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Gramian based model reduction for descriptor systems

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Abstract

Model reduction is of fundamental importance in many control applications. We consider model reduction methods for linear time-invariant continuous-time descriptor systems. These methods are based on the balanced truncation technique and closely related to the controllability and observability Gramians and Hankel singular values of descriptor systems. The Gramians can be computed by solving generalized Lyapunov equations with special right-hand sides. Numerical examples are given.

Key words: Descriptor systems, Gramians, Hankel singular values, model reduction, balanced truncation.

1 Introduction

Consider a linear time-invariant continuous-time system

$$\begin{array}{rcl} E\,\dot{x}(t) & = & A\,x(t) + B\,u(t), \quad x(0) = x^0, \\ y(t) & = & C\,x(t), \end{array} \tag{1.1}$$

where $E,\ A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output and $x^0 \in \mathbb{R}^n$ is the initial value. The number of state variables n is called the *order* of system (1.1). If $I = E$, then (1.1) is a *standard state space system*. Otherwise, (1.1) is a *descriptor system* or *generalized state space system*. Such systems arise naturally in many applications such as multibody dynamics, electrical circuit simulation and semidiscretization of partial differential equations, see [6, 9, 19].

We will assume throughout the paper that the pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. In this case $\lambda E - A$ can be reduced to the Weierstrass canonical form [33]. There exist nonsingular matrices W and T such that

$$E = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} T, \qquad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T, \tag{1.2}$$

where I_k is the identity matrix of order k, J is the Jordan block corresponding to the finite eigenvalues of $\lambda E - A$, N is nilpotent and corresponds to the eigenvalue at infinity. The index of nilpotency of N, denoted by ν, is called the *index* of the pencil $\lambda E - A$. Representation (1.2)

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defines the decomposition of $\mathbb{R}^n$ into complementary deflating subspaces of dimensions n_f and n_∞ corresponding to the finite and infinite eigenvalues of the pencil $\lambda E - A$, respectively. The matrices

$$P_r = T^{-1}\begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T \qquad \text{and} \qquad P_l = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \tag{1.3}$$

are the *spectral projections* onto the right and left deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues. The pencil $\lambda E - A$ is called *c-stable* if it is regular and all the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane.

Descriptor systems arising, e.g., from the spatial discretization of partial differential equation have usually large order n, while the number m of inputs and the number p of outputs are small compared to n. Simulation or real time controller design for such large-scale systems becomes difficult because of storage requirements and expensive computations. In this case model order reduction plays an important role. It consists in an approximation of the descriptor system (1.1) by a reduced order system

$$\begin{aligned} \widetilde{E}\,\dot{\widetilde{x}}(t) &= \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \quad \widetilde{x}(0) = \widetilde{x}^0, \\ \widetilde{y}(t) &= \widetilde{C}\,\widetilde{x}(t), \end{aligned} \tag{1.4}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$ and $\ell \ll n$. Note that systems (1.1) and (1.4) have the same input $u(t)$. We require for the approximate system (1.4) to preserve properties of the original system (1.1) like regularity and stability. It is also desirable for the approximation error to be small. Moreover, the computation of the reduced order system should be numerically stable and efficient.

There exist various model reduction approaches for standard state space systems such as balanced truncation [15, 24, 27, 31, 37, 38], moment matching approximation [1, 14, 18] and optimal Hankel norm approximation [15]. Surveys on system approximation and model reduction can be found in [1, 13]. One of the most effective and well studied model reduction techniques is balanced truncation which is closely related to the two Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T = -BB^T, \qquad A^T\mathcal{Q} + \mathcal{Q}A = -C^TC.$$

The solutions $\mathcal{P}$ and $\mathcal{Q}$ of these equations are called the *controllability* and *observability Gramians*, respectively. The balanced truncation approach consists in transforming the state space system into a balanced form whose the controllability and observability Gramians become diagonal and equal, together with a truncation of states that are both difficult to reach and to observe, see [27] for details. Balanced truncation model reduction for descriptor systems has been considered in [26, 30]. The algorithms presented there are based on computing the Weierstrass canonical form (1.2) of the pencil $\lambda E - A$. However, it is well known [33] that this computational problem is, in general, ill-conditioned in the sense that small perturbations in E and A may lead to an inaccurate numerical result.

In this paper we generalize controllability and observability Gramians as well as Hankel singular values for descriptor systems (Section 2). In Section 3 we present an extension of balanced truncation methods [24, 37, 38] to descriptor systems. These methods are based on computing the generalized Schur form of the pencil $\lambda E - A$ and solving the generalized Sylvester and Lyapunov equations using numerically stable algorithms. Section 4 contains numerical examples.

