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Input-Output Invariants for Descriptor Systems

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Abstract

We study continuous-time and discrete-time descriptor systems from the time domain and frequency domain points of view. We present some input-output invariants for descriptor systems like transfer function, impulse and frequency responses, convolution and Hankel operators, Hankel singular values. These invariants are of great importance in robust control and approximation theory. Some norms for descriptor systems are introduced and their representations via the different input-output invariants are given.

Key words. descriptor system, transfer function, impulse response, frequency response, controllability and observability Gramians, convolution operator, Hankel operator, Hankel singular values, system norm.

AMS subject classification. 30H05, 34A09, 46J15, 47A30, 47B35, 93B, 93C, 93D20,

1 Introduction

Consider a linear time-invariant descriptor system

$$\begin{array}{rcl} E(\mathcal{D}x(t)) & = & Ax(t) + Bu(t), \quad x(0) = x^0, \\ y(t) & = & Cx(t) + Du(t), \end{array} \tag{1.1}$$

where $\mathcal{D}x(t) = \dot{x}(t)$, $t \in \mathbb{R}$, in the continuous-time case and $\mathcal{D}x(t) = x_{t+1}$, $t \in \mathbb{Z}$, in the discrete-time case. Here E, $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output and $x^0 \in \mathbb{R}^n$ is the initial value.

For $E = I$, system (1.1) is *standard state space system.* Such a system has been extensively studied for a long time, see [11, 15, 35] and the references therein. Descriptor systems (or *generalized state space systems*) with singular E arise naturally in many applications [6, 8, 22] and have been investigated in [5, 6, 8, 19, 20, 23, 32, 33].

The main goal of this paper is to analyze linear time-invariant descriptor systems from the time domain and frequency domain viewpoints. We consider some important linear system concepts for such systems including fundamental solution matrices, state transition matrices,

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controllability and observability, stability, transfer functions and realizations. These concepts are essential for system analysis and design. The continuous-time and discrete-time case are treated in parallel. We present generalizations for descriptor systems of impulse and frequency responses, controllability and observability Gramians, convolution operators, Hankel operators and closely related Hankel singular values. The Gramians and the Hankel singular values play an important role in model reduction via balanced truncation methods [11, 24, 29]. System norms for (1.1) are also introduced and their features are studied.

We will assume without loss of generality that $D = 0$ in (1.1). If $D \neq 0$, then we may consider an extended descriptor system

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} (\mathcal{D}\xi(t)) &= \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \xi(t) + \begin{bmatrix} B \\ D_2 \end{bmatrix} u(t), \\ y(t) &= [\,C,\ -D_1\,]\,\xi(t), \end{aligned} \tag{1.2}$$

where $D = D_1 D_2$ is a factorization of D, for example, $D_1 = I$ and $D_2 = D$. System (1.1) is equivalent to (1.2) in the sense that $x(t)$ is the solution of (1.1) with a given input $u(t)$ if and only if $\xi(t) = \begin{bmatrix} x(t) \\ -D_2 u(t) \end{bmatrix}$ satisfies (1.2).

Throughout the paper we will denote by $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ the spaces of $n \times m$ real and complex matrices, respectively. The imaginary axis is denoted by $i\mathbb{R}$ and the unit circle is denoted by $\mathbb{T}$. The matrix A^T stands for the transpose of real A, the matrix A^* denotes the complex conjugate transpose of complex A, and $A^{-T} = (A^{-1})^T$. An identity matrix of order m is denoted by I_m. The matrix A is *positive definite* (*positive semidefinite*) if $x^T A x > 0$ ($x^T A x \geq 0$) for all nonzero $x \in \mathbb{R}^n$, and A is *positive definite on a subspace* $\mathcal{X} \subset \mathbb{R}^n$ if $x^T A x > 0$ for all nonzero $x \in \mathcal{X}$. The largest singular value of a matrix $A \in \mathbb{R}^{n,m}$ is denoted by $\sigma_{\max}(A)$ and the trace of $A \in \mathbb{R}^{n,n}$ is designated by $\mathrm{tr}(A)$. We will denote by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ the Euclidean vector norm of $x \in \mathbb{R}^n$, by $\|A\|_2 = \sigma_{\max}(A)$ the spectral matrix norm and by $\|A\|_F = \sqrt{\mathrm{tr}(A^T A)}$ the Frobenius matrix norm of $A \in \mathbb{R}^{n,m}$.

2 Discrete-time descriptor systems

Since the results for the continuous-time case are partly related to the discrete-time case, we begin our discussion with the discrete-time descriptor system

$$\begin{aligned} E x_{k+1} &= A x_k + B u_k, \quad x_0 = x^0, \\ y_k &= C x_k. \end{aligned} \tag{2.1}$$

Assume that the matrix pencil $\lambda E - A$ is *regular*, that is, $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. In this case $\lambda E - A$ can be reduced to the Weierstrass canonical form [28]. There exist nonsingular matrices W and T such that

$$E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T \qquad \text{and} \qquad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T, \tag{2.2}$$

where J and N are matrices in Jordan canonical form and N is nilpotent with index of nilpotency ν. The numbers n_f and n_∞ are the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively, and ν is the *index* of the pencil $\lambda E - A$. The matrices

$$P_r = T^{-1}\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T, \qquad P_l = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \tag{2.3}$$

are the *spectral projections* onto the right and left deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues.

Using the Weierstrass canonical form (2.2), we obtain the following Laurent expansion at infinity for the generalized resolvent

$$(\lambda E - A)^{-1} = \sum_{k=-\infty}^{\infty} F_k \lambda^{-k-1}, \tag{2.4}$$

where the coefficients F_k have the form

$$F_k = \begin{cases} T^{-1}\begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, & k = 0, 1, 2 \ldots, \\ T^{-1}\begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix} W^{-1}, & k = -1, -2, \ldots . \end{cases} \tag{2.5}$$

Note that $F_k = 0$ for $k < -\nu$, where ν is the index of the pencil $\lambda E - A$. The matrices F_k are said to be *fundamental matrices*. They play an essential role for the discrete-time descriptor system (2.1).

It is well known [8, 19] that if the pencil $\lambda E - A$ is regular and if the initial value x^0 is *consistent*, that is, it satisfies

$$(I - P_r)x^0 = \sum_{j=0}^{\nu-1} F_{-j-1} B u_j,$$

then the discrete-time descriptor system (2.1) has a unique solution x_k for all $k \geq 0$. Using the fundamental matrices F_k this solution can be written as

$$x_k = F_k E x^0 + \sum_{j=0}^{k+\nu-1} F_{k-j-1} B u_j, \qquad k \geq 0. \tag{2.6}$$

One can see that this solution belongs to a manifold

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \quad : \quad (I - P_r)x = \sum_{j=0}^{\nu-1} F_{-j-1} B w_j, \ \ w_j \in \mathbb{R}^m \right\} \tag{2.7}$$

that is called the *solution space* for the descriptor system (2.1). Moreover, equation (2.6) shows that to determine x_k we need not only past inputs u_j, $j \leq k$, but also future inputs u_j, $k < j \leq k + \nu - 1$. This concept is often called *non-causality* of discrete-time descriptor systems. For a *causal* system (2.1), the state x_k is determined completely by the initial vector x^0 and control inputs u_0, u_1, …, u_k. Clearly, system (2.1) is causal if the pencil $\lambda E - A$ is of index at most one.

2.1 The transfer function and realizations

Let $\mathbb{Z}$ denote the set of integers. Consider a two-sided $\mathbf{Z}$-transform [18] that maps a sequence $\{f_k\}_{k\in\mathbb{Z}}$ with $f_k \in \mathbb{R}^n$ into the function $\mathbf{f}(z)$ of complex variable z defined via

$$\mathbf{f}(z) = \mathbf{Z}[f_k] = \sum_{k=-\infty}^{\infty} f_k\, z^{-k}.$$

The complex variable z is called *frequency* in the discrete-time case. Applying the $\mathbf{Z}$-transform to the descriptor system (2.1), we obtain that $\mathbf{y}(z) = C(zE - A)^{-1}B\mathbf{u}(z)$, where $\mathbf{u}(z)$ and $\mathbf{y}(z)$ are the $\mathbf{Z}$-transforms of the sequences $\{u_k\}_{k\in\mathbb{Z}}$ and $\{y_k\}_{k\in\mathbb{Z}}$, respectively. The rational matrix-valued function $\mathbf{G}(z) = C(zE - A)^{-1}B$ is called the *transfer function* of the discrete-time descriptor system (2.1). It gives a transfer relation between the $\mathbf{Z}$-transforms of the input u_k and the output y_k. In other words, the transfer function $\mathbf{G}(z)$ describes the input-output behaviour of system (2.1) in the frequency domain.

For any rational matrix-valued function $\mathbf{G}(z)$, there exist matrices E, A, B and C such that $\mathbf{G}(z) = C(zE - A)^{-1}B$, see [8]. A descriptor system (2.1) with these matrices is called a *realization* of $\mathbf{G}(z)$. We will also denote a realization of $\mathbf{G}(z)$ by $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$. Note that the realization of $\mathbf{G}(z)$ is, in general, not unique [8].

Definition 2.1. Two realizations $[\,E,\, A,\, B,\, C\,]$ and $[\,\check{E},\, \check{A},\, \check{B},\, \check{C}\,]$ are *restricted system equivalent* if there exist nonsingular matrices $\check{W}$ and $\check{T}$ such that

$$E = \check{W}\check{E}\check{T}, \qquad A = \check{W}\check{A}\check{T}, \qquad B = \check{W}\check{B}, \qquad C = \check{C}\check{T}.$$

The pair $(\check{W}, \check{T})$ is called *system equivalence transformation.*

The notion of restricted system equivalence is consistent with [25]. A characteristic quantity of system (2.1) is said to be *input-output invariant* if it is preserved under a system equivalence transformation. The transfer function $\mathbf{G}(z)$ is input-output invariant, since

$$\mathbf{G}(z) = C(zE - A)^{-1}B = \check{C}\check{T}\check{T}^{-1}(z\check{E} - \check{A})^{-1}\check{W}^{-1}\check{W}\check{B} = \check{C}(z\check{E} - \check{A})^{-1}\check{B}.$$

Other important results from the theory of rational functions and realization theory may be found in [8, 14, 32].

2.2 Controllability and observability

In contrast to standard state space systems, for discrete-time descriptor systems, there are several different notions of controllability and observability, see [5, 8, 19, 20, 32] and the references therein.

Definition 2.2. System (2.1) and the triplet (E, A, B) are called R-*controllable* if

$$\operatorname{rank}\,[\,\lambda E - A,\ B] = n \quad \text{for all finite } \ \lambda \in \mathbb{C}. \tag{2.8}$$

System (2.1) and the triplet (E, A, B) are called I-*controllable* if

$$\operatorname{rank}\,[\,E,\ AK_E,\ B] = n, \ \text{ where the columns of } K_E \text{ span } \operatorname{Ker} E.$$

System (2.1) and the triplet (E, A, B) are called C-*controllable* if (2.8) holds and $\operatorname{rank}[E, B]{=}n$.

C-controllability implies that for any given initial state $x^0 \in \mathbb{R}^n$ and final state $x_f \in \mathbb{R}^n$, there exists a control input u_k that transfers the system from x^0 to x_f in finite time. This notion follows [5, 33] and is consistent with the definition of *controllability* given in [8].

R-controllability ensures that for any initial and final states x^0, $x_f \in \mathcal{X}$ with $\mathcal{X}$ as in (2.7), there exists a control input that transfers the system from x^0 to x_f in finite time. In the case of $E = I$, R-controllability coincides with C-controllability and is the usual controllability of standard state space systems [15].

I-controllability means that for any given initial state $x^0 \in \mathbb{R}^n$, there exists a state feedback control $u_k = Fx_k + v_k$ with a feedback matrix $F \in \mathbb{R}^{m,n}$ and a new control input $v_k \in \mathbb{R}^m$ such that the closed-loop system $Ex_{k+1} = (A + BF)x_k + Bv_k$ is causal [8]. Note that the descriptor system (2.1) with the pencil $\lambda E - A$ of index at most one is I-controllable.

Observability is a dual property of controllability. System (2.1) and the triplet (E, A, C) are called R (I, C)-*observable* if (E^T, A^T, C^T) is R (I, C)-controllable.

For equivalent algebraic and geometric characterizations of different concepts of controllability and observability for descriptor systems, see [8, 20, 29, 33].

It should be noted that the controllability and observability conditions for the descriptor system (2.1) are input-output invariant.

2.3 Stability

We now present some results on the asymptotic stability for the descriptor system (2.1).

Definition 2.3. The discrete-time descriptor system (2.1) is called *asymptotically stable* if $\lim_{k\to\infty} x_k = 0$ for all solutions x_k of the system $Ex_{k+1} = Ax_k$.

The following theorem gives equivalent conditions for system (2.1) to be asymptotically stable.

Theorem 2.4. [8, 31] *Consider the discrete-time descriptor system* (2.1)*, where the pencil* $\lambda E - A$ *is regular. Let* P_l *and* P_r *be the spectral projections as in* (2.3)*. The following statements are equivalent.*

1. *System* (2.1) *is asymptotically stable.*
2. *All finite eigenvalues of the pencil* $\lambda E - A$ *lie inside the unit circle.*
3. *The projected generalized discrete-time Lyapunov equation*
$$A^T XA - E^T XE = -P_r^T QP_r, \qquad X = XP_l \tag{2.9}$$
has a unique Hermitian, positive semidefinite solution X *for every Hermitian positive definite matrix* Q*.*
4. *For all matrices* C *such that the triplet* (E, A, C) *is R-observable, the projected generalized discrete-time Lyapunov equation* (2.9) *with* $Q = C^T C$ *has a unique solution* X *which is Hermitian and positive definite on the subspace* $\operatorname{Im} P_l$*.*

We see that asymptotic stability of the descriptor system (2.1) can be characterized in terms of the generalized spectrum of the pencil $\lambda E - A$. Note that although the eigenvalue at infinity lies outside the unit circle, it has no effect on the asymptotic stability of system (2.1). In the following, the pencil $\lambda E - A$ will be called *d-stable* if $\lambda E - A$ is regular and all finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

2.4 Impulse and frequency responses

The purpose of this subsection is to generalize the impulse and frequency responses [17, 35] for discrete-time descriptor systems.

