)^i \|A\| = \alpha - \varepsilon(1 + 4d)^h \|A\|. \tag{33}$$

In this case, it is obvious that the parameters $\widetilde{\delta}$ and $\widetilde{\alpha}$ correctly determine a quasinilpotent block of $\widetilde{A}$; moreover, the condition $\widetilde{\delta} < \widetilde{\alpha}$ is guaranteed by the inequality

$$\varepsilon < \frac{1}{2d(1 + 4d)^{h-1}(1 + 2d)} \tag{34}$$

that is trivially derived from (32) and (33). The theorem is proven.

As was mentioned in the introduction, the considered algorithm for reducing a matrix to block triangular form by means of the left singular vectors is a modification of the Kublanovskaya algorithm for constructing an orthogonal basis for the subspace generated by the zero eigenvalue [7]. We will compare the estimate obtained in [8] for the norm of the additional perturbation matrix E needed for calculating a quasinilpotent block of the original matrix in the Kublanovskaya algorithm with the analogous estimate in our algorithm. To begin with, we give some necessary definitions and basic inequalities of [8].

Definition 3 . *Suppose that a matrix A is normalized by the condition $\|A\| = 1$. Then $\lambda = 0$ is the numerically multiple eigenvalue of A with characteristics $m_1, \ldots, m_h$ if the inequalities*

$$\sigma_{m_i}(A^i) \le \xi < \eta \le \sigma_{m_i+1}(A^i), \quad i = 1, \ldots, h,$$

hold for some ξ and η.

The Kublanovskaja algorithm can be represented in terms of orthogonal projections as follows [8, formula (5.3)]:

$$\begin{aligned} A^{(1)} &:= A, \\ A^{(i+1)} &:= P^{(i)} A P^{(i)}, \quad i = 1, \ldots, h, \end{aligned}$$

where $P^{(i)}$ is the orthogonal projection to the range of the matrix $(A - E^{(i)})^i$, $E^{(i)}$ being the matrix of the perturbation caused by nullifying the corresponding small elements in i steps of the algorithm. The matrix $E^{(h)}$ satisfies estimate (5.13) of [8]:

$$\|E^{(h)}\| \le 5(2.12)^{h-1} h! \eta^{-h} \xi = \gamma,$$

which holds under the condition $h\gamma < 0.1$.

We obtain an estimate for the norm of the additional perturbation matrix E in our algorithm. Denote by $\overline{E}^{(i)}$ the perturbation that is caused by nullifying the corresponding elements at the $(i+1)$th step of the algorithm. Then

$$\|E\| \le \sum_{i=0}^{h-1} \|\overline{E}^{(i)}\| \le \sum_{i=0}^{h-1} \sigma_{m_i}(A^{(i)}) \le h\delta. \tag{35}$$

If we put $m_i = N - A^i$ for the matrix A having an exactly zero eigenvalue then we obtain $\|E\| = 0$ in both the algorithms. Moreover, we obviously have $\delta = \xi = 0$, $\alpha > 0$, and $\eta > 0$. Furthermore, involving the estimate

$$\sigma_j(A^{(i+1)}) \ge \sigma_j(A^{i+1}) - 2((1 + \|E^{(i)}\|)^i - 1 + 2\sigma_{m_i}(A^i))\sigma_{m_i+1}^{-1}(A^i),$$

(see [8, inequality (5.11)]) and recalling that $\|E^{(i)}\| = 0$ and $\sigma_{m_i}(A^i) = 0$, we can easily infer

$$\alpha \ge \eta. \qquad (3\ \ )$$

Now, suppose that $\tilde{A} = A + X$, where $\|X\| \le \varepsilon$. In this case (see [8], formulas (3.4) and (3.5)) we have

$$\tilde{\xi} = 1.0\ \ h\varepsilon, \qquad \tilde{\eta} = \eta - 1.0\ \ h\varepsilon.$$

Consequently, to reconstruct the nilpotent block by using the Kublanovskaya algorithm, we must additionally introduce a perturbation $\tilde{E}$ in the matrix $\tilde{A}$ with the estimate

$$\|\tilde{E}\| \le 5.3(2.12)^{h-1} h! H(\eta - 1.0\ \ h\varepsilon)^{-h} \varepsilon = \beta_1.$$

In our algorithm, by (32) we have $\tilde{\delta} \le \varepsilon 5^{h-1} \alpha^{-(h-1)}$, since $\alpha \le 1$. Applying inequality (3), from (35) we now derive

$$\|\tilde{E}\| \le 5^{h-1} h \eta^{-(h-1)} \varepsilon = \beta_2.$$

Below we present the ratios of the estimates for the norm of the perturbation matrix $\widetilde{E}$ in the Kublanovskaya algorithm (β_1) and the proposed algorithm (β_2) which are obtained at various values of h; the calculations were carried out for fixed $\eta = 1$:

Table 1.

h	2	3	4	5		7	8	9	10
β_1/β_2	4.494	5.717	9. 9	20.55	52.29	155.20	52 .45	2008.95	8517.97

In [10], there was proposed a more economic algorithm for calculating the nilpotent block of a degenerate matrix. In the case when the matrix in question has only the zero eigenvalue and all its eigenvectors are linearly independent, the algorithm has the complexity of $O(N^3)$ arithmetic operations, whereas the proposed algorithm, $O(N^4)$. The difference becomes essential for higher-order matrices. However, we confine ourselves to considering matrices with a nilpotent block of small order, since (as is seen from inequality (34)) we cannot otherwise guarantee accuracy for the result.