2 Descriptor systems

Consider the continuous-time descriptor system (1.1). It is well known that if the pencil $\lambda E - A$ is regular, $u(t)$ is ν times continuously differentiable and x^0 is consistent, i.e., it belongs to the *set of consistent initial conditions*

$$\mathcal{X}_0 = \left\{ x^0 \in \mathbb{R}^n \quad : \quad (I - P_r)x^0 = \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(0) \right\},$$

then the descriptor system (1.1) has a unique continuously differentiable solution $x(t)$, see [9], that is given by

$$x(t) = \mathcal{F}(t)Ex^0 + \int_0^t \mathcal{F}(t-\tau)Bu(\tau)\,d\tau + \sum_{k=0}^{\nu-1} F_{-k-1}Bu^{(k)}(t).$$

Here

$$\mathcal{F}(t) = T^{-1} \begin{bmatrix} e^{tJ} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \tag{2.1}$$

is a *fundamental solution matrix* of the descriptor system (1.1), and the matrices F_k have the form

$$F_k = T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix} W^{-1}, \qquad k = -1, -2, \ldots. \tag{2.2}$$

Clearly, $F_k = 0$ for $k < -\nu$.

If the initial condition x^0 is inconsistent or the input $u(t)$ is not sufficiently smooth (for example, in most control problems u(t) is only piecewise continuous), then the solution of the descriptor system (1.1) may have impulsive modes [8, 9].

The rational matrix-valued function $\mathbf{G}(s) = C(sE - A)^{-1}B$ is called the *transfer function* of the descriptor system (1.1). A quadruple of matrices $[\,E,\ A,\ B,\ C\,]$ is a *realization* of $\mathbf{G}(s)$. We will also often denote a realization of $\mathbf{G}(s)$ by

$$\left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array}\right].$$

Two realizations $[\,E,\ A,\ B,\ C\,]$ and $[\,\check{E},\ \check{A},\ \check{B},\ \check{C}\,]$ are *restricted system equivalent* if there exist nonsingular matrices $\check{W}$ and $\check{T}$ such that

$$\left[\begin{array}{c|c} s\check{E} - \check{A} & \check{B} \\ \hline \check{C} & 0 \end{array}\right] = \left[\begin{array}{c|c} s\check{W}E\check{T} - \check{W}A\check{T} & \check{W}B \\ \hline C\check{T} & 0 \end{array}\right].$$

A pair $(\check{W}, \check{T})$ is called *system equivalence transformation.* A characteristic quantity of system (1.1) is *system invariant* if it is preserved under a system equivalence transformation. The transfer function $\mathbf{G}(s)$ is system invariant, since

$$\mathbf{G}(s) = C(sE - A)^{-1}B = \check{C}\check{T}^{-1}\check{T}(s\check{E} - \check{A})^{-1}\check{W}\check{W}^{-1}\check{B} = \check{C}(s\check{E} - \check{A})^{-1}\check{B}.$$

The transfer function $\mathbf{G}(s)$ is called *proper* if $\lim\limits_{s\to\infty} \mathbf{G}(s) < \infty$, and $\mathbf{G}(s)$ is called *strictly proper* if $\lim\limits_{s\to\infty} \mathbf{G}(s) = 0$.

Other important results from the theory of rational functions and realization theory may be found in [9].

2.1 Controllability and observability

For descriptor systems there are various concepts of controllability and observability, e.g., [8, 9, 42].

Definition 2.1. System (1.1) and the triplet (E, A, B) are called *controllable on the reachable set* (*R-controllable*) if

$$\operatorname{rank}\,[\,\lambda E - A,\ B\,] = n \quad \text{for all finite } \lambda \in \mathbb{C}. \tag{2.3}$$

System (1.1) and the triplet (E, A, B) are called *impulse controllable* (*I-controllable*) if

$$\operatorname{rank}\,[\,E,\ AK_E,\ B\,] = n, \ \text{ where the columns of } K_E \text{ span } \ker E. \tag{2.4}$$

System (1.1) and the triplet (E, A, B) are called *completely controllable* (*C-controllable*) if (2.3) holds and

$$\operatorname{rank}\,[\,E,\ B\,] = n. \tag{2.5}$$

Observability is a dual property of controllability.

Definition 2.2. System (1.1) and the triplet (E, A, C) are called *observable on the reachable set* (*R-observable*) if

$$\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \text{for all finite } \lambda \in \mathbb{C}. \tag{2.6}$$

System (1.1) and the triplet (E, A, C) are called *impulse observable* (*I-observable*) if

$$\operatorname{rank}\begin{bmatrix} E \\ K_{E^T}^T A \\ C \end{bmatrix} = n, \quad \text{where the columns of } K_{E^T} \text{ span } \ker E^T. \tag{2.7}$$

System (1.1) and the triplet (E, A, C) are called *completely observable* (*C-observable*) if (2.6) holds and

$$\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{2.8}$$

Clearly, conditions (2.4) and (2.7) are weaker than (2.5) and (2.8), respectively. Equivalent algebraic characterizations of various concepts of controllability and observability for descriptor systems are presented in [8, 9, 42].