Using (2.4) the transfer function $\mathbf{G}(z) = C(zE - A)^{-1}B$ can be expanded into a Laurent series [18] around $z = \infty$ as follows

$$\mathbf{G}(z) = \sum_{k=-\infty}^{\infty} G_k z^{-k}, \tag{2.10}$$

where $G_k = CF_{k-1}B$ and F_k are as in (2.5). The sequence $\{G_k\}_{k\in\mathbb{Z}}$ defines an *impulse response* of the discrete-time descriptor system (2.1). We see that the transfer function $\mathbf{G}(z)$ is just the $\mathbf{Z}$-transform of the impulse response. Observe that $G_k = 0$ for $k \le -\nu$, where ν is the index of the pencil $\lambda E - A$. Physically the impulse response of (2.1) can be interpreted as follows.

Consider the system of difference equations

$$EX_{k+1} = AX_k + BU_k, \qquad Y_k = CX_k, \tag{2.11}$$

where $X_k \in \mathbb{R}^{n,m}$, $U_k \in \mathbb{R}^{m,m}$ and $Y_k \in \mathbb{R}^{p,m}$. For an impulsive input $U_k = \delta_{0,k}I$, where $\delta_{j,k}$ is the *Kronecker delta*, system (2.11) has the solution $X_k = F_{k-1}B$ for $k \in \mathbb{Z}$. In this case the output of (2.11) has the form $Y_k = CF_{k-1}B = G_k$. Thus, the elements G_k of the impulse response of system (2.1) coincide with the output matrices Y_k of the matrix difference system (2.11) produced by the impulsive input.

Definition 2.5. A transfer function $\mathbf{G}(z)$ is said to be *proper* if $\lim\limits_{z\to\infty} \mathbf{G}(z) < \infty$, and *improper*, otherwise. If $\lim\limits_{z\to\infty} \mathbf{G}(z) = 0$, then $\mathbf{G}(z)$ is said to be *strictly proper*.

Taking into account (2.10), the transfer function $\mathbf{G}(z)$ can be additively decomposed as $\mathbf{G}(z) = \mathbf{G}_{sp}(z) + \mathbf{P}(z)$, where $\mathbf{G}_{sp}(z) = \sum\limits_{k=1}^{\infty} G_k z^{-k}$ and $\mathbf{P}(z) = \sum\limits_{k=0}^{\nu-1} G_{-k} z^k$ are, respectively, the *strictly proper part* and the *polynomial part* of $\mathbf{G}(z)$. The transfer function $\mathbf{G}(z)$ is strictly proper if and only if $G_k = 0$ for $k \le 0$. Moreover, $\mathbf{G}(z)$ is proper if and only if $G_k = 0$ for $k \le -1$. Obviously, if the pencil $\lambda E - A$ is of index at most one, then $\mathbf{G}(z)$ is proper.

Remark 2.6. Note that the causal descriptor system (2.1) has the proper transfer function $\mathbf{G}(z)$. However, system (2.1) with proper $\mathbf{G}(z)$ is not necessarily causal.

Example 2.7. The descriptor system (2.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = C^T$$

is non-causal, although the transfer function $\mathbf{G}(z) = (2 - z)/(z - 1)$ is proper.

As in the standard state space case [17], the *frequency response* of the discrete-time descriptor system (2.1) is given by the values of the transfer function on the unit circle $\mathbf{G}(e^{i\omega})$. It follows from (2.10) that

$$\mathbf{G}(e^{i\omega}) = \sum_{k=-\infty}^{\infty} G_k e^{-i\omega k}, \tag{2.12}$$

that is, the impulse response $\{G_k\}_{k\in\mathbb{Z}}$ is a sequence of the Fourier coefficients [9, 18] of the frequency response $\mathbf{G}(e^{i\omega})$. System (2.1) with an input sequence $\{e^{i\omega k}u_0\}_{k\in\mathbb{Z}}$, where $\omega \in \mathbb{R}$ and $u_0 \in \mathbb{R}^m$, has the output

$$y_k = \sum_{j=-\infty}^{\infty} CF_{k-j-1}Be^{i\omega j}u_0 = \left(\sum_{j=-\infty}^{\infty} G_j e^{-i\omega j}\right)\left(e^{i\omega k}u_0\right) = \mathbf{G}(e^{i\omega})\left(e^{i\omega k}u_0\right).$$

Thus, the frequency response $\mathbf{G}(e^{i\omega})$ gives a transfer relation between the input sequence $u_k = e^{i\omega k}u_0$ and the output sequence y_k of system (2.1).

It should be noted that the impulse response $\{G_k\}_{k\in\mathbb{Z}}$ and the frequency response $\mathbf{G}(e^{i\omega})$ are input-output invariants of the discrete-time descriptor system (2.1).

2.5 Controllability and observability Gramians

Consider now the *causal controllability matrix* and the *causal observability matrix* given by

$$\mathbf{C}_+ = [\,F_0B,\,\ldots,\,F_kB\,\ldots] \quad \text{and} \quad \mathbf{O}_+ = [\,F_0^TC^T,\,\ldots,\,F_k^TC^T,\,\ldots]^T, \tag{2.13}$$

respectively, where the matrices F_k are as in (2.5). If the pencil $\lambda E - A$ is d-stable, then the *causal controllability Gramian* of the descriptor system (2.1) is defined via

$$\mathcal{G}_{dcc} = \mathbf{C}_+\mathbf{C}_+^T = \sum_{k=0}^{\infty} F_kBB^TF_k^T \tag{2.14}$$

and the *causal observability Gramian* of system (2.1) has the form

$$\mathcal{G}_{dco} = \mathbf{O}_+^T\mathbf{O}_+ = \sum_{k=0}^{\infty} F_k^TC^TCF_k, \tag{2.15}$$

see [3, 29]. The *non-causal controllability matrix* and the *non-causal observability matrix* are given by

$$\mathbf{C}_- = [F_{-\nu}B,\ \ldots,\ F_{-1}B\,] \quad \text{and} \quad \mathbf{O}_- = [F_{-\nu}^TC^T,\ \ldots,\ F_{-1}^TC^T\,]^T, \tag{2.16}$$

respectively. The matrix

$$\mathcal{G}_{dnc} = \mathbf{C}_-\mathbf{C}_-^T = \sum_{k=-\nu}^{-1} F_kBB^TF_k^T$$

is called the *non-causal controllability Gramian* of the discrete-time descriptor system (2.1) and the matrix

$$\mathcal{G}_{dno} = \mathbf{O}_-^T\mathbf{O}_- = \sum_{k=-\nu}^{-1} F_k^TC^TCF_k$$

is called the *non-causal observability Gramian* of (2.1). In summary, the *controllability Gramian* of the discrete-time descriptor system (2.1) is defined via $\mathcal{G}_{dc} = \mathcal{G}_{dcc} + \mathcal{G}_{dnc}$, and the *observability Gramian* of system (2.1) is given by $\mathcal{G}_{do} = \mathcal{G}_{dco} + \mathcal{G}_{dno}$.

If $E = I$, then $\mathcal{G}_{dcc} = \mathcal{G}_{dc}$ and $\mathcal{G}_{dco} = \mathcal{G}_{do}$ are the usual controllability and observability Gramians for standard state space systems [11, 35].

The following theorem shows that the Gramians of system (2.1) satisfy generalized discrete-time Lyapunov equations with special right-hand sides.

Theorem 2.8. *Consider the discrete-time descriptor system* (2.1)*, where the pencil $\lambda E - A$ is d-stable.*

1. *The causal controllability and observability Gramians $\mathcal{G}_{dcc}$ and $\mathcal{G}_{dco}$ are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations*
$$\begin{aligned} &A\mathcal{G}_{dcc}A^T - E\mathcal{G}_{dcc}E^T = -P_l BB^T P_l^T, \\ &\mathcal{G}_{dcc} = P_r \mathcal{G}_{dcc} P_r^T \end{aligned} \tag{2.17}$$
and
$$\begin{aligned} &A^T\mathcal{G}_{dco}A - E^T\mathcal{G}_{dco}E = -P_r^T C^T C P_r, \\ &\mathcal{G}_{dco} = P_l^T \mathcal{G}_{dco} P_l, \end{aligned} \tag{2.18}$$
respectively.

2. *The non-causal controllability and observability Gramians $\mathcal{G}_{dnc}$ and $\mathcal{G}_{dno}$ are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations*
$$\begin{aligned} &A\mathcal{G}_{dnc}A^T - E\mathcal{G}_{dnc}E^T = (I - P_l)BB^T(I - P_l)^T, \\ &P_r\mathcal{G}_{dnc}P_r^T = 0 \end{aligned} \tag{2.19}$$
and
$$\begin{aligned} &A^T\mathcal{G}_{dno}A - E^T\mathcal{G}_{dno}E = (I - P_r)^T C^T C(I - P_r), \\ &P_l^T\mathcal{G}_{dno}P_l = 0, \end{aligned} \tag{2.20}$$
respectively.

3. *The controllability and observability Gramians $\mathcal{G}_{dc}$ and $\mathcal{G}_{do}$ are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations*
$$\begin{aligned} &A\mathcal{G}_{dc}A^T - E\mathcal{G}_{dc}E^T = -P_l BB^T P_l^T + (I - P_l)BB^T(I - P_l)^T, \\ &\mathcal{G}_{dc} = (I - P_r)\mathcal{G}_{dc}(I - P_r)^T \end{aligned} \tag{2.21}$$
and
$$\begin{aligned} &A^T\mathcal{G}_{do}A - E^T\mathcal{G}_{do}E = -P_r^T C^T C P_r + (I - P_r)^T C^T C(I - P_r), \\ &\mathcal{G}_{do} = (I - P_l)^T\mathcal{G}_{do}(I - P_l), \end{aligned} \tag{2.22}$$
respectively.

Proof. See [29, 31]. □

The controllability and observability Gramians can be used to characterize controllability and observability properties of system (2.1).

Theorem 2.9. [3, 29] *Consider the discrete-time descriptor system* (2.1)*. Assume that the pencil $\lambda E - A$ is d-stable.*

1. *System* (2.1) *is R-controllable if and only if the causal controllability Gramian $\mathcal{G}_{dcc}$ is positive definite on the subspace* $\operatorname{Im} P_r^T$.

2. *System* (2.1) *is I-controllable if the non-causal controllability Gramian $\mathcal{G}_{dnc}$ is positive definite on the subspace* $\operatorname{Ker} P_r^T$.

3. *System* (2.1) *is C-controllable if and only if the controllability Gramian* $\mathcal{G}_{dc}$ *is positive definite.*

4. *System* (2.1) *is R-observable if and only if the causal observability Gramian* $\mathcal{G}_{dco}$ *is positive definite on the subspace* $\operatorname{Im} P_l$.

5. *System* (2.1) *is I-observable if the non-causal observability Gramian* $\mathcal{G}_{dno}$ *is positive definite on the subspace* $\operatorname{Ker} P_l$.

6. *System* (2.1) *is C-observable if and only if the observability Gramian* $\mathcal{G}_{do}$ *is positive definite.*

The following example shows that I-controllability of (2.1) does not imply that $\mathcal{G}_{dnc}$ is positive definite on $\operatorname{Ker} P_r^T$ and the I-observable system (2.1) may have the non-causal observability Gramian $\mathcal{G}_{dnc}$ that is not positive definite on $\operatorname{Ker} P_l$.

Example 2.10. The descriptor system (2.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = C^T$$

is I-controllable and I-observable. The improper controllability and observability Gramians have the form

$$\mathcal{G}_{dnc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathcal{G}_{dno},$$

and $\operatorname{Ker} P_l = \operatorname{Ker} P_r^T$. We get $v^T \mathcal{G}_{dnc} v = v^T \mathcal{G}_{dno} v = 0$ for $v = [0, 0, 1]^T \in \operatorname{Ker} P_l$.

2.6 Causal and non-causal Hankel singular values

The controllability and observability Gramians of the descriptor system (2.1) are not input-output invariants. However, we know that for standard state space systems the spectrum of the product of the controllability and observability Gramians does not change under the system equivalence transformation [35]. For the discrete-time descriptor system (2.1), we will show that the spectrum of a matrix $\Phi_d = \mathcal{G}_{dcc} E^T \mathcal{G}_{dco} E$ is an input-output invariant. Indeed, under a system equivalence transformation $(\check{W}, \check{T})$ the causal controllability Gramian $\mathcal{G}_{dcc}$ and the causal observability Gramian $\mathcal{G}_{dco}$ are transformed to $\check{\mathcal{G}}_{dcc} = \check{T} \mathcal{G}_{dcc} \check{T}^T$ and $\check{\mathcal{G}}_{dco} = \check{W}^T \mathcal{G}_{dco} \check{W}$, respectively. Then

$$\check{\Phi}_d = \check{\mathcal{G}}_{dcc} \check{E}^T \check{\mathcal{G}}_{dco} \check{E} = \check{T} \mathcal{G}_{dcc} E^T \mathcal{G}_{dco} E \check{T}^{-1} = \check{T} \Phi_d \check{T}^{-1}.$$

Moreover, we can prove that the matrix Φ_d has the real and non-negative spectrum.