4 An m- uasi-Inverse Matrix

The calculation of a quasi-inverse matrix might encounter difficulties due to the extreme sensitivity of the eigenvalues of a matrix to perturbations of its entries and the fact that the nondegenerate block in the Schur orthogonal decomposition could be ill-conditioned. The latter means that not all "almost zero" eigenvalues are eliminated. To overcome the difficulties, we introduce the definition of an m-quasi-inverse matrix $A^{\#}_m$. To this end, we use the concept of a pseudo-inverse matrix A^+ (see, for instance, [12]). It is worth noting that $A^{\#}_m$ is defined if a collection $m = (m_0, \dots, m_{h-1}, m_h)$ correctly determines a quasinilpotent block of A, i.e., if inequality (2) holds for some δ and α and Remark 4 (§ 3) is valid.

Definition 4 . *Given an arbitrary matrix A, the m-quasi-inverse matrix is defined to be the matrix*

$$A^{\#}_m = (A^{(h)})^+. \tag{37}$$

It is easy to see that $A^{\#}_m$ is a quasi-inverse matrix of $A+E$ and is determined uniquely for the given collection m.

We obtain an estimate for the continuous dependence of the m-quasi-inverse matrix on perturbations of the matrix A. Suppose that $\|\widetilde{A} - A\| \leq \varepsilon\|A\|$. Inequalities (31) yield

$$\|\widetilde{A}^{(h)} - A^{(h)}\| < \varepsilon(1+4d)^h\|A\|.$$

Using an error estimate for the pseudoinverse matrix [12], we obtain (for $\varepsilon(1+4d)^h\|A\|\|A^{\#}_m\| < 1$)

$$\|\widetilde{A}^{\#}_m - A^{\#}_m\| \leq \frac{2\varepsilon(1+4d)^h\|A\|\|A^{\#}_m\|^2}{1-\varepsilon(1+4d)^h\|A\|\|A^{\#}_m\|}.$$

Hence, we can conclude that the m-quasi-inverse matrix $A^{\#}_m$ is stably determined if the quantity $d = \|A\|/(\alpha - \delta)$ is not too large.

We turn to describing the algorithm for calculating the m-quasi-inverse matrix.

Step 1. Reduction of the matrix A to Schur block triangular form. As a result of it, we obtain some matrix

$$F = \begin{pmatrix} C & D \\ 0 & B \end{pmatrix}$$

and an orthogonal transformation U.

Step 2. Calculation of the inverse matrix of C. Observe that this procedure is always realizable if in reducing the matrix to block triangular form the value of the parameter σ^* (see § 3) is taken to be the least level of the minimal singular value above which the inverse matrix of C can be calculated with admissible accuracy.

Step 3. Inverse transformations. Without counting the round-off errors, we obtain

$$A_m^{\#} = U \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

The above algorithms for reducing a matrix to block triangular form and calculating the m-quasi-inverse matrix admit thorough stability analysis of round-off errors. Unfortunately, the analysis requires quite laborious arguments which cannot be exhibited.

5 Numeric Examples

We expose some numeric tests of reducing a matrix to block triangular form. All tests were carried out on an IBM PC AT with double precision.

Example 1. Consider the following matrix of sixth order:

$$\begin{bmatrix} 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where ε is a small parameter. The following table represents the values of the basic parameters δ, α, and d, the Euclidean norm of the additional perturbation matrix $\|E\|_E$, and the number h of steps of the algorithm at various values of ε (the calculations were carried out with $\sigma^* = 10^{-7}$):

Table 2.

ε	0	10^{-10}	10^{-8}	$5 \cdot 10^{-8}$	10^{-7}
δ	$7.11 \cdot 10^{-15}$	10^{-10}	10^{-8}	$5 \cdot 10^{-8}$	0
α	100	100	100	100	10^{-7}
d	1	1	1	1	10^9
$\|E\|_E$	4. $\cdot 10^{-15}$	10^{-10}	10^{-8}	$3.91 \cdot 10^{-8}$	0
h					0

From the table we can see the influence of the value of the parameter d on the quality of the numeric results.

Example 2. Consider the bidiagonal matrix

$$A = \begin{bmatrix} d_1 & b_2 & & & & \\ & d_2 & b_3 & & O & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & O & & & d_{\ -1} & b \\ & & & & & d \end{bmatrix},$$

where $N = 20$,

$$d_j = \begin{cases} 0.01(j-1), & 1 \le j \le \ , \\ 0.0\ \ + 0.07(j - \ ), & 7 \le j \le 15, \\ 0.92 + 0.02(j - 1\ ), & 1\ \ \le j \le 20, \end{cases} \qquad b_j = \begin{cases} 0.5, & 2 \le j \le 7, \\ 0.05j, & 8 \le j \le 1\ , \\ 0.5, & 17 \le j \le 20. \end{cases}$$

Since the matrix is upper triangular, its eigenvalues are the diagonal elements. We isolate a quasinilpotent block of A of fourth order and obtain the following values of the parameters:

$$\delta = 8.453 \cdot 10^{-6}, \qquad \alpha = 1.8\ \ \cdot 10^{-4}, \qquad d = 80\ \ .7, \qquad \|E\|_E = 8.455 \cdot 10^{-6}.$$

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