2.2 Controllability and observability Gramians

Assume that the pencil $\lambda E - A$ is *c-stable*. Then the integrals

$$\mathcal{G}_{pc} = \int_0^\infty \mathcal{F}(t) B B^T \mathcal{F}^T(t)\, dt \tag{2.9}$$

and

$$\mathcal{G}_{po} = \int_0^\infty \mathcal{F}^T(t) C^T C \mathcal{F}(t)\, dt \tag{2.10}$$

exist, where $\mathcal{F}(t)$ is as in (2.1). The matrix $\mathcal{G}_{pc}$ is called the *proper controllability Gramian* and the matrix $\mathcal{G}_{po}$ is called the *proper observability Gramian* of the continuous-time descriptor system (1.1), see [3, 34]. The *improper controllability Gramian* of system (1.1) is defined by

$$\mathcal{G}_{ic} = \sum_{k=-\nu}^{-1} F_k B B^T F_k^T,$$

and the *improper observability Gramian* of system (1.1) is defined by

$$\mathcal{G}_{io} = \sum_{k=-\nu}^{-1} F_k^T C^T C F_k,$$

where F_k are as in (2.2). Note that the improper controllability and observability Gramians $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ are, up to the sign, the same as those defined in [3]. If $E = I$, then $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ are the usual controllability and observability Gramians for standard state space systems [15, 43].

The proper controllability and observability Gramians are the unique symmetric, positive semidefinite solutions of the *projected generalized continuous-time Lyapunov equations*

$$\begin{aligned} &E\mathcal{G}_{pc}A^T + A\mathcal{G}_{pc}E^T = -P_l B B^T P_l^T, \\ &\mathcal{G}_{pc} = P_r \mathcal{G}_{pc} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^T C^T C P_r, \\ &\mathcal{G}_{po} = \mathcal{G}_{po} P_l, \end{aligned} \tag{2.12}$$

respectively, where P_l and P_r are given in (1.3), see [34]. If $\lambda E - A$ is in Weierstrass canonical form (1.2) and if the matrices

$$W^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad \text{and} \qquad CT^{-1} = [\, C_1, \;\; C_2 \,]$$

are partitioned in blocks conformally to E and A, then we can show that

$$\mathcal{G}_{pc} = T^{-1} \begin{bmatrix} G_{1c} & 0 \\ 0 & 0 \end{bmatrix} T^{-T}, \qquad \mathcal{G}_{po} = W^{-T} \begin{bmatrix} G_{1o} & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \tag{2.13}$$

where G_{1c} and G_{1o} satisfy the standard continuous-time Lyapunov equations

$$\begin{aligned} JG_{1c} + G_{1c}J^T &= -B_1 B_1^T, \\ J^T G_{1o} + G_{1o} J &= -C_1^T C_1. \end{aligned}$$

The improper controllability and observability Gramians are the unique symmetric, positive semidefinite solutions of the *projected generalized discrete-time Lyapunov equations*

$$\begin{aligned} &A\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T = (I - P_l) B B^T (I - P_l)^T, \\ &P_r \mathcal{G}_{ic} = 0 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = (I - P_r)^T C^T C (I - P_r), \\ &\mathcal{G}_{io} P_l = 0, \end{aligned} \tag{2.15}$$

respectively [34]. They can be represented as

$$\mathcal{G}_{ic} = T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & G_{2c} \end{bmatrix} T^{-T}, \qquad \mathcal{G}_{io} = W^{-T} \begin{bmatrix} 0 & 0 \\ 0 & G_{2o} \end{bmatrix} W^{-1}, \tag{2.16}$$

where G_{2c} and G_{2o} satisfy the standard discrete-time Lyapunov equations

$$G_{2c} - NG_{2c}N^T = B_2B_2^T,$$
$$G_{2o} - N^TG_{2o}N = C_2^TC_2.$$

The controllability and observability Gramians can be used to characterize controllability and observability properties of system (1.1).

Theorem 2.3. [3, 34] *Consider the descriptor system* (1.1)*. Assume that* $\lambda E - A$ *is c-stable.*

1. *System* (1.1) *is R-controllable if and only if the proper controllability Gramian* $\mathcal{G}_{pc}$ *is positive definite on the subspace* $\operatorname{im} P_r^T$*.*

2. *System* (1.1) *is I-controllable if the improper controllability Gramian* $\mathcal{G}_{ic}$ *is positive definite on the subspace* $\ker P_r^T$*.*

3. *System* (1.1) *is C-controllable if and only if* $\mathcal{G}_{pc} + \mathcal{G}_{ic}$ *is positive definite.*

4. *System* (1.1) *is R-observable if and only if the proper observability Gramian* $\mathcal{G}_{po}$ *is positive definite on the subspace* $\operatorname{im} P_l$*.*

5. *System* (1.1) *is I-observable if the improper observability Gramian* $\mathcal{G}_{io}$ *is positive definite on the subspace* $\ker P_l$*.*

6. *System* (1.1) *is C-observable if and only if* $\mathcal{G}_{po} + \mathcal{G}_{io}$ *is positive definite.*

Note that the I-controllability (I-observability) of (1.1) does not imply that the improper controllability (observability) Gramian is positive definite on $\ker P_r^T$ (on $\ker P_l$).