Lemma 2.11. *Let* $\lambda E - A$ *be d-stable and let* $\Phi_d = \mathcal{G}_{dcc} E^T \mathcal{G}_{dco} E$. *Then all eigenvalues of* Φ_d *are real and non-negative.*

Proof. It follows from (2.14) and (2.15) that the matrices $\mathcal{G}_{dcc}$ and $E^T \mathcal{G}_{dco} E$ are symmetric and positive semidefinite. In this case there exists a nonsingular matrix $\check{T}$ such that

$$\check{T} \mathcal{G}_{dcc} \check{T}^T = \begin{bmatrix} \Sigma_1 & & & 0 \\ & \Sigma_2 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}, \qquad \check{T}^{-T} E^T \mathcal{G}_{dco} E \check{T}^{-1} = \begin{bmatrix} \Sigma_1 & & & 0 \\ & 0 & & \\ & & \Sigma_3 & \\ 0 & & & 0 \end{bmatrix},$$

where Σ_1, Σ_2, Σ_3 are diagonal matrices with positive diagonal elements [35]. Then

$$\check{T}\Phi_d\check{T}^{-1} = \check{T}\mathcal{G}_{dcc}E^T\mathcal{G}_{dco}E\check{T}^{-1} = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix},$$

i.e., Φ_d is similar to the diagonal matrix with real non-negative diagonal elements. □

An analogous result holds for the matrix $\Psi_d = \mathcal{G}_{dnc}A^T\mathcal{G}_{dno}A$.

Lemma 2.12. *All eigenvalues of the matrix $\Psi_d = \mathcal{G}_{dnc}A^T\mathcal{G}_{dno}A$ are real and non-negative.*

The matrices Φ_d and Ψ_d play the same role for descriptor systems as the product of the controllability and observability Gramians for standard state space systems [11]. Using these matrices we can define the causal and non-causal Hankel singular values for the descriptor system (2.1) as follows.

Definition 2.13. Let the pencil $\lambda E - A$ be d-stable and let n_f and n_∞ be the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The square roots of the n_f largest eigenvalues of the matrix Φ_d, denoted by ς_j, are called the *causal Hankel singular values* of the discrete-time descriptor system (2.1). The square roots of the n_∞ largest eigenvalues of the matrix Ψ_d, denoted by θ_j, are called the *non-causal Hankel singular values* of the discrete-time descriptor system (2.1).

The causal and non-causal Hankel singular values together form the set of Hankel singular values of the discrete-time descriptor system (2.1). Clearly, the causal Hankel singular values are defined only for asymptotically stable descriptor systems. Since the spectra of Φ_d and Ψ_d do not change under system equivalence transformations, the causal and non-causal Hankel singular values are input-output invariants of system (2.1). For $E = I$, the causal Hankel singular values are the classical Hankel singular values of standard discrete-time state space systems [11].

Since the causal and non-causal controllability and observability Gramians are symmetric and positive semidefinite, there exist full fank factorizations

$$\begin{aligned} \mathcal{G}_{dcc} &= R_cR_c^T, & \mathcal{G}_{dco} &= L_c^TL_c, \\ \mathcal{G}_{dnc} &= R_nR_n^T, & \mathcal{G}_{dno} &= L_n^TL_n, \end{aligned} \tag{2.23}$$

where the matrices R_c, L_c^T, R_n, L_n^T are full column rank factors [16]. The following lemma gives a connection between the proper and improper Hankel singular values and the standard singular values of the matrices L_cER_c and L_nAR_n.

Lemma 2.14. *Let $\lambda E - A$ be a d-stable pencil. Consider the full rank factorizations* (2.23) *of the causal and non-causal Gramians of the descriptor system* (2.1)*. Then the non-zero causal Hankel singular values are the non-zero singular values of the matrix L_cER_c, while the non-zero non-causal Hankel singular values are the non-zero singular values of the matrix L_nAR_n.*

Proof. We have

$$\begin{aligned} \varsigma_j^2 &= \lambda_j(R_cR_c^TE^TL_c^TL_cE) = \lambda_j(R_c^TE^TL_c^TL_cER_c) = \sigma_j^2(L_cER_c), \\ \theta_j^2 &= \lambda_j(R_nR_n^TA^TL_n^TL_nA) = \lambda_j(R_n^TA^TL_n^TL_nAR_n) = \sigma_j^2(L_nAR_n), \end{aligned}$$

where $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, the eigenvalues and singular values of the corresponding matrices. □

As a consequence of Theorem 2.9 and Lemma 2.14 we obtain the following result.

Corollary 2.15. *Consider the discrete-time descriptor system* (2.1)*. Assume that the pencil* $\lambda E - A$ *is d-stable.*

1. *All causal Hankel singular values of* (2.1) *are non-zero if and only if system* (2.1) *is R-controllable and R-observable.*
2. *All non-causal Hankel singular values of* (2.1) *are non-zero if and only if*
$$\operatorname{rank}[\,E,\ B\,] = n \qquad \text{and} \qquad \operatorname{rank}[\,E^T,\ C^T\,] = n. \tag{2.24}$$
3. *All causal and non-causal Hankel singular values of* (2.1) *are non-zero if and only if system* (2.1) *is C-controllable and C-observable.*

2.7 System norms

In this subsection we generalize convolution and Hankel operators [1, 17] to the discrete-time descriptor system (2.1). Moreover, we extend some known system norms [2, 35] to (2.1) and establish their connection with the controllability and observability Gramians $\mathcal{G}_{dc}$ and $\mathcal{G}_{do}$, the matrices Φ_d and Ψ_d, the convolution and Hankel operators as well the Hankel singular values. System norms are important in robust control and system approximation [1, 2, 11, 35].

2.7.1 $\mathbb{L}_2^{p,m}(\mathbb{\Gamma})$-norm and $\mathbb{h}_2$-norm

Let $\mathbb{L}_2^{p,m}(\mathbb{\Gamma})$ be the Hilbert space of matrix-valued functions $\mathbf{F} : \mathbb{\Gamma} \longrightarrow \mathbb{C}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(\mathbb{\Gamma})$-norm
$$\|\mathbf{F}\|_{\mathbb{L}_2^{p,m}(\mathbb{\Gamma})} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\Big(\mathbf{F}^*(e^{i\omega})\mathbf{F}(e^{i\omega})\Big)\,d\omega\right)^{1/2}. \tag{2.25}$$
A subspace of $\mathbb{L}_2^{p,m}(\mathbb{\Gamma})$ which consists of all rational transfer functions that have no poles in the exterior of the closed unit disk is denoted by $\mathbb{h}_2$. The $\mathbb{h}_2$-*norm* of a transfer function $\mathbf{G}(z) \in \mathbb{h}_2$ is defined by
$$\|\mathbf{G}\|_{\mathbb{h}_2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\Big(\mathbf{G}^*(e^{i\omega})\mathbf{G}(e^{i\omega})\Big)\,d\omega\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} \|\mathbf{G}(e^{i\omega})\|_F^2\,d\omega\right)^{1/2}.$$

If $\mathbf{G}(z)$ is strictly proper and $\lambda E - A$ is d-stable, then $\mathbf{G}(z) = C(zE - A)^{-1}B \in \mathbb{h}_2$. On the other hand, if $\mathbf{G}(z) \in \mathbb{h}_2$, then $\mathbf{G}(z)$ is strictly proper, but the pencil $\lambda E - A$ is not necessarily d-stable.

Example 2.16. Let
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = C^T.$$

We have $\mathbf{G}(z) = 2/(2z-1) \in \mathbb{h}_2$, but the pencil $\lambda E - A$ is not d-stable.

The transfer function $\mathbf{G}(z)$ of the descriptor system (2.1) may be improper. In this case, if the pencil $\lambda E - A$ has no eigenvalues on the unit circle, then $\mathbf{G}(z) \in \mathbb{L}_2^{p,m}(\Gamma)$.

Consider now the Hilbert space $\mathbb{l}_2^{p,m}(\mathbb{Z})$ containing matrix-valued sequences $S = \{S_k\}_{k\in\mathbb{Z}}$, $S_k \in \mathbb{R}^{p,m}$, that have bounded $\mathbb{l}_2^{p,m}(\mathbb{Z})$-norm

$$\|S\|_{\mathbb{l}_2^{p,m}(\mathbb{Z})} = \left(\sum_{k=-\infty}^{\infty} \operatorname{tr}\left(S_k^T S_k\right) \right)^{1/2} = \left(\sum_{k=-\infty}^{\infty} \|S_k\|_F^2 \right)^{1/2}.$$

By Parseval's identity [26] we find from (2.12) that

$$\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \|G\|_{\mathbb{l}_2^{p,m}(\mathbb{Z})} = \left(\sum_{k=-\infty}^{\infty} \|G_k\|_F^2 \right)^{1/2}, \tag{2.26}$$

where $G = \{G_k\}_{k\in\mathbb{Z}}$ is the impulse response of the descriptor system (2.1). Moreover, if the pencil $\lambda E - A$ is d-stable, then substituting $G_k = CF_{k-1}B$ in (2.26) we get

$$\begin{aligned} \|\mathbf{G}\|^2_{\mathbb{L}_2^{p,m}(\Gamma)} &= \sum_{k=-\infty}^{\infty} \operatorname{tr}\left(B^T F_{k-1}^T C^T C F_{k-1} B\right) = \sum_{k=-\infty}^{\infty} \operatorname{tr}\left(C F_{k-1} B B^T F_{k-1}^T C^T\right) \\ &= \operatorname{tr}\left(B^T \mathcal{G}_{do} B\right) = \operatorname{tr}\left(C \mathcal{G}_{dc} C^T\right). \end{aligned}$$

These relations give a simple numerical algorithm for computing the $\mathbb{L}_2^{p,m}(\Gamma)$-norm of $\mathbf{G}(z)$ with the d-stable pencil $\lambda E - A$. Note that we do not need to calculate the controllability or observability Gramian explicitly. It is sufficient to determine the full rank factorization $\mathcal{G}_{dc} = RR^T$ or $\mathcal{G}_{do} = L^T L$, where $\mathcal{G}_{dc}$ and $\mathcal{G}_{do}$ satisfy the projected generalized Lyapunov equation (2.21) and (2.22), respectively. Then the $\mathbb{L}_2^{p,m}(\Gamma)$-norm of $\mathbf{G}(z)$ can be computed as $\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \|LB\|_F = \|CR\|_F$.

In summary, we have the following algorithm to compute the $\mathbb{L}_2^{p,m}(\Gamma)$-norm of $\mathbf{G}(z)$ using, for example, the full rank factor of the controllability Gramian $\mathcal{G}_{dc}$.

Algorithm 2.17. *Computing the $\mathbb{L}_2^{p,m}(\Gamma)$-norm of the transfer function $\mathbf{G}(z)$.*
Input: *A realization $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$, where the pencil $\lambda E - A$ is d-stable.*
Output: *The $\mathbb{L}_2^{p,m}(\Gamma)$-norm of the transfer function $\mathbf{G}(z) = C(zE - A)^{-1}B$.*
1. *Use the generalized Schur-Hammarling method* [29] *to compute the full rank factor R of the controllability Gramian $\mathcal{G}_{dc} = RR^T$ which satisfies* (2.21).
2. *Compute $\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} = \|CR\|_F$.*

2.7.2 $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm and $\mathbb{h}_\infty$-norm

Let $\mathbb{L}_\infty^{p,m}(\Gamma)$ be the Banach space of matrix-valued functions $\mathbf{F} : \mathbb{C} \longrightarrow \mathbb{C}^{p,m}$ that are (essentially) bounded on Γ. The $\mathbb{L}_\infty^{p,m}(\Gamma)$-norm is defined by

$$\|\mathbf{F}\|_{\mathbb{L}_\infty^{p,m}(\Gamma)} = \operatorname*{ess\,sup}_{\omega\in[0,2\pi]} \sigma_{\max}\left(\mathbf{F}(e^{i\omega})\right) = \operatorname*{ess\,sup}_{\omega\in[0,2\pi]} \|\mathbf{F}(e^{i\omega})\|_2.$$

The subspace of $\mathbb{L}_\infty^{p,m}(\Gamma)$ denoted by $\mathbb{h}_\infty$ consists of all rational transfer functions that are analytic and bounded in the exterior of the closed unit disk. The $\mathbb{h}_\infty$-*norm* of the transfer function $\mathbf{G}(z) \in \mathbb{h}_\infty$ is defined by

$$\|\mathbf{G}\|_{\mathbb{h}_\infty} = \sup_{|z|>1} \|\mathbf{G}(z)\|_2 = \sup_{\omega\in[0,2\pi]} \|\mathbf{G}(e^{i\omega})\|_2.$$

Clearly, the $\mathbb{h}_\infty$-norm of $\mathbf{G}(z)$ is finite only if $\mathbf{G}(z) \in \mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})$ is proper.

Consider a *convolution operator* $\mathcal{K}_d$ for the discrete-time descriptor system (2.1) that maps the inputs u_k into the outputs y_k defined via

$$y_k = (\mathcal{K}_d u)_k = (G * u)_k = \sum_{j=-\infty}^{\infty} G_{k-j} u_j = \sum_{j=-\infty}^{k+\nu-1} G_{k-j} u_j. \tag{2.27}$$

Writing the sequences $\{y_k\}_{k\in\mathbb{Z}}$ and $\{u_k\}_{k\in\mathbb{Z}}$ as the column vectors

$$y = \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix} \qquad \text{and} \qquad u = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix},$$

relation (2.27) can be represented as a linear system $y = \mathcal{K}_d u$, where

$$\mathcal{K}_d = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & G_0 & G_{-1} & G_{-2} & \cdots \\ \cdots & G_1 & G_0 & G_{-1} & \cdots \\ \cdots & G_2 & G_1 & G_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is the matrix representation of the convolution operator. We see that the operator $\mathcal{K}_d$ has block Toeplitz structure and gives an input-output relationship in the time domain. Note that $\mathcal{K}_d$ is input-output invariant.

If the pencil $\lambda E - A$ is d-stable, then $\mathcal{K}_d$ is the bounded linear operator that maps $\mathbb{l}_2^m(\mathbb{Z})$ into $\mathbb{l}_2^p(\mathbb{Z})$. The spectral norm of $\mathcal{K}_d$ is given by

$$\|\mathcal{K}_d\|_2 = \sup_{u\neq 0} \frac{\|\mathcal{K}_d u\|_{\mathbb{l}_2^p(\mathbb{Z})}}{\|u\|_{\mathbb{l}_2^m(\mathbb{Z})}}.$$

By Parseval's identity [26] we have

$$\|\mathcal{K}_d\|_2 = \sup_{u\neq 0} \frac{\|\mathcal{K}_d u\|_{\mathbb{l}_2^p(\mathbb{Z})}}{\|u\|_{\mathbb{l}_2^m(\mathbb{Z})}} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(\mathbb{\Gamma})}}{\|\mathbf{u}\|_{\mathbb{L}_2^m(\mathbb{\Gamma})}} = \|\mathbf{G}\|_{\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})}.$$

Thus, the $\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})$-norm of the transfer function $\mathbf{G}(z)$ can be interpreted as a ratio of the output energy to the input energy of the descriptor system (2.1).

For computing the $\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})$-norm of $\mathbf{G}(z)$ we can use a midpoint rule [13, 21, 27] or a cubic interpolation method [10] that are based on the fact that $\|\mathbf{G}\|_{\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})} < \gamma$ for some $\gamma > 0$ if and only if a matrix pencil

$$\lambda E_\gamma - A_\gamma = \lambda \begin{bmatrix} E & -\gamma^{-1}BB^T \\ 0 & -A^T \end{bmatrix} - \begin{bmatrix} A & 0 \\ \gamma^{-1}C^TC & -E^T \end{bmatrix}$$

has no eigenvalues on the unit circle [27]. These iterative methods have quadratic convergence and provide lower and upper bounds on the $\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})$-norm of the transfer function $\mathbf{G}(z)$.

2.7.3 Hilbert-Schmidt-Hankel norm

Let $\mathbb{Z}^-$ and $\mathbb{Z}_0^+$ denote the sets of negative and non-negative integers, respectively. A *causal Hankel operator* $\mathcal{H}_c : \mathbb{l}_2^m(\mathbb{Z}^-) \longrightarrow \mathbb{l}_2^p(\mathbb{Z}_0^+)$ for the discrete-time descriptor system (2.1) with the d-stable pencil $\lambda E - A$ is defined via

$$y_k = (\mathcal{H}_c u)_k = \sum_{j=-\infty}^{-1} G_{k-j} u_j, \qquad k \geq 0. \tag{2.28}$$

If we set

$$y_+ = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix} \qquad \text{and} \qquad u_- = \begin{bmatrix} u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix},$$

then (2.28) can be written as a linear system $y_+ = \mathcal{H}_c u_-$, where

$$\mathcal{H}_c = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.29}$$

has block Hankel structure and is the matrix representation of the causal Hankel operator. The operator $\mathcal{H}_c$ maps past inputs ($u_k = 0$ for $k \geq 0$) to present and future outputs ($y_k = 0$ for $k < 0$).

A *non-causal Hankel operator* $\mathcal{H}_n$ for the discrete-time descriptor system (2.1) is given by

$$y_k = (\mathcal{H}_n u)_k = \sum_{j=0}^{\infty} G_{k-j+1} u_j, \qquad k < 0.$$

For

$$y_- = \begin{bmatrix} \vdots \\ y_{-2} \\ y_{-1} \end{bmatrix} \qquad \text{and} \qquad u_+ = \begin{bmatrix} \vdots \\ u_1 \\ u_0 \end{bmatrix},$$

we have a linear system $y_- = \mathcal{H}_n u_+$, where

$$\mathcal{H}_n = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & G_{-4} & G_{-3} & G_{-2} \\ \cdots & G_{-3} & G_{-2} & G_{-1} \\ \cdots & G_{-2} & G_{-1} & G_0 \end{bmatrix} \tag{2.30}$$

is the matrix representation of the non-causal Hankel operator. We see that the operator $\mathcal{H}_n$ also has block Hankel structure and maps present and future inputs ($u_k = 0$ for $k < 0$) to past outputs ($y_k = 0$ for $k \geq 0$). Clearly, the causal and non-causal Hankel operators are input-output invariants of system (2.1).

We will now establish a connection between the singular values of the causal Hankel operator $\mathcal{H}_c$ and the causal Hankel singular values of (2.1).

Lemma 2.18. *Consider the discrete-time descriptor system* (2.1), *where the pencil* $\lambda E - A$ *is d-stable. Let* $\mathcal{H}_c$ *be a causal Hankel operator as in* (2.29). *Then* $\mathcal{H}_c$ *has a finite set of non-zero singular values* $\sigma_j(\mathcal{H}_c)$ *that coincide with the non-zero causal Hankel singular values of* (2.1).

Proof. Note that $\sigma_j(\mathcal{H}_c) = \sqrt{\lambda_j(\mathcal{H}_c^T\mathcal{H}_c)}$, where $\lambda_j(\mathcal{H}_c^T\mathcal{H}_c)$ denote the eigenvalues of $\mathcal{H}_c^T\mathcal{H}_c$. Consider the matrices $\mathbf{C}_+$ and $\mathbf{O}_+$ as in (2.13). Using the Weierstrass canonical form (2.2) and (2.5), we obtain that $F_jEF_k = F_{j+k}$ for $j,k \geq 0$. Then $\mathbf{O}_+E\mathbf{C}_+ = \mathcal{H}_c$. Hence, $\varsigma_j^2 = \lambda_j(\mathbf{C}_+\mathbf{C}_+^TE^T\mathbf{O}_+^T\mathbf{O}_+E) = \sigma_j^2(\mathbf{O}_+E\mathbf{C}_+) = \sigma_j^2(\mathcal{H}_c)$. □

An analogous result holds for the singular values of the non-causal Hankel operator $\mathcal{H}_n$ and the non-causal Hankel singular values of system (2.1).

Lemma 2.19. *Consider the descriptor system* (2.1). *Let* $\mathcal{H}_n$ *be a non-causal Hankel operator as in* (2.30). *Then* $\mathcal{H}_n$ *has a finite set of non-zero singular values* $\sigma_j(\mathcal{H}_n)$ *that are the non-zero non-causal Hankel singular values of* (2.1).

It immediately follows from Corollary 2.15 and Lemmas 2.18, 2.19 that $\operatorname{rank}(\mathcal{H}_c) \leq n_f$ and $\operatorname{rank}(\mathcal{H}_n) \leq n_\infty$, where n_f and n_∞ are the dimensions of the deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. We have $\operatorname{rank}(\mathcal{H}_c) = n_f$ if and only if system (2.1) is R-controllable and R-observable. Furthermore, $\operatorname{rank}(\mathcal{H}_n) = n_\infty$ if and only if relations (2.24) hold.

Let the causal and non-causal Hankel singular values of system (2.1) be ordered decreasingly, that is, $\varsigma_1 \geq \ldots \geq \varsigma_{n_f}$ and $\theta_1 \geq \ldots \geq \theta_{n_\infty}$. The Frobenius and spectral norms of the causal and non-causal Hankel operators are computed as

$$\begin{aligned} \|\mathcal{H}_c\|_F &= \sqrt{\varsigma_1^2 + \ldots + \varsigma_{n_f}^2}, & \|\mathcal{H}_n\|_F &= \sqrt{\theta_1^2 + \ldots + \theta_{n_\infty}^2}, \\ \|\mathcal{H}_c\|_2 &= \varsigma_1, & \|\mathcal{H}_n\|_2 &= \theta_1. \end{aligned}$$

A *Hilbert-Schmidt-Hankel norm* or *HSH-norm* of the transfer function $\mathbf{G}(z)$ is defined via

$$\|\mathbf{G}\|_{HSH} = \sqrt{\operatorname{tr}\left(\mathcal{H}_c^T\mathcal{H}_c + \mathcal{H}_n^T\mathcal{H}_n\right)} = \sqrt{\|\mathcal{H}_c\|_F^2 + \|\mathcal{H}_n\|_F^2} = \sqrt{\sum_{j=1}^{n_f}\varsigma_j^2 + \sum_{j=1}^{n_\infty}\theta_j^2}.$$

Using (2.29) and (2.30) we obtain

$$\begin{aligned} \|\mathbf{G}\|_{HSH}^2 &= \sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\operatorname{tr}\left(G_k^TG_k + G_{-k+1}^TG_{-k+1}\right) \\ &= \sum_{k=1}^{\infty} k\left(\|G_k\|_F^2 + \|G_{-k+1}\|_F^2\right). \end{aligned} \tag{2.31}$$

Furthermore, from Lemmas 2.18 and 2.19 it follows that the squared non-zero singular values of the causal and non-causal Hankel operators $\mathcal{H}_c$ and $\mathcal{H}_n$ coincide with the non-zero eigenvalues of the matrices Φ_d and Ψ_d, respectively. In this case we get

$$\|\mathbf{G}\|_{HSH}^2 = \sum_{j=1}^{n_f}\lambda_j(\Phi_d) + \sum_{j=1}^{n_\infty}\lambda_j(\Phi_d) = \operatorname{tr}(\Phi_d + \Psi_d). \tag{2.32}$$

2.7.4 Hankel norm

Assume that the pencil $\lambda E - A$ is d-stable. The *Hankel norm* of the transfer function $\mathbf{G}(z)$ is defined via

$$\|\mathbf{G}\|_H = \max(\|\mathcal{H}_c\|_2, \|\mathcal{H}_n\|_2) = \max(\varsigma_1, \theta_1). \tag{2.33}$$

From the definition of the causal and non-causal Hankel singular values we find that

$$\|\mathbf{G}\|_H = \sqrt{\lambda_{\max}(\Phi_d + \Psi_d)}.$$

To compute the HSH-norm and the Hankel norm of the transfer function $\mathbf{G}(z)$ we need the Hankel singular values. Using the generalized Schur-Hammarling method [29] we can solve the projected generalized Lyapunov equations (2.17) – (2.20) for the full rank factors R_c, L_c, R_n and L_n as in (2.23). By Lemma 2.14 the non-zero causal and non-causal Hankel singular values of (2.1) are the non-zero singular values of the matrices L_cER_c and L_nAR_n, respectively, and, hence,

$$\begin{aligned} \|L_cER_c\|_F^2 &= \varsigma_1^2 + \ldots + \varsigma_{n_f}^2, \qquad & \|L_nAR_n\|_F^2 &= \theta_1^2 + \ldots + \theta_{n_\infty}^2, \\ \|L_cER_c\|_2 &= \varsigma_1, & \|L_nAR_n\|_2 &= \theta_1. \end{aligned}$$

Then $\|\mathbf{G}\|_{HSH}^2 = \|L_cER_c\|_F^2 + \|L_nAR_n\|_F^2$ and $\|\mathbf{G}\|_H = \max(\|L_cER_c\|_2, \|L_nAR_n\|_2)$. Thus, we have the following algorithm for computing the HSH-norm and the Hankel norm of the transfer function $\mathbf{G}(z)$.

Algorithm 2.20. *Computing the HSH-norm or the Hankel norm of* $\mathbf{G}(z)$.
Input: *A realization* $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$, *where the pencil* $\lambda E - A$ *is d-stable.*
Output: *The HSH-norm or the Hankel norm of* $\mathbf{G}(z) = C(zE - A)^{-1}B$.
1. *Use the generalized Schur-Hammarling method* [29] *to compute the full rank factors* R_c *and* L_c *of the causal Gramians* $\mathcal{G}_{dcc} = R_cR_c^T$ *and* $G_{dco} = L_c^TL_c$ *that satisfy equations* (2.17) *and* (2.18), *respectively.*
2. *Use the generalized Schur-Hammarling method* [29] *to compute the full rank factors* R_n *and* L_n *of the non-causal Gramians* $\mathcal{G}_{dnc} = R_nR_n^T$ *and* $G_{dno} = L_n^TL_n$ *that satisfy equations* (2.19) *and* (2.20), *respectively.*
3. *Compute* $\|\mathbf{G}\|_{HSH} = \sqrt{\|L_cER_c\|_F^2 + \|L_nAR_n\|_F^2}$ *or* $\|\mathbf{G}\|_H = \max(\|L_cER_c\|_2, \|L_nAR_n\|_2)$.

We summarize the considered norms for the asymptotically stable discrete-time descriptor system (2.1) in Table 1.