Example 2.4. The descriptor system (1.1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [\,1, \;\; 0\,]$$

is I-controllable and I-observable. We have $\mathcal{G}_{ic} = \mathcal{G}_{io} = 0$ and $P_r^T = P_l$, i.e., neither $\mathcal{G}_{ic}$ nor $\mathcal{G}_{io}$ are positive definite on $\ker P_r^T = \ker P_l$.

Corollary 2.5. *Consider the descriptor system* (1.1)*, where the pencil* $\lambda E - A$ *is c-stable.*

1. *System* (1.1) *is R-controllable and R-observable if and only if*

$$\operatorname{rank}(\mathcal{G}_{pc}) = \operatorname{rank}(\mathcal{G}_{po}) = \operatorname{rank}(\mathcal{G}_{pc}E^T\mathcal{G}_{po}E) = n_f. \tag{2.17}$$

2. *System* (1.1) *is I-controllable and I-observable if*

$$\operatorname{rank}(\mathcal{G}_{ic}) = \operatorname{rank}(\mathcal{G}_{io}) = \operatorname{rank}(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A) = n_\infty. \tag{2.18}$$

3. *System* (1.1) *is C-controllable and C-observable if and only if* (2.17) *and* (2.18) *hold.*

Proof. The result follows from Theorem 2.3 and representations (1.2), (2.13) and (2.16). □

2.3 Hankel singular values

The proper controllability and observability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ as well as the improper controllability and observability Gramians $\mathcal{G}_{ic}$ and $\mathcal{G}_{io}$ are not system invariant. Indeed, under a system equivalence transformation $(\check{W}, \check{T})$ the proper and improper controllability Gramians $\mathcal{G}_{pc}$ and $\mathcal{G}_{ic}$ are transformed to $\check{\mathcal{G}}_{pc} = \check{T}^{-1}\mathcal{G}_{pc}\check{T}^{-T}$ and $\check{\mathcal{G}}_{ic} = \check{T}^{-1}\mathcal{G}_{ic}\check{T}^{-T}$, respectively, whereas the proper and improper observability Gramians $\mathcal{G}_{po}$ and $\mathcal{G}_{io}$ are transformed to $\check{\mathcal{G}}_{po} = \check{W}^{-T}\mathcal{G}_{po}\check{W}^{-1}$ and $\check{\mathcal{G}}_{io} = \check{W}^{-T}\mathcal{G}_{io}\check{W}^{-1}$, respectively. However, it follows from

$$\begin{aligned} \check{\mathcal{G}}_{pc}\check{E}^T\check{\mathcal{G}}_{po}\check{E} &= \check{T}^{-1}\mathcal{G}_{pc}E^T\mathcal{G}_{po}E\check{T}, \\ \check{\mathcal{G}}_{ic}\check{A}^T\check{\mathcal{G}}_{io}\check{A} &= \check{T}^{-1}\mathcal{G}_{ic}A^T\mathcal{G}_{io}A\check{T} \end{aligned}$$

that the spectra of the matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ are system invariant. These matrices play the same role for descriptor systems as the product of the controllability and observability Gramians for standard state space systems [15, 43]. We have the following result.

Theorem 2.6. *Let $\lambda E - A$ be c-stable. Then the matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ have real and non-negative eigenvalues.*

Proof. It follows from (2.9) and (2.10) that $\mathcal{G}_{pc}$ and $E^T\mathcal{G}_{po}E$ are symmetric and positive semidefinite. In this case there exists a nonsingular matrix $\check{T}$ such that

$$\check{T}^{-1}\mathcal{G}_{pc}\check{T}^{-T} = \begin{bmatrix} \Sigma_1 & & & 0 \\ & \Sigma_2 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix}, \qquad \check{T}^TE^T\mathcal{G}_{po}E\check{T} = \begin{bmatrix} \Sigma_1 & & & 0 \\ & 0 & & \\ & & \Sigma_3 & \\ 0 & & & 0 \end{bmatrix},$$

where Σ_1, Σ_2 and Σ_3 are diagonal matrices with positive diagonal elements [43, p. 76]. Then

$$\check{T}^{-1}\mathcal{G}_{pc}E^T\mathcal{G}_{po}E\check{T} = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ is diagonalizable and it has real and non-negative eigenvalues.

Similarly, we can show that the eigenvalues of $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ are real and non-negative. ☐

Definition 2.7. Let n_f and n_∞ be the dimensions of the deflating subspaces of the c-stable pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The square roots of the largest n_f eigenvalues of the matrix $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$, denoted by ς_j, are called the *proper Hankel singular values* of the continuous-time descriptor system (1.1). The square roots of the largest n_∞ eigenvalues of the matrix $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$, denoted by θ_j, are called the *improper Hankel singular values* of system (1.1).