In the remainder of this section we establish a connection among different system norms. It follows from (2.26) and (2.31) that

$$\|\mathbf{G}\|_{\mathbb{L}_2^{p,m}(\Gamma)} \le \|\mathbf{G}\|_{HSH}.$$

Furthermore, from (2.32) and (2.33) we have

$$\|\mathbf{G}\|_H \le \|\mathbf{G}\|_{HSH} \le \sqrt{n}\|\mathbf{G}\|_H.$$

Taking into account the matrix representations of the convolution operator and the causal and non-causal Hankel operators, we get

$$\|\mathbf{G}\|_H \le \|\mathbf{G}\|_{\mathbb{L}_\infty^{p,m}(\Gamma)} \le \|\mathbf{G}_{sp}\|_{\mathbb{h}_\infty} + \|\mathbf{P}\|_{\mathbb{L}_\infty^{p,m}(\Gamma)},$$

$\mathbf{G}(z) = C(zE-A)^{-1}B$ $zE-A$ is d-stable	$\Vert\mathbf{G}\Vert_{\mathbb{L}_2^{p,m}(\mathbf{\Gamma})}$	$\Vert\mathbf{G}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbf{\Gamma})}$
$\mathbf{G}(e^{i\omega})$	$\left(\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\Vert\mathbf{G}(e^{i\omega})\Vert_F^2\,d\omega\right)^{\frac{1}{2}}$	$\sup\limits_{\omega\in\mathbb{R}}\Vert\mathbf{G}(e^{i\omega})\Vert_2$
G_k	$\left(\displaystyle\sum_{k=-\infty}^{\infty}\Vert G_k\Vert_F^2\right)^{\frac{1}{2}}$	
$\mathcal{G}_{dc} = RR^T$	$\sqrt{\mathrm{tr}(C\mathcal{G}_{dc}C^T)} = \Vert CR\Vert_F$	
$\mathcal{G}_{do} = L^TL$	$\sqrt{\mathrm{tr}(B^T\mathcal{G}_{do}B)} = \Vert LB\Vert_F$	
$\mathcal{K}_d$		$\Vert\mathcal{K}_d\Vert_2$
$\mathbf{G}(z) = C(zE-A)^{-1}B$ $zE-A$ is d-stable	$\Vert\mathbf{G}\Vert_{HSH}$	$\Vert\mathbf{G}\Vert_H$
G_k	$\left(\displaystyle\sum_{k=1}^{\infty}k\left(\Vert G_k\Vert_F^2+\Vert G_{-k+1}\Vert_F^2\right)\right)^{\frac{1}{2}}$	
$\mathcal{H}_c,\quad \mathcal{H}_n$	$\sqrt{\Vert\mathcal{H}_c\Vert_F^2+\Vert\mathcal{H}_n\Vert_F^2}$	$\max(\Vert\mathcal{H}_c\Vert_2, \Vert\mathcal{H}_n\Vert_2)$
$\mathcal{G}_{dcc} = R_cR_c^T,\ \mathcal{G}_{dco} = L_c^TL_c$ $\mathcal{G}_{dnc} = R_nR_n^T,\ \mathcal{G}_{dno} = L_n^TL_n$	$\sqrt{\Vert L_cER_c\Vert_F^2+\Vert L_nAR_n\Vert_F^2}$	$\max(\Vert L_cER_c\Vert_2, \Vert L_nAR_n\Vert_2)$
$\Phi_d,\quad \Psi_d$	$\sqrt{\mathrm{tr}(\Phi_d+\Psi_d)}$	$\sqrt{\lambda_{max}(\Phi_d+\Psi_d)}$
$\varsigma_1\ge\ldots\ge\varsigma_{n_f}$ $\theta_1\ge\ldots\ge\theta_{n_\infty}$	$\sqrt{\varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2}$	$\max(\varsigma_1,\,\theta_1)$

Table 1: Generalized norms for asymptotically stable discrete-time descriptor systems.

where $\mathbf{G}_{sp}(z)$ and $\mathbf{P}(z)$ are the strictly proper and polynomial parts of $\mathbf{G}(z)$. As in the standard state space case [11], we have an estimate $\Vert\mathbf{G}_{sp}\Vert_{\mathbb{h}_\infty}\le 2(\varsigma_1+\ldots+\varsigma_{n_f})$.

Consider now a transfer function $\mathbf{G}_0(z) = -\frac{1}{z}\mathbf{P}(\frac{1}{z})$ that is strictly proper and has only zero poles. Clearly, $\mathbf{G}_0(z)$ and $\mathbf{P}(z)$ have the same Hankel singular values that are just the improper Hankel singular values θ_j of system (2.1). In this case we have

$$\begin{aligned}\Vert\mathbf{P}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})} &= \sup_{\omega\in[0,2\pi]}\Vert\mathbf{P}(e^{i\omega})\Vert_2 = \sup_{\omega\in[0,2\pi]}\Vert e^{-i\omega}\mathbf{P}(e^{-i\omega})\Vert_2 \\ &= \Vert\mathbf{G}_0\Vert_{\mathbb{h}_\infty}\le 2(\theta_1+\ldots+\theta_{n_\infty}).\end{aligned}$$

Hence, $\Vert\mathbf{G}\Vert_{\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})}\le 2(\varsigma_1+\ldots+\varsigma_{n_f}+\theta_1+\ldots+\theta_{n_\infty})\le 2n\Vert\mathbf{G}\Vert_H$. Thus, the $\mathbb{L}_\infty^{p,m}(\mathbb{\Gamma})$-norm, the HSH-norm and the Hankel norm of the asymptotically stable discrete-time descriptor system (2.1) are equivalent.

3 Continuous-time descriptor systems

In this section we consider the continuous-time descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0, \\ y(t) &= Cx(t). \end{aligned} \tag{3.1}$$

Although there are differences between the continuous-time and discrete-time descriptor systems, some linear system concepts are similar. Therefore, to avoid repetition, results for system (3.1) are only listed without proof unless necessary.

It is well known [6, 8] that system (3.1) has a unique continuously differentiable solution $x(t)$ if the pencil $\lambda E - A$ is regular, the input $u(t)$ is ν times continuously differentiable and the initial value x^0 is *consistent*, that is, it satisfies

$$(I - P_r)x^0 = \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(0),$$

where the matrices F_k are as in (2.5). This solution is given by

$$x(t) = \mathcal{F}(t)Ex^0 + \int_0^t \mathcal{F}(t-\tau)Bu(\tau)\,d\tau + \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(t), \quad t \geq 0,$$

where the matrix function $\mathcal{F}(t)$ is the *fundamental solution matrix* of (3.1) defined by

$$\mathcal{F}(t) = T^{-1}\begin{bmatrix} e^{tJ} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \tag{3.2}$$

see [8, 29] for details.

If the initial condition x^0 is inconsistent or the input $u(t)$ is not sufficiently smooth (for example, in most control problems $u(t)$ is only piecewise continuous), then the solution of system (3.1) may have impulsive modes [6, 8]. Such a solution exists in the distributional sense and has the form

$$\begin{aligned} x(t) &= \mathcal{F}(t)Ex^0 + \int_0^t \mathcal{F}(t-\tau)Bu(\tau)\,d\tau \\ &+ \sum_{k=1}^{\nu-1} \delta^{(k-1)}(t)F_{-k}Ex^0 + \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(t), \end{aligned} \tag{3.3}$$

where $\delta(t)$ denote the *Dirac delta function*, $\delta^{(k)}(t)$ and $u^{(k)}(t)$ are distributional derivatives [8, 9]. It follows from (3.3) that if $x^0 \in \operatorname{Ker} E$ and $F_{-k-1}B = 0$ for $k > 0$, then system (3.1) has no impulsive solutions for every piecewise continuous input $u(t)$. Moreover, impulsive solutions in (3.1) do not arise if the pencil $\lambda E - A$ is of index at most one.

Similarly to the standard state space case [15], we define a state transition matrix of the descriptor system (3.1) as follows.

Definition 3.1. A matrix-valued function $\mathcal{T}(t,\tau)$ defined for all t, $\tau \in \mathbb{R}$ is called a *state transition matrix* of the continuous-time descriptor system (3.1) if it satisfies the matrix differential equation

$$E\frac{\partial}{\partial t}\mathcal{T}(t,\tau) = A\mathcal{T}(t,\tau), \qquad \mathcal{T}(\tau,\tau) = P_r,$$

where P_r is the spectral projection as in (2.3).

Using the Weierstrass canonical form (2.2) and representation (3.2), we can show that there exists a unique state transition matrix $\mathcal{T}(t,\tau)$ given by

$$\mathcal{T}(t,\tau) = \mathcal{F}(t-\tau)E. \tag{3.4}$$

In this case a general solution of the homogeneous system $E\dot{x}(t) = Ax(t)$ has the form $x(t) = \mathcal{T}(t,\tau)x(\tau)$.

It immediately follows from (2.2), (3.2) and (3.4) that the state transition matrix $\mathcal{T}(t,\tau)$ satisfies the semigroup property $\mathcal{T}(t,t_0) = \mathcal{T}(t,\tau)\mathcal{T}(\tau,t_0)$ for $t \geq \tau \geq t_0$. Note that if the matrix E is singular, then the state transition matrix $\mathcal{T}(t,\tau)$ is also singular and the zero eigenvalue of $\mathcal{T}(t,\tau)$ is simple for all t and τ. In this case there exists a group pseudoinverse $\mathcal{T}^{\#}(t,\tau)$ of $\mathcal{T}(t,\tau)$, see [7]. It is unique [7] and can be computed as $\mathcal{T}^{\#}(t,\tau) = \mathcal{T}(\tau,t)$.

3.1 The transfer function

Consider the *Laplace transform* of a function $f(t)$, $t \in \mathbb{R}$, given by

$$\mathbf{f}(s) = \mathfrak{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, \tag{3.5}$$

where s is a complex variable called *frequency* in the continuous-time case. A discussion of the convergence region of the integral (3.5) in the complex plane and properties of the Laplace transform may be found in [9, 18]. If we apply the Laplace transform to (3.1), then we obtain that

$$\mathbf{y}(s) = C(sE-A)^{-1}B\mathbf{u}(s) + C(sE-A)^{-1}Ex(0), \tag{3.6}$$

where $\mathbf{u}(s)$ and $\mathbf{y}(s)$ are the Laplace transforms of $u(t)$ and $y(t)$, respectively. The rational matrix-valued function $\mathbf{G}(s) = C(sE-A)^{-1}B$ is called the *transfer function* of the continuous-time descriptor system (3.1). Equation (3.6) shows that if $Ex(0) = 0$, then $\mathbf{G}(s)$ gives the relation between the Laplace transforms of the input $u(t)$ and the output $y(t)$. In other words, $\mathbf{G}(s)$ describes the input-output behaviour of system (3.1) in the frequency domain.

3.2 Stability

In this subsection we collect some results on the asymptotic stability for the continuous-time descriptor system (3.1).

Definition 3.2. The continuous-time descriptor system (3.1) is called *asymptotically stable* if $\lim\limits_{t\to\infty} x(t) = 0$ for all solutions $x(t)$ of the homogeneous system $E\dot{x}(t) = Ax(t)$.

The following theorem gives equivalent conditions for system (3.1) to be asymptotically stable.

Theorem 3.3. [8, 31] *Consider the continuous-time descriptor system* (3.1) *with a regular pencil $\lambda E - A$. The following statements are equivalent.*

1. *System* (3.1) *is asymptotically stable.*
2. *All finite eigenvalues of the pencil $\lambda E - A$ lie in the open left half-plane.*

3. *The projected generalized continuous-time Lyapunov equation*

$$E^T XA + A^T XE = -P_r^T Q P_r, \qquad X = XP_l \tag{3.7}$$

has a unique Hermitian, positive semidefinite solution X for every Hermitian, positive definite matrix Q.

4. *For all matrices C such that the triplet (E, A, C) is R-observable, the projected generalized continuous-time Lyapunov equation* (3.7) *with $Q = C^T C$ has a unique solution X which is Hermitian and positive definite on the subspace* $\operatorname{Im} P_l$.

In the sequel, the pencil $\lambda E - A$ will be called *c-stable* if $\lambda E - A$ is regular and all finite eigenvalues of $\lambda E - A$ have negative real part.

3.3 Impulse and frequency responses

An *impulse response* of the continuous-time descriptor system (3.1) is defined via

$$G(t) = \mathfrak{L}^{-1}[\mathbf{G}(s)] = C\mathcal{F}(t)Bh_0(t) + \sum_{k=0}^{\nu-1} CF_{-k-1}B\delta^{(k)}(t), \tag{3.8}$$

where

$$h_0(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0 \end{cases}$$

is the *Heaviside function*. We will show that the impulse response $G(t)$ is the output matrix $Y(t)$ of the matrix system

$$\begin{array}{rcl} E\dot{X}(t) & = & AX(t) + BU(t), \quad EX(0) = 0, \\ Y(t) & = & CX(t) \end{array} \tag{3.9}$$

with the distribution matrix input $U(t) = \delta(t) I_m$. Indeed, the solution of (3.9) has the form

$$X(t) = \int_0^t \mathcal{F}(t-\tau)B\delta(\tau)\, d\tau + \sum_{k=0}^{\nu-1} F_{-k-1}B\delta^{(k)}(t), \qquad t \geq 0,$$

and, hence,

$$Y(t) = C\mathcal{F}(t)Bh_0(t) + \sum_{k=0}^{\nu-1} CF_{-k-1}B\delta^{(k)}(t) = G(t), \qquad t \geq 0.$$

A *frequency response* of the continuous-time descriptor system (3.1) is given by $\mathbf{G}(i\omega)$, i.e., the values of the transfer function on the imaginary axis. From (3.5) and (3.8) we obtain that

$$\mathbf{G}(i\omega) = \mathfrak{L}[G(t)] = \int_{-\infty}^{\infty} e^{-i\omega t} G(t)\, dt.$$

Therefore, the frequency response $\mathbf{G}(i\omega)$ is just the Fourier transform [9] of the impulse response $G(t)$.

If we take an input function $u(t) = e^{i\omega t}u_0$ with $\omega \in \mathbb{R}$ and $u_0 \in \mathbb{R}^m$, then we get

$$y(t) = \int_{-\infty}^{\infty} G(t-\tau)e^{i\omega\tau}u_0\,d\tau = \left(\int_{-\infty}^{\infty} e^{-i\omega\tau}G(\tau)d\tau\right)\left(e^{i\omega t}u_0\right) = \mathbf{G}(i\omega)\left(e^{i\omega t}u_0\right).$$

Thus, the frequency response gives a transfer relation from the input $u(t) = e^{i\omega t}u_0$ into the output $y(t)$.

Note that the impulse response $G(t)$ and the frequency response $\mathbf{G}(i\omega)$ are input-output invariants of system (3.1). If $E = I$, then $G(t)$ and $\mathbf{G}(i\omega)$ are the classical impulse and frequency responses for standard continuous-time state space systems [1, 17].