We will assume that the proper and improper Hankel singular values are ordered decreasingly, i.e.,

$$\varsigma_1 \geq \varsigma_2 \geq \ldots \geq \varsigma_{n_f} \geq 0, \qquad \theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n_\infty} \geq 0.$$

The proper and improper Hankel singular values form the set of Hankel singular values of the continuous-time descriptor system (1.1). For $E = I$, the proper Hankel singular values are the classical Hankel singular values of standard state space systems [15, 27].

Since the proper and improper controllability and observability Gramians are symmetric and positive semidefinite, there exist Cholesky factorizations

$$\begin{aligned} \mathcal{G}_{pc} &= R_p R_p^T, & \mathcal{G}_{po} &= L_p^T L_p, \\ \mathcal{G}_{ic} &= R_i R_i^T, & \mathcal{G}_{io} &= L_i^T L_i, \end{aligned} \tag{2.19}$$

where the matrices R_p^T, L_p, R_i^T, $L_i \in \mathbb{R}^{n,n}$ are upper triangular Cholesky factors. The following lemma gives a connection between the proper and improper Hankel singular values of system (1.1) and the standard singular values of the matrices $L_p E R_p$ and $L_i A R_i$.

Lemma 2.8. *Assume that the pencil $\lambda E - A$ in system* (1.1) *is c-stable. Consider the Cholesky factorizations* (2.19) *of the proper and improper Gramians of* (1.1)*. Then the proper Hankel singular values of system* (1.1) *are the n_f largest singular values of the matrix $L_p E R_p$, and the improper Hankel singular values of system* (1.1) *are the n_∞ largest singular values of the matrix $L_i A R_i$.*

Proof. We have

$$\begin{aligned} \varsigma_j^2 &= \lambda_j(\mathcal{G}_{pc} E^T \mathcal{G}_{po} E) = \lambda_j(R_p R_p^T E^T L_p^T L_p E) = \lambda_j(R_p^T E^T L_p^T L_p E R_p) = \sigma_j^2(L_p E R_p), \\ \theta_j^2 &= \lambda_j(\mathcal{G}_{ic} A^T \mathcal{G}_{io} A) = \lambda_j(R_i R_i^T A^T L_i^T L_i A) = \lambda_j(R_i^T A^T L_i^T L_i A R_i) = \sigma_j^2(L_i A R_i), \end{aligned}$$

where $\lambda_j(\cdot)$ and $\sigma_j(\cdot)$ denote, respectively, the eigenvalues and the singular values ordered decreasingly. □

3 Model reduction

In this section we consider the problem of reducing the order of the descriptor system (1.1).

3.1 Balanced realizations

For a given transfer function $\mathbf{G}(s)$, there are many different realizations [9]. Here we are interested only in particular realizations that are useful in applications.

Definition 3.1. A realization $[\,E, A, B, C\,]$ of the transfer function $\mathbf{G}(s)$ is called *minimal* if the triplet (E, A, B) is C-controllable and the triplet (E, A, C) is C-observable.

Definition 3.2. A realization $[\,E, A, B, C\,]$ of the transfer function $\mathbf{G}(s)$ is called *balanced* if

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix}$$

with $\Sigma = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{n_f})$ and $\Theta = \operatorname{diag}(\theta_1, \ldots, \theta_{n_\infty})$.

We will show that for a minimal realization $[\,E, A, B, C\,]$ with the c-stable pencil $\lambda E - A$, there exists a system equivalence transformation (W_b^T, T_b) such that the realization

$$[\,W_b^T E T_b,\; W_b^T A T_b,\; W_b^T B,\; C T_b\,] \tag{3.1}$$

is balanced.

Consider the Cholesky factorizations (2.19) of the controllability and observability Gramians. We may assume without loss of generality that the Cholesky factors R_p^T, L_p, R_i^T and L_i have full row rank. If (E, A, B) is C-controllable and (E, A, C) is C-observable, then it follows from Corollary 2.5 and Lemma 2.8 that $\varsigma_j = \sigma_j(L_pER_p) > 0$, $j = 1, \ldots, n_f$, and $\theta_j = \sigma_j(L_iAR_i) > 0$, $j = 1, \ldots, n_\infty$. Hence, the matrices $L_pER_p \in \mathbb{R}^{n_f,n_f}$ and $L_iAR_i \in \mathbb{R}^{n_\infty,n_\infty}$ are nonsingular.

Let

$$L_pER_p = U_p\Sigma V_p^T, \qquad L_iAR_i = U_i\Theta V_i^T, \tag{3.2}$$

be singular value decompositions of L_pER_p and L_iAR_i, where U_p, V_p, U_i and V_i are orthogonal, $\Sigma = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{n_f})$ and $\Theta = \operatorname{diag}(\theta_1, \ldots, \theta_{n_\infty})$ are nonsingular. Consider the matrices

$$W_b = \left[\, L_p^TU_p\Sigma^{-1/2},\;\; L_i^TU_i\Theta^{-1/2} \,\right], \qquad W_b' = \left[\, ER_pV_p\Sigma^{-1/2},\;\; AR_iV_i\Theta^{-1/2} \,\right] \tag{3.3}$$

and

$$T_b = \left[\, R_pV_p\Sigma^{-1/2},\;\; R_iV_i\Theta^{-1/2} \,\right], \qquad T_b' = \left[\, E^TL_p^TU_p\Sigma^{-1/2},\;\; A^TL_i^TU_i\Theta^{-1/2} \,\right]. \tag{3.4}$$

Since

$$\begin{aligned}
(I-P_r)R_pR_p^T(I-P_r)^T = (I-P_r)\mathcal{G}_{pc}(I-P_r)^T = 0,\\
(I-P_l)^TL_p^TL_p(I-P_l) = (I-P_l)^T\mathcal{G}_{po}(I-P_l) = 0,\\
P_rR_iR_i^TP_r^T = P_r\mathcal{G}_{pc}P_r^T = 0, \quad P_l^TL_i^TL_iP_l = P_l^T\mathcal{G}_{po}P_l = 0,\\
P_lE = EP_r, \qquad P_lA = AP_r,
\end{aligned}$$

we obtain that

$$L_pER_i = 0 \qquad \text{and} \qquad L_iAR_p = 0. \tag{3.5}$$

Then

$$(T_b')^TT_b = \begin{bmatrix} \Sigma^{-1/2}U_p^TL_pER_pV_p\Sigma^{-1/2} & \Sigma^{-1/2}U_p^TL_pER_iV_i\Theta^{-1/2} \\ \Theta^{-1/2}U_i^TL_iAR_pV_p\Sigma^{-1/2} & \Theta^{-1/2}U_i^TL_iAR_iV_i\Theta^{-1/2} \end{bmatrix} = I_n,$$

i.e., the matrices T_b and T_b' are nonsingular and $(T_b')^T = T_b^{-1}$. Similarly, we can show that the matrices W_b and W_b' are also nonsingular and $(W_b')^T = W_b^{-1}$.

Using (2.19) and (3.2)-(3.5), we obtain that the proper and improper controllability and observability Gramians of the transformed system (3.1) have the form

$$\begin{aligned}
T_b^{-1}\mathcal{G}_{pc}T_b^{-T} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = W_b^{-1}\mathcal{G}_{po}W_b^{-T},\\
T_b^{-1}\mathcal{G}_{ic}T_b^{-T} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix} = W_b^{-1}\mathcal{G}_{io}W_b^{-T}.
\end{aligned}$$

Thus, (W_b^T, T_b) with W_b and T_b as in (3.3) and (3.4), respectively, is the balancing transformation and realization (3.1) is balanced.

Note that just as for standard state space systems [15, 27], the balancing transformation for descriptor systems is not unique.

From (3.2)-(3.4) we find

$$E_b = W_b^TET_b = \begin{bmatrix} I_{n_f} & 0 \\ 0 & E_2 \end{bmatrix}, \qquad A_b = W_b^TAT_b = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

where $A_1 = \Sigma^{-1/2}U_p^TL_pAR_pV_p\Sigma^{-1/2}$ and $E_2 = \Theta^{-1/2}U_i^TL_iER_iV_i\Theta^{-1/2}$. Thus, the pencil $\lambda E_b - A_b$ is in Weierstrass-like canonical form. Clearly, it is regular, c-stable and has the same index as $\lambda E - A$.

3.2 Balanced truncation

In the previous subsection we have considered a reduction of a minimal realization to a balanced form. However, computing the balanced realization may be ill-conditioned as soon as Σ or Θ in (3.2) has small singular values. In addition, if the realization is not minimal, then the matrix Σ or Θ is singular. In the similar situation for standard state space systems one performs a model reduction by truncating the state components corresponding to the zero and small Hankel singular values without significant changes of the system properties, see, e.g., [15, 27, 37]. This procedure is known as *balanced truncation*. It can also be applied to the descriptor system (1.1).

The proper controllability and observability Gramians can be used to describe the future output energy

$$\mathbf{E}_y = \int_0^\infty y^T(t)y(t)\,dt$$

and the past input energy

$$\mathbf{E}_u = \int_{-\infty}^0 u^T(t)u(t)\,dt$$

that is needed to reach from $x(-\infty) = 0$ the state $x(0) = x^0 \in \operatorname{im} P_r$ when no input is applied for $t \geq 0$.