3.4 Controllability and observability Gramians

Assume that the pencil $\lambda E - A$ is *c-stable*. Then the *proper controllability Gramian* of the continuous-time descriptor system (3.1) is defined via

$$\mathcal{G}_{cpc} = \int_0^{\infty} \mathcal{F}(t)BB^T\mathcal{F}^T(t)\,dt$$

and the *proper observability Gramian* $\mathcal{G}_{cpo}$ of (3.1) is given by

$$\mathcal{G}_{cpo} = \int_0^{\infty} \mathcal{F}^T(t)C^TC\mathcal{F}(t)\,dt,$$

where $\mathcal{F}(t)$ is the fundamental solution matrix of (3.1), see [3, 29]. The matrix

$$\mathcal{G}_{cic} = \sum_{k=-\nu}^{-1} F_kBB^TF_k^T$$

is the *improper controllability Gramian* of system (3.1) and the matrix

$$\mathcal{G}_{cio} = \sum_{k=-\nu}^{-1} F_k^TC^TCF_k$$

is the *improper observability Gramian* of (3.1). Note that the improper controllability and observability Gramians can be also written as $\mathcal{G}_{cic} = \mathbf{C}_-\mathbf{C}_-^T$ and $\mathcal{G}_{cio} = \mathbf{O}_-^T\mathbf{O}_-$, where $\mathbf{C}_-$ and $\mathbf{O}_-$ are as in (2.16). In summary, the *controllability Gramian* of the continuous-time descriptor system (3.1) is defined by $\mathcal{G}_{cc} = \mathcal{G}_{cpc} + \mathcal{G}_{cic}$ and the *observability Gramian* of the continuous-time descriptor system (3.1) is defined by $\mathcal{G}_{co} = \mathcal{G}_{cpo} + \mathcal{G}_{cio}$.

Theorem 3.4. *Consider the continuous-time descriptor system* (3.1)*, where the pencil* $\lambda E - A$ *is c-stable.*

1. *The proper controllability and observability Gramians of* (3.1) *are the unique symmetric, positive semidefinite solutions of the projected generalized continuous-time Lyapunov equations*
$$\begin{aligned} &E\mathcal{G}_{cpc}A^T + A\mathcal{G}_{cpc}E^T = -P_lBB^TP_l^T, \\ &\mathcal{G}_{cpc} = P_r\mathcal{G}_{cpc} \end{aligned} \tag{3.10}$$
and
$$\begin{aligned} &E^T\mathcal{G}_{cpo}A + A^T\mathcal{G}_{cpo}E = -P_r^TC^TCP_r, \\ &\mathcal{G}_{cpo} = \mathcal{G}_{cpo}P_l, \end{aligned} \tag{3.11}$$
respectively.

2. *The improper controllability and observability Gramians of* (3.1) *are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations*
$$\begin{aligned} &A\mathcal{G}_{cic}A^T - E\mathcal{G}_{cic}E^T = (I-P_l)BB^T(I-P_l)^T, \\ &P_r\mathcal{G}_{cic}P_r^T = 0 \end{aligned} \tag{3.12}$$
and
$$\begin{aligned} &A^T\mathcal{G}_{cio}A - E^T\mathcal{G}_{cio}E = (I-P_r)^TC^TC(I-P_r), \\ &P_l^T\mathcal{G}_{cio}P_l = 0, \end{aligned} \tag{3.13}$$
respectively.

Proof. See [29, 31] for details. □

Unfortunately, it is not clear whether the controllability and observability Gramians $\mathcal{G}_{cc}$ and $\mathcal{G}_{co}$ can be expressed as solutions of equations of Lyapunov type.

As in the discrete-time case, one can show that the matrices
$$\Phi_c = \mathcal{G}_{cpc}E^T\mathcal{G}_{cpo}E \qquad \text{and} \qquad \Psi_c = \mathcal{G}_{cic}A^T\mathcal{G}_{cio}A$$
are diagonalizable and have non-negative eigenvalues, see also [29].

Definition 3.5. Let the pencil $\lambda E - A$ be c-stable and let n_f and n_∞ be the dimensions of the deflating subspaces of $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The square roots of the n_f largest eigenvalues of the matrix Φ_c, denoted by ς_j, are called the *proper Hankel singular values* of the c-stable continuous-time descriptor system (3.1). The square roots of the n_∞ largest eigenvalues of the matrix Ψ_c, denoted by θ_j, are called the *improper Hankel singular values* of system (3.1).

The proper and improper Hankel singular values are input-output invariants of system (3.1). For $E = I$, the proper Hankel singular values are the classical Hankel singular values of standard continuous-time state space systems [11].

The continuous-time counterparts to Lemma 2.14 and Corollary 2.15 can also be stated for system (3.1).

Lemma 3.6. *Consider the continuous-time descriptor system* (3.1), *where the pencil* $\lambda E - A$ *is c-stable. Let*
$$\begin{aligned} \mathcal{G}_{cpc} &= R_pR_p^T, \qquad & \mathcal{G}_{cpo} &= L_p^TL_p, \\ \mathcal{G}_{cic} &= R_iR_i^T, \qquad & \mathcal{G}_{cio} &= L_i^TL_i. \end{aligned} \tag{3.14}$$
be full rank factorizations of the proper and improper controllability and obsevability Gramians of (3.1). *Then the non-zero proper and improper Hankel singular values of* (3.1) *are the non-zero singular values of the matrices* L_pER_p *and* L_iAR_i, *respectively.*

Corollary 3.7. *Consider the continuous-time descriptor system* (3.1), *where the pencil* λE–A *is c-stable.*

1. *All proper Hankel singular values of* (3.1) *are non-zero if and only if the triplet* (E, A, B) *is R-controllable and the triplet* (E, A, C) *is R-observable.*

2. *All improper Hankel singular values of* (3.1) *are non-zero if and only if relations* (2.24) *hold.*

3. *All proper and imprioper Hankel singular values of* (3.1) *are non-zero if and only if the triplet* (E, A, B) *is C-controllable and the triplet* (E, A, C) *is C-observable.*

3.5 System norms

In this subsection we present convolution and Hankel operators for the continuous-time descriptor system (3.1). Moreover, we introduce system norms for (3.1) and establish their connection with the frequency response $G(t)$, the controllability and observability Gramians, the matrices Φ_c and Ψ_c, the convolution and Hankel operators as well as the Hankel singular values of (3.1).

3.5.1 $\mathbb{L}_2^{p,m}(i\mathbb{R})$-norm, $\mathbb{H}_2$-norm and $\mathbb{HL}_2$-norm

Let $\mathbb{L}_2^{p,m}(i\mathbb{R})$ be the Hilbert space of matrix-valued functions $\mathbf{F} : i\mathbb{R} \longrightarrow \mathbb{C}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(i\mathbb{R})$-norm

$$\|\mathbf{F}\|_{\mathbb{L}_2^{p,m}(i\mathbb{R})} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}\Big(\mathbf{F}^*(i\omega)\mathbf{F}(i\omega)\Big)\, d\omega \right)^{1/2}.$$

By definition, the subspace $\mathbb{H}_2$ of $\mathbb{L}_2^{p,m}(i\mathbb{R})$ consists of all strictly proper rational transfer functions that are analytic in the closed right half-plane. The $\mathbb{H}_2$-*norm* of a transfer fiction $\mathbf{G}(s) \in \mathbb{H}_2$ is defined by

$$\|\mathbf{G}\|_{\mathbb{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{tr}\Big(\mathbf{G}^*(i\omega)\mathbf{G}(i\omega)\Big)\, d\omega \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathbf{G}(i\omega)\|_F^2\, d\omega \right)^{1/2}.$$

If the pencil $\lambda E - A$ is c-stable and the transfer function $\mathbf{G}(s)$ of (3.1) is strictly proper, then $\mathbf{G}(s) \in \mathbb{H}_2$. However, the condition $\mathbf{G}(s) \in \mathbb{H}_2$ does not imply that $\lambda E - A$ is c-stable.

Example 3.8. Let

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad C = [\,1,\ 1,\ 0\,].$$

Then $\mathbf{G}(s) = C(sE - A)^{-1}B = (s+1)^{-1} \in \mathbb{H}_2$, but the pencil $\lambda E - A$ is not c-stable.

It should be noted that the improper transfer function $\mathbf{G}(s)$ does not belong to the space $\mathbb{L}_2^{p,m}(i\mathbb{R})$ even if the pencil $\lambda E - A$ is c-stable.

Consider an additive decomposition of the transfer function $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where

$$\mathbf{G}_{sp}(s) = \sum_{k=1}^{\infty} M_k s^{-k} \qquad \text{and} \qquad \mathbf{P}(s) = \sum_{k=-\nu+1}^{0} M_k s^{-k} \tag{3.15}$$

are, respectively, the strictly proper part and the polynomial part of $\mathbf{G}(s)$, and $M_k = CF_{k-1}B$ are the coefficients of the Laurent expansion at infinity for $\mathbf{G}(s)$. The matrices M_k are called the *Markov parameters* of the descriptor system (3.1). We denote by $\mathbb{HL}_2$ the space of transfer functions $\mathbf{G}(s)$ such that $\mathbf{G}_{sp}(s) \in \mathbb{H}_2$. The $\mathbb{HL}_2$-*norm* of the transfer function $\mathbf{G}(s) \in \mathbb{HL}_2$ is defined via

$$\|\mathbf{G}\|_{\mathbb{HL}_2} = \sqrt{\|\mathbf{G}_{sp}\|_{\mathbb{H}_2}^2 + \|\mathbf{P}\|_{\mathbb{L}_2^{p,m}(\Gamma)}^2},$$

where $\|\cdot\|_{\mathbb{L}_2^{p,m}(\Gamma)}$ is as in (2.25).

Let $\mathbb{I}$ denote either $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^- = (-\infty, 0)$ or $\mathbb{R}_0^+ = [0, \infty)$. Consider the Hilbert space $\mathbb{L}_2^{p,m}(\mathbb{I})$ of matrix-valued functions $F : \mathbb{I} \longrightarrow \mathbb{R}^{p,m}$ that have bounded $\mathbb{L}_2^{p,m}(\mathbb{I})$-norm

$$\|F\|_{\mathbb{L}_2^{p,m}(\mathbb{I})} = \left(\int_{\mathbb{I}} \mathrm{tr}\Big(F^T(t) F(t) \Big)\, dt \right)^{1/2} = \left(\int_{\mathbb{I}} \|F(t)\|_F^2\, dt \right)^{1/2}.$$

Note that the space $\mathbb{L}_2^{p,m}(\mathbb{R}_0^+)$ is isomorphic to $\mathbb{H}_2$ under the Fourier transform [35]. Setting $G_{sp}(t) = \mathfrak{L}^{-1}[\mathbf{G}_{sp}(s)] = C\mathcal{F}(t)Bh_0(t)$, we have from Parseval's identity [26] that

$$\|\mathbf{G}_{sp}\|_{\mathbb{H}_2} = \|G_{sp}\|_{\mathbb{L}_2^{p,m}(\mathbb{R}_0^+)} = \left(\int_0^\infty \|G_{sp}(t)\|_F^2\, dt \right)^{1/2}.$$

Furthermore, taking into account the fact that $\|\mathbf{P}\|^2_{\mathbb{L}_2^{p,m}(\Gamma)} = \sum\limits_{k=1}^{\nu} \|M_{-k+1}\|_F^2$, we get

$$\|\mathbf{G}\|^2_{\mathbb{HL}_2} = \int_0^\infty \|G_{sp}(t)\|_F^2\, dt + \sum_{k=1}^{\nu} \|M_{-k+1}\|_F^2.$$

The following relations can be used to compute the $\mathbb{HL}_2$-norm of the transfer function $\mathbf{G}(s) = C(sE - A)^{-1}B$ with the c-stable pencil $\lambda E - A$. We have

$$\begin{aligned} \|\mathbf{G}_{sp}\|^2_{\mathbb{H}_2} &= \int_0^\infty \mathrm{tr}\Big(B^T \mathcal{F}^T(t) C^T C \mathcal{F}(t) B \Big)\, dt = \int_0^\infty \mathrm{tr}\Big(C\mathcal{F}(t) B B^T \mathcal{F}^T(t) C^T \Big)\, dt \\ &= \mathrm{tr}\left(B^T \mathcal{G}_{cpo} B \right) = \mathrm{tr}\left(C \mathcal{G}_{cpc} C^T \right). \end{aligned}$$

As in the discrete-time case, we obtain that $\|\mathbf{P}\|^2_{\mathbb{L}_2^{p,m}(\Gamma)} = \mathrm{tr}\left(B^T \mathcal{G}_{cio} B \right) = \mathrm{tr}\left(C \mathcal{G}_{cic} C^T \right)$ and, hence,

$$\|\mathbf{G}\|^2_{\mathbb{HL}_2} = \mathrm{tr}\left(B^T \mathcal{G}_{co} B \right) = \mathrm{tr}\left(C \mathcal{G}_{cc} C^T \right) = \|LB\|_F^2 = \|CR\|_F^2,$$

where R and L are the full rank factors of the controllability and observability Gramians $\mathcal{G}_{cc} = RR^T$ and $\mathcal{G}_{co} = L^T L$. These factors can be computed from the QR-factorizations

$$\begin{bmatrix} R_p^T \\ R_i^T \end{bmatrix} = Q_R \begin{bmatrix} R^T \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} L_p \\ L_i \end{bmatrix} = Q_L \begin{bmatrix} L \\ 0 \end{bmatrix},$$

where Q_R and Q_L are orthogonal, R^T and L have full row rank [12], and R_p, R_i, L_p and L_i are the full rank factors as in (3.14).

Thus, we have the following algorithm for computing the $\mathbb{HL}_2$-norm of $\mathbf{G}(s)$ using, for example, the proper and improper observability Gramians.

Algorithm 3.9. *Computing the $\mathbb{HL}_2$-norm of the transfer function $\mathbf{G}(s)$.*
Input: *A realization $\mathbf{G} = [\,E,\, A,\, B,\, C\,]$, where the pencil $\lambda E - A$ is c-stable.*
Output: *The $\mathbb{HL}_2$-norm of the transfer function $\mathbf{G}(s) = C(sE - A)^{-1}B$.*
1. *Use the generalized Schur-Hammarling method* [29, 30] *to compute the full rank factors L_p and L_i of the proper and improper observability Gramians $\mathcal{G}_{cpo} = L_p^T L_p$ and $G_{cio} = L_i^T L_i$ that satisfy the projected generalized Lyapunov equations* (3.11) *and* (3.13)*, respectively.*
2. *Compute the QR-factorization* $\begin{bmatrix} L_p \\ L_i \end{bmatrix} = Q_L \begin{bmatrix} L \\ 0 \end{bmatrix}$.
3. *Compute* $\|\mathbf{G}\|_{\mathbb{HL}_2} = \|LB\|_F$.

3.5.2 $\mathbb{L}_\infty^{p,m}(i\mathbb{R})$-norm, $\mathbb{H}_\infty$-norm and $\mathbb{HL}_\infty$-norm

Let $\mathbb{L}_\infty^{p,m}(i\mathbb{R})$ be the Banach space of matrix-valued functions that are (essentially) bounded on $i\mathbb{R}$. The subspace of $\mathbb{L}_\infty^{p,m}(i\mathbb{R})$ denoted by $\mathbb{H}_\infty$ consists of all proper rational transfer functions that are analytic and bounded in the closed right half-plane. The $\mathbb{H}_\infty$-*norm* of $\mathbf{G}(s) \in \mathbb{H}_\infty$ is defined via

$$\|\mathbf{G}\|_{\mathbb{H}_\infty} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\|\mathbf{u}\|_{\mathbb{L}_2^m(i\mathbb{R})}} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(\mathbf{G}(i\omega)) = \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(i\omega)\|_2.$$

We denote by $\mathbb{HL}_\infty$ the space of all rational transfer functions $\mathbf{G}(s)$ with the proper part $\mathbf{G}_p(s) = \mathbf{G}_{sp}(s) + M_0 \in \mathbb{H}_\infty$. Let $\mathbb{L}_{2,l}^m(i\mathbb{R})$ be the space of all vector-valued functions $\mathbf{f} : i\mathbb{R} \longrightarrow \mathbb{C}^m$ that have bounded $\mathbb{L}_{2,l}^m(i\mathbb{R})$-norm

$$\|\mathbf{f}\|_{\mathbb{L}_{2,l}^m(i\mathbb{R})} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{k=0}^{l} |\omega|^{2k} \right) \|\mathbf{f}(i\omega)\|^2 d\omega \right)^{1/2}.$$

The $\mathbb{HL}_\infty$-*norm* of the transfer function $\mathbf{G}(s) \in \mathbb{HL}_\infty$ is defined via

$$\|\mathbf{G}\|_{\mathbb{HL}_\infty} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\|\mathbf{u}\|_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}}.$$

The following lemma gives an upper bound on the $\mathbb{HL}_\infty$-norm of $\mathbf{G}(s)$.

Lemma 3.10. *Consider the transfer function* $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, *where* $\mathbf{G}_{sp}(s)$ *and* $\mathbf{P}(s)$ *are as in* (3.15). *We have*

$$\|\mathbf{G}\|_{\mathbb{HL}_\infty} \le \left(\|\mathbf{G}_p\|_{\mathbb{H}_\infty}^2 + \sum_{k=1}^{\nu-1} \|M_{-k}\|_2^2 \right)^{1/2}. \tag{3.16}$$

Proof. For any $\mathbf{u} \in \mathbb{L}_{2,\nu-1}^m(i\mathbb{R})$, we obtain

$$\begin{aligned}
\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \left(\mathbf{G}_{sp}(i\omega) + M_0 + M_{-1}i\omega + \ldots + M_{-\nu+1}(i\omega)^{\nu-1} \right) \mathbf{u}(i\omega)\|^2 d\omega \\
&\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\|\mathbf{G}_{sp}(i\omega) + M_0\|_2^2 + \sum_{k=1}^{\nu-1} \|M_{-k}\|_2^2 \right) \sum_{k=0}^{\nu-1} |\omega|^{2k} \|\mathbf{u}(i\omega)\|^2 d\omega \\
&\le \left(\|\mathbf{G}_p\|_{\mathbb{H}_\infty}^2 + \sum_{k=1}^{\nu-1} \|M_{-k}\|_2^2 \right) \|\mathbf{u}\|_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}^2.
\end{aligned}$$

Thus, estimate (3.16) holds. □

Note that if the transfer function $\mathbf{G}(s) = \mathbf{G}_p(s)$ is proper, then we have the equality in (3.16).

For the continuous-time descriptor system (3.1), we consider a *convolution operator* $\mathcal{K}_c$ that maps the input $u(t)$ into the output $y(t)$ and is defined by

$$y(t) = (\mathcal{K}_c u)(t) = (G * u)(t) = \int_{-\infty}^{\infty} G(t-\tau)u(\tau)\, d\tau. \tag{3.17}$$

The convolution operator $\mathcal{K}_c$ describes the input-output behavior of the descriptor system (3.1) in the time domain. Substituting (3.8) in (3.17) and taking into account that

$$\int_{-\infty}^{\infty} \delta^{(k)}(t-\tau)u(\tau)\,d\tau = u^{(k)}(t), \qquad k = 0, 1, \ldots,$$

where $u^{(k)}(t)$ are the distributional derivatives, we find that

$$(\mathcal{K}_c u)(t) = \int_{-\infty}^{t} C\mathcal{F}(t-\tau)Bu(\tau)\,d\tau + \sum_{k=0}^{\nu-1} CF_{-k-1}Bu^{(k)}(t).$$

Let $\mathbb{L}_{2,l}^m(\mathbb{R})$ be the Sobolev space consisting of vector-valued functions $f : \mathbb{R} \longrightarrow \mathbb{R}^m$ such that $f^{(k)}(t) \in \mathbb{L}_2^m(\mathbb{R})$ for $k = 0, 1, \ldots, l$. The $\mathbb{L}_{2,l}^m(\mathbb{R})$-norm is defined via

$$\|f\|_{\mathbb{L}_{2,l}^m(\mathbb{R})} = \left(\sum_{k=0}^{l} \|f^{(k)}\|^2_{\mathbb{L}_2^m(\mathbb{R})} \right)^{1/2}.$$

If the pencil $\lambda E - A$ is c-stable, then $\mathcal{K}_c$ is the bounded operator mapping $\mathbb{L}_{2,\nu-1}^m(\mathbb{R})$ into $\mathbb{L}_2^p(\mathbb{R})$. In this case the spectral norm of the convolution operator $\mathcal{K}_c$ is given by

$$\|\mathcal{K}_c\|_2 = \sup_{u\neq 0} \frac{\|\mathcal{K}_c u\|_{\mathbb{L}_2^p(\mathbb{R})}}{\|u\|_{\mathbb{L}_{2,\nu-1}^m(\mathbb{R})}}.$$

Using the Fourier transform [18] the time domain relation $y(t) = (\mathcal{K}_c u)(t)$ is expressed in the frequency domain via $\mathbf{y}(i\omega) = \mathbf{G}(i\omega)\mathbf{u}(i\omega)$. Since the Fourier transform gives an isometric isomorphism between the Sobolev space $\mathbb{L}_{2,\nu-1}^m(\mathbb{R})$ and the space $\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})$, we obtain by Parseval's identity [26] that

$$\|\mathcal{K}_c\|_2 = \sup_{u\neq 0} \frac{\|G * u\|_{\mathbb{L}_2^p(\mathbb{R})}}{\|u\|_{\mathbb{L}_{2,\nu-1}^m(\mathbb{R})}} = \sup_{\mathbf{u}\neq 0} \frac{\|\mathbf{G}\mathbf{u}\|_{\mathbb{L}_2^p(i\mathbb{R})}}{\|\mathbf{u}\|_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}} = \|\mathbf{G}\|_{\mathbb{HL}_\infty}.$$

The $\mathbb{H}_\infty$-norm of the proper transfer function $\mathbf{G}(s)$ can be computed by using the midpoint rule [4] or the cubic interpolation method [10] that are based on the fact that $\|\mathbf{G}\|_{\mathbb{H}_\infty} < \gamma$ for some $\gamma > 0$ if and only if a matrix pencil

$$\lambda E_\gamma - A_\gamma = \lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & \gamma^{-1}BB^T \\ -\gamma^{-1}C^TC & -A^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis. These quadraticaly convergent iterative methods provide lower and upper bounds on the $\mathbb{H}_\infty$-norm of proper $\mathbf{G}(s)$, see [4, 10] for details.

Computing the $\mathbb{HL}_\infty$-norm of the improper transfer function $\mathbf{G}(s)$ is still an open problem.

3.5.3 Hilbert-Schmidt-Hankel norm

Assume that the pencil $\lambda E - A$ is c-stable. The *Hilbert-Schmidt-Hankel norm* or *HSH-norm* of the transfer function $\mathbf{G}(s)$ is given by

$$\begin{aligned}\|\mathbf{G}\|_{HSH} &= \left(\int_0^\infty\!\!\int_0^\infty \mathrm{tr}\Big(G_{sp}^T(t+\tau)G_{sp}(t+\tau)\Big)\,dt\,d\tau + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{tr}\Big(M_{-j-k}^T M_{-j-k}\Big) \right)^{1/2} \\ &= \left(\int_0^\infty\!\!\int_0^\infty \|G_{sp}(t+\tau)\|_F^2\,dt\,d\tau + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \|M_{-j-k}\|_F^2 \right)^{1/2},\end{aligned}$$

where $G_{sp}(t) = C\mathcal{F}(t)Bh_0(t)$ and $M_{-k} = CF_{-k-1}B$. Note that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|M_{-j-k}\|_F^2 = \sum_{j=0}^{\nu-1}\sum_{k=j}^{\nu-1}\|M_{-k}\|_F^2 = \sum_{k=1}^{\nu}k\|M_{-k+1}\|_F^2.$$

Furthermore, taking into account that $\mathcal{F}(t+\tau) = \mathcal{F}(t)E\mathcal{F}(\tau)$, we have

$$\int_0^{\infty}\int_0^{\infty}\|G_{sp}(t+\tau)\|_F^2\,dt\,d\tau = \operatorname{tr}\Bigl(\mathcal{G}_{cpc}E^T\mathcal{G}_{cpo}E\Bigr) = \operatorname{tr}(\Phi_c).$$

Using the Weierstrass canonical form (2.2) and representations (2.5), we get

$$F_{-j-k-1} = -F_{-j-1}AF_{-k-1}, \qquad j,k \geq 0. \tag{3.18}$$

In this case we have

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|M_{-j-k}\|_F^2 = \operatorname{tr}\Bigl(\mathcal{G}_{cic}A^T\mathcal{G}_{cio}A\Bigr) = \operatorname{tr}(\Psi_c).$$

Hence,

$$\|\mathbf{G}\|_{HSH} = \sqrt{\operatorname{tr}(\Phi_c+\Psi_c)} = \sqrt{\varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2}, \tag{3.19}$$

where ς_j and θ_j are the proper and improper Hankel singular values of (3.1).

For the continuous-time descriptor system (3.1), we define a *proper Hankel operator* $\mathcal{H}_p$ transforming the past inputs $u_-(t)$ ($u_-(t) = 0$ for $t \geq 0$) into the present and future outputs $y_+(t)$ ($y_+(t) = 0$ for $t < 0$) through the state $x(0) \in \operatorname{Im} P_r$ via

$$y_+(t) = (\mathcal{H}_p u_-)(t) = \int_{-\infty}^{0} G_{sp}(t-\tau)u_-(\tau)\,d\tau, \qquad t \geq 0. \tag{3.20}$$

If the pencil $\lambda E - A$ is c-stable, then $\mathcal{H}_p$ acts from $\mathbb{L}_2^m(\mathbb{R}^-)$ into $\mathbb{L}_2^p(\mathbb{R}_0^+)$.

The following lemma gives a connection between the proper Hankel singular values and the singular values of the proper Hankel operator.

Lemma 3.11. *Consider the descriptor system* (3.1)*, where the pencil* $\lambda E - A$ *is c-stable. The non-zero proper Hankel singular values* ς_j *of* (3.1) *are the non-zero singular values of the proper Hankel operator* $\mathcal{H}_p$.

Proof. Compute the adjoint operator $\mathcal{H}_p^*$ of the proper Hankel operator $\mathcal{H}_p$ that satisfies $\langle\mathcal{H}_p u, y\rangle_{\mathbb{L}_2^p(\mathbb{R}_0^+)} = \langle u, \mathcal{H}_p^* y\rangle_{\mathbb{L}_2^m(\mathbb{R}^-)}$ for any $u(t) \in \mathbb{L}_2^m(\mathbb{R}^-)$ and $y(t) \in \mathbb{L}_2^p(\mathbb{R}_0^+)$. We have

$$\begin{aligned}\langle\mathcal{H}_p u, y\rangle_{\mathbb{L}_2^p(\mathbb{R}_0^+)} &= \int_0^{\infty}\int_{-\infty}^{0} u^T(\tau)B^T\mathcal{F}^T(t-\tau)C^Ty(t)\,d\tau\,dt \\ &= \int_{-\infty}^{0} u^T(\tau)\int_0^{\infty} B^T\mathcal{F}^T(t-\tau)C^Ty(t)\,dt\,d\tau = \langle u, \mathcal{H}_p^* y\rangle_{\mathbb{L}_2^m(\mathbb{R}^-)},\end{aligned}$$

where

$$(\mathcal{H}_p^* y)(\tau) = \int_0^{\infty} B^T\mathcal{F}^T(t-\tau)C^Ty(t)\,dt.$$

Let $\sigma \neq 0$ be a singular value of $\mathcal{H}_p$ and let $u(t) \in \mathbb{L}_2^m(\mathbb{R}^-)$ be a non-zero right singular vector corresponding to σ. Then

$$\begin{aligned}\sigma^2 u(t) &= (\mathcal{H}_p^T \mathcal{H}_p u)(t) \\ &= \int_0^\infty \int_{-\infty}^0 B^T \mathcal{F}^T(\tau - t) C^T C \mathcal{F}(\tau - \xi) B u(\xi)\, d\xi\, d\tau \\ &= \int_0^\infty \int_{-\infty}^0 B^T \mathcal{F}^T(-t) E^T \mathcal{F}^T(\tau) C^T C \mathcal{F}(\tau) E \mathcal{F}(-\xi) B u(\xi)\, d\xi\, d\tau.\end{aligned} \tag{3.21}$$

It follows from (3.21) that

$$v = \int_{-\infty}^0 \mathcal{F}(-\xi) B u(\xi)\, d\xi \neq 0.$$

Multiplying (3.21) from the left by $\mathcal{F}(-t)B$ and integrating on $\mathbb{R}^-$ gives

$$\begin{aligned}\sigma^2 v &= \int_{-\infty}^0 \int_0^\infty \mathcal{F}(-t) B B^T \mathcal{F}^T(-t) E^T \mathcal{F}^T(\tau) C^T C \mathcal{F}(\tau) E v\, d\tau\, dt \\ &= \left(\int_0^\infty \mathcal{F}(t) B B^T \mathcal{F}^T(t) dt \right) E^T \left(\int_0^\infty \mathcal{F}^T(\tau) C^T C \mathcal{F}(\tau) d\tau \right) E v \\ &= \mathcal{G}_{cpc} E^T \mathcal{G}_{cpo} E v = \Phi_c v,\end{aligned} \tag{3.22}$$

i.e., v is an eigenvector of the matrix Φ_c corresponding to the eigenvalue σ^2.