Theorem 3.3. *Consider a descriptor system* (1.1). *Assume that the pencil $\lambda E - A$ is c-stable and the triplet (E, A, B) is R-controllable. Let $\mathcal{G}_{pc}$ and $\mathcal{G}_{po}$ be the proper controllability and observability Gramians of* (1.1). *If $x^0 \in \operatorname{im} P_r$ and $u(t) = 0$ for $t \geq 0$, then*

$$\mathbf{E}_y = (x^0)^T E^T \mathcal{G}_{po} E x^0.$$

Moreover, for $u_{opt}(t) = B^T \mathcal{F}^T(-t)\mathcal{G}_{pc}^- x^0$, we have

$$\mathbf{E}_{u_{opt}} = \min_{u \in \mathbb{L}_2^m(\mathbb{R}^-)} \mathbf{E}_u = (x^0)^T \mathcal{G}_{pc}^- x^0,$$

where $\mathbb{L}_2^m(\mathbb{R}^-)$ is the Hilbert space of all m dimensional vector-functions that are square integrable on $\mathbb{R}^- = (-\infty, 0)$ and the matrix $\mathcal{G}_{pc}^-$ satisfies

$$\mathcal{G}_{pc}\mathcal{G}_{pc}^-\mathcal{G}_{pc} = \mathcal{G}_{pc}, \qquad \mathcal{G}_{pc}^-\mathcal{G}_{pc}\mathcal{G}_{pc}^- = \mathcal{G}_{pc}^-, \qquad (\mathcal{G}_{pc}^-)^T = \mathcal{G}_{pc}^-. \tag{3.6}$$

Proof. System (1.1) with $x^0 \in \operatorname{im} P_r$ and $u(t) = 0$ for $t \geq 0$ has a unique solution given by $x(t) = \mathcal{F}(t)Ex^0$, $t \geq 0$. Then $y(t) = C\mathcal{F}(t)Ex^0$ for $t \geq 0$ and, hence,

$$\mathbf{E}_y = \int_0^\infty (x^0)^T E^T \mathcal{F}^T(t) C^T C \mathcal{F}(t) E x^0\,dt = (x^0)^T E^T \mathcal{G}_{po} E x^0.$$

Consider now the minimization of $\mathbf{E}_u$ for $u(t) \in \mathbb{L}_2^m(\mathbb{R}^-)$. Note that the state $x(0) = x^0$ of the descriptor system (1.1) with $u(t) = 0$ for $t \geq 0$ satisfies the constraint equation

$$x^0 = \int_{-\infty}^0 \mathcal{F}(-t)Bu(t)\,dt. \tag{3.7}$$

Since system (1.1) is R-controllable, the matrix G_{1c} in (2.13) is nonsingular and, hence, $\operatorname{im} P_r = \operatorname{im} \mathcal{G}_{pc}$. In this case there exists a vector $v \in \mathbb{R}^n$ such that $\mathcal{G}_{pc} v = x^0$ with $x^0 \in \operatorname{im} P_r$. Therefore,

$$\int_{-\infty}^{0} \mathcal{F}(-t) B B^T \mathcal{F}^T(-t)\, \mathcal{G}_{pc}^- x^0 \, dt = \mathcal{G}_{pc} \mathcal{G}_{pc}^- \mathcal{G}_{pc} v = \mathcal{G}_{pc} v = x^0. \tag{3.8}$$

For any input $u(t) = u_{opt}(t) + \hat{u}(t) \in \mathbb{L}_2^m(\mathbb{R}^-)$ with $u_{opt}(t) = B^T \mathcal{F}^T(-t) \mathcal{G}_{pc}^- x^0 \in \mathbb{L}_2^m(\mathbb{R}^-)$, it follows from (3.7) and (3.8) that

$$\int_{-\infty}^{0} \mathcal{F}(-t) B \hat{u}(t)\, dt = 0.$$

Then

$$\begin{aligned} \mathbf{E}_u &= \int_{-\infty}^{0} u_{opt}^T(t) u_{opt}(t)\, dt + 2(x^0)^T (\mathcal{G}_{pc}^-)^T \int_{-\infty}^{0} \mathcal{F}(-t) B \hat{u}(t)\, dt + \int_{-\infty}^{0} \hat{u}^T(t) \hat{u}(t)\, dt \\ &\geq \int_{-\infty}^{0} u_{opt}^T(t) u_{opt}(t)\, dt. \end{aligned}$$

Thus, $u_{opt}(t)$ minimizes $\mathbf{E}_u$ among all inputs that transfer the system from $x(-\infty) = 0$ to $x(0) = x^0 \in \operatorname{im} P_r$. Using the second and third equations in (3.6), we find

$$\mathbf{E}_{u_{opt}} = \int_{-\infty}^{0} (x^0)^T (\mathcal{G}_{pc}^-)^T \mathcal{F}(-t) B B^T \mathcal{F}^T(-t) \mathcal{G}_{pc}^- x^0 \, dt = (x^0)^T \mathcal{G}_{pc}^- x^0. \qquad \square$$

Remark 3.4. Equations (3.6) imply that $\mathcal{G}_{pc}^-$ is a symmetric $(1,2)$-pseudoinverse [7] of $\mathcal{G}_{pc}$. It is, in general, not unique, but $u_{opt}(t) = B^T \mathcal{F}^T(-t) \mathcal{G}_{pc}^- x^0$ and $\mathbf{E}_{u_{opt}} = (x^0)^T \mathcal{G}_{pc}^- x^0$ with $x^0 \in \operatorname{im} P_r$ are uniquely defined.

Unfortunately, we were unable to find a similar energy interpretation for the improper controllability and observability Gramians.