On the other hand, consider an eigenvalue $\sigma^2 \neq 0$ and a corresponding eigenvector $v \neq 0$ of the matrix Φ_c. Then (3.22) holds. Set

$$u(\tau) = \int_0^\infty B^T \mathcal{F}^T(\xi - \tau) C^T C \mathcal{F}(\xi) E v\, d\xi, \qquad \tau < 0.$$

Clearly, $u \neq 0$ and $u(\tau) \in \mathbb{L}_2^m(\mathbb{R}^-)$. If we multiply (3.22) from the left by the matrix

$$\int_0^\infty B^T \mathcal{F}^T(\xi - \tau) C^T C \mathcal{F}(\xi) E d\xi,$$

then we obtain that

$$\sigma^2 u(\tau) = \int_0^\infty \int_{-\infty}^0 B^T \mathcal{F}^T(\xi - \tau) C^T C \mathcal{F}(\xi - t) B u(t)\, dt\, d\xi = (\mathcal{H}_p^* \mathcal{H}_p u)(\tau).$$

Since the proper Hankel operator of the asymptotically stable system (3.1) is the Hilbert-Schmidt operator [34], it is compact. In this case $\mathcal{H}_p$ has a discrete set of non-zero singular values and they coincide with the square roots of non-zero eigenvalues of the matrix Φ_c that are, in fact, the non-zero proper Hankel singular values. □

The Frobenius norm and the spectral norm of the proper Hankel operator $\mathcal{H}_p$ are given by $\|\mathcal{H}_p\|_F = \sqrt{\varsigma_1^2 + \ldots + \varsigma_{n_f}^2}$ and $\|\mathcal{H}_p\|_2 = \varsigma_1$, respectively, where the proper Hankel singular values ς_j are ordered decreasingly.

The proper Hankel operator is closely related to a proper Hankel matrix given by

$$\mathbf{H}_p = \begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \cdots \\ M_3 & M_4 & M_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Consider the Taylor series expansion of $G_{sp}(t)$ at $t = 0^+$ given by

$$G_{sp}(t) = \sum_{k=0}^{\infty} CF_k B \, \frac{t^k}{k!} = \sum_{k=0}^{\infty} M_{k+1} \frac{t^k}{k!}. \tag{3.23}$$

If we substitute (3.23) in (3.20), then we obtain

$$y_+(t) = \sum_{k=0}^{\infty} M_{k+1} \int_{-\infty}^{0} \frac{(t-\tau)^k}{k!} u_-(\tau)\, d\tau = \sum_{k=0}^{\infty} M_{k+1} \mathcal{J}_k[u_-](t), \quad t \geq 0,$$

where

$$\begin{aligned} \mathcal{J}_k[u_-](t) &= \int_{-\infty}^{0} \frac{(t-\tau)^k}{k!} u_-(\tau) d\tau \\ &= \int_{-\infty}^{t} \int_{-\infty}^{\tau_{k+1}} \ldots \int_{-\infty}^{\tau_2} u_-(\tau_1)\, d\tau_1 \ldots d\tau_k d\tau_{k+1}, \quad k = 0, 1, \ldots . \end{aligned}$$

The j-th derivative of $y_+(t)$ is computed as

$$y_+^{(j)}(t) = \sum_{k=0}^{\infty} M_{k+j+1} \mathcal{J}_k[u_-](t)$$

and, hence,

$$\begin{bmatrix} y_+(t) \\ \dot{y}_+(t) \\ \ddot{y}_+(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \cdots \\ M_3 & M_4 & M_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \mathcal{J}_0[u_-](t) \\ \mathcal{J}_1[u_-](t) \\ \mathcal{J}_2[u_-](t) \\ \vdots \end{bmatrix} = \mathbf{H}_p \mathbf{I}[u_-](t).$$

For some $\tau > 0$ there exists an interval $(a, b) \subset [0, \infty)$ such that $\tau \in (a, b)$ and for all $t \in (a, b)$ we have

$$y_+(t) = \sum_{j=0}^{\infty} y_+^{(j)}(\tau) \, \frac{(t-\tau)^j}{j!} = \mathbf{T}(t, \tau) \mathbf{H}_p \mathbf{I}[u_-](\tau),$$

where $\mathbf{T}(t, \tau) = [\,1,\ t-\tau,\ (t-\tau)^2/(2!),\ \ldots\,] \otimes I_p$ and $\otimes$ denotes the Kronecker product.

Remark 3.12. It should be noted that the proper Hankel singular values of the continuous-time descriptor system (3.1) are not equal to the singular values of the proper Hankel matrix $\mathbf{H}_p$. However, as the following lemma shows, the non-zero improper Hankel singular values coincide with the classical non-zero singular values of the improper Hankel matrix

$$\mathbf{H}_i = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & M_{-4} & M_{-3} & M_{-2} \\ \cdots & M_{-3} & M_{-2} & M_{-1} \\ \cdots & M_{-2} & M_{-1} & M_0 \end{bmatrix}$$

associated with system (3.1).

Lemma 3.13. *The non-zero improper Hankel singular values of the continuous-time descriptor system* (3.1) *are the non-zero singular values of the improper Hankel matrix* $\mathbf{H}_i$.

Proof. Consider the improper controllability and observability matrices $\mathbf{C}_-$ and $\mathbf{O}_-$ as in (2.16). Using (3.18) we obtain that $\mathbf{O}_- A\mathbf{C}_- = -\mathbf{H}_i$. Then by the definition of the improper Hankel singular values we have $\theta_j^2 = \lambda_j(\mathbf{C}_-\mathbf{C}_-^T A^T \mathbf{O}_-^T \mathbf{O}_- A) = \sigma_j^2(-\mathbf{O}_- A\mathbf{C}_-) = \sigma_j^2(\mathbf{H}_i)$. $\square$

As a consequence of Lemmas 3.6, 3.11 and 3.13 we obtain from (3.19) that

$$\|\mathbf{G}\|_{HSH}^2 = \|\mathcal{H}_p\|_F^2 + \|\mathbf{H}_i\|_F^2 = \|L_p E R_p\|_F^2 + \|L_i A R_i\|_F^2,$$

where R_p, L_p, R_i and L_i are as in (3.14).

3.5.4 Hankel norm

Let the pencil $\lambda E - A$ be c-stable. The *Hankel norm* of the transfer function $\mathbf{G}(s)$ is defined by $\|\mathbf{G}\|_H = \max(\|\mathcal{H}_p\|_2, \|\mathbf{H}_i\|_2) = \max(\varsigma_1, \theta_1)$, where ς_1 and θ_1 are the largest proper and improper Hankel singular values of the descriptor system (3.1). From the definition of the Hankel singular values and Lemma 3.6 we find that

$$\|\mathbf{G}\|_H = \sqrt{\lambda_{\max}(\Phi_c + \Psi_c)} = \max(\|L_p E R_p\|_2, \|L_i A R_i\|_2).$$

To compute the the HSH-norm and the Hankel norm of the transfer function $\mathbf{G}(s)$ we can use the following algorithm.

Algorithm 3.14. *Computing the HSH-norm or the Hankel norm of* $\mathbf{G}(s)$.
Input: *A realization* $\mathbf{G} = [\,E,\ A,\ B,\ C\,]$, *where the pencil* $\lambda E - A$ *is c-stable.*
Output: *The HSH-norm or the Hankel norm of* $\mathbf{G}(s) = C(sE - A)^{-1}B$.
1. *Use the generalized Schur-Hammarling method* [29, 30] *to compute the full rank factors* R_p *and* L_p *of the proper controllability and observability Gramians* $\mathcal{G}_{cpc} = R_p R_p^T$ *and* $G_{cpo} = L_p^T L_p$ *that satisfy the projected generalized continuous-time Lyapunov equations* (3.10) *and* (3.11), *respectively.*
2. *Use the generalized Schur-Hammarling method* [29] *to compute the full rank factors* R_i *and* L_i *of the improper controllability and observability Gramians* $\mathcal{G}_{cic} = R_i R_i^T$ *and* $G_{cio} = L_i^T L_i$ *that satisfy the projected generalized discrete-time Lyapunov equations* (3.12) *and* (3.13), *respectively.*
3. *Compute* $\|\mathbf{G}\|_{HSH} = \sqrt{\|L_p E R_p\|_F^2 + \|L_i A R_i\|_F^2}$ *or* $\|\mathbf{G}\|_H = \max(\|L_p E R_p\|_2, \|L_i A R_i\|_2)$.

We summarize system norms for the asymptotically stable continuous-time descriptor system (3.1) in Table 2.

4 Conclusion

In this paper we have discussed input-output invariants for linear continuous-time and discrete-time descriptor systems. We have generalized for both systems the impulse and frequency responses, convolution and Hankel operators as well Hankel singular values. The latter are useful in balanced truncation model reduction.

Various norms for descriptor systems have been introduced and their different representations in time domain and frequency domain have been given. System norms play an important role in control design and system approximation. We have also discussed the computation of norms for descriptor systems.

$\mathbf{G}(s) = C(sE-A)^{-1}B$ $sE-A$ is c-stable	$\Vert\mathbf{G}\Vert_{\mathbb{HL}_2}$	$\Vert\mathbf{G}\Vert_{\mathbb{HL}_\infty}$
$\mathbf{G}(i\omega)=\mathbf{G}_{sp}(i\omega)+\mathbf{P}(i\omega)$	$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\Vert\mathbf{G}_{sp}(i\omega)\Vert_F^2 d\omega+\frac{1}{2\pi}\int_{0}^{2\pi}\Vert\mathbf{P}(e^{i\omega})\Vert_F^2 d\omega\right)^{\frac{1}{2}}$	$\sup_{\mathbf{u}\neq 0}\frac{\Vert\mathbf{G}\mathbf{u}\Vert_{\mathbb{L}_2^p(i\mathbb{R})}}{\Vert\mathbf{u}\Vert_{\mathbb{L}_{2,\nu-1}^m(i\mathbb{R})}}$
$G_{sp}(t), \quad M_k$	$\left(\int_0^{\infty}\Vert G_{sp}(t)\Vert_F^2 dt+\sum_{k=1}^{\nu}\Vert M_{-k+1}\Vert_F^2\right)^{\frac{1}{2}}$	
$\mathcal{G}_{cc} = RR^T$	$\sqrt{\mathrm{tr}(C\mathcal{G}_{cpc}C^T)} = \Vert CR\Vert_F$	
$\mathcal{G}_{co} = L^TL$	$\sqrt{\mathrm{tr}(B^T\mathcal{G}_{cpo}B)} = \Vert LB\Vert_F$	
$\mathcal{K}_c$		$\Vert\mathcal{K}_c\Vert_2$
$\mathbf{G}(s) = C(sE-A)^{-1}B$ $sE-A$ is c-stable	$\Vert\mathbf{G}\Vert_{HSH}$	$\Vert\mathbf{G}\Vert_{H}$
$G_{sp}(t), \quad M_k$	$\left(\int_0^{\infty}\int_0^{\infty}\Vert G_{sp}(t+\tau)\Vert_F^2 dt d\tau+\sum_{k=1}^{\nu}k\Vert M_{-k+1}\Vert_F^2\right)^{\frac{1}{2}}$	
$\mathcal{H}_p, \quad \mathbf{H}_i$	$\sqrt{\Vert\mathcal{H}_p\Vert_F^2+\Vert\mathbf{H}_i\Vert_F^2}$	$\max(\Vert\mathcal{H}_p\Vert_2,\Vert\mathbf{H}_i\Vert_2)$
$\mathcal{G}_{cpc}=R_pR_p^T,\ \mathcal{G}_{cpo}=L_p^TL_p$ $\mathcal{G}_{cic}=R_iR_i^T,\ \mathcal{G}_{cio}=L_i^TL_i$	$\sqrt{\Vert L_pER_p\Vert_F^2+\Vert L_iAR_i\Vert_F^2}$	$\max(\Vert L_pER_p\Vert_2,\Vert L_iAR_i\Vert_2)$
$\Phi_c, \quad \Psi_c$	$\sqrt{\mathrm{tr}(\Phi_c+\Psi_c)}$	$\sqrt{\lambda_{\max}(\Phi_c+\Psi_c)}$
$\varsigma_1\geq\ldots\geq\varsigma_{n_f}$ $\theta_1\geq\ldots\geq\theta_{n_\infty}$	$\sqrt{\varsigma_1^2+\ldots+\varsigma_{n_f}^2+\theta_1^2+\ldots+\theta_{n_\infty}^2}$	$\max(\varsigma_1,\ \theta_1)$

Table 2: Generalized norms for asymptotically stable continuous-time descriptor systems.

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