If the descriptor system (1.1) is not minimal, then it has states that are uncontrollable or/and unobservable. These states correspond to the zero proper and improper Hankel singular values and can be truncated without changing the input-output relation in the system. Note that the number of non-zero improper Hankel singular values of (1.1) is equal to $\operatorname{rank}(\mathcal{G}_{ic} A^T \mathcal{G}_{io} A)$ which can in turn be estimated as

$$\operatorname{rank}(\mathcal{G}_{ic} A^T \mathcal{G}_{io} A) \leq \min(\nu m, \nu p, n_\infty).$$

This estimate shows that if the index ν of the pencil $\lambda E - A$ times the number m of inputs or the number p of outputs is much smaller than the dimension n_∞ of the deflating subspace of $\lambda E - A$ corresponding to the infinite eigenvalues, then the order of system (1.1) can be reduced significantly.

Furthermore, Theorem 3.3 implies that a large input energy $\mathbf{E}_u$ is required to reach from $x(-\infty) = 0$ the state $x(0) = P_r x^0$ which lies in an invariant subspace of the proper controllability Gramian $\mathcal{G}_{pc}$ corresponding to its small non-zero eigenvalues. Moreover, if x^0

is contained in an invariant subspace of the matrix $E^T\mathcal{G}_{po}E$ corresponding to its small non-zero eigenvalues, then the initial value $x(0) = x^0$ has a small effect on the output energy $\mathbf{E}_y$. For the balanced system, $\mathcal{G}_{pc}$ and $E^T\mathcal{G}_{po}E$ are equal and, hence, they have the same invariant subspaces. In this case the truncation of the states related to the small proper Hankel singular values does not change system properties essentially.

Unfortunately, this does not hold for the improper Hankel singular values. If we truncate the states that correspond to the small non-zero improper Hankel singular values, then the pencil of the reduced order system may get finite eigenvalues in the closed right half-plane, see [26]. In this case the approximation will be inaccurate.

Let $[\,E, A, B, C\,]$ be a realization (not necessarily minimal) of the transfer function $\mathbf{G}(s)$. Assume that the pencil $\lambda E - A$ is c-stable. Consider the Cholesky factorizations (2.19). Let

$$\begin{aligned} L_pER_p &= [\,U_1,\, U_2\,]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}[\,V_1,\, V_2\,]^T, \\ L_iAR_i &= U_3\Theta_3V_3^T \end{aligned}$$

be 'thin' singular value decompositions of L_pER_p and L_iAR_i, where $[\,U_1,\, U_2\,]$, $[\,V_1,\, V_2\,]$, U_3 and V_3 have orthonormal columns, $\Sigma_1 = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{\ell_f})$ and $\Sigma_2 = \operatorname{diag}(\varsigma_{\ell_f+1}, \ldots, \varsigma_r)$ with $\varsigma_1 \geq \varsigma_2 \geq \ldots \geq \varsigma_{\ell_f} > \varsigma_{\ell_f+1} \geq \ldots \geq \varsigma_r > 0$ and $r = \operatorname{rank}(L_pER_p) \leq n_f$, $\Theta_3 = \operatorname{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ with $\ell_\infty = \operatorname{rank}(L_iAR_i)$. Then the reduced order realization can be computed as

$$\left[\begin{array}{c|c} s\widetilde{E} - \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C} & 0 \end{array}\right] = \left[\begin{array}{c|c} W_\ell^T(sE - A)T_\ell & W_\ell^T B \\ \hline CT_\ell & 0 \end{array}\right], \tag{3.9}$$

where

$$\begin{aligned} W_\ell &= \left[\, L_p^TU_1\Sigma_1^{-1/2},\;\; L_i^TU_3\Theta_3^{-1/2} \,\right] \in \mathbb{R}^{n,\ell}, \\ T_\ell &= \left[\, R_pV_1\Sigma_1^{-1/2},\;\; R_iV_3\Theta_3^{-1/2} \,\right] \in \mathbb{R}^{n,\ell} \end{aligned} \tag{3.10}$$

and $\ell = \ell_f + \ell_\infty$.

Note that computing the reduced order descriptor system can be interpreted as performing a system equivalence transformation $(\check{W}, \check{T})$ such that

$$\left[\begin{array}{c|c} \check{W}(sE - A)\check{T} & \check{W}B \\ \hline C\check{T} & 0 \end{array}\right] = \left[\begin{array}{cc|c} sE_f - A_f & & B_f \\ & sE_\infty - A_\infty & B_\infty \\ \hline C_f & C_\infty & 0 \end{array}\right],$$

where the pencil $\lambda E_f - A_f$ has the finite eigenvalues only, all the eigenvalues of $\lambda E_\infty - A_\infty$ are infinite, and then reducing the order of the subsystems $[E_f, A_f, B_f, C_f]$ and $[A_\infty, E_\infty, B_\infty, C_\infty]$ with nonsingular E_f and A_∞ using classical balanced truncation methods for continuous-time and discrete-time state space systems, respectively. Clearly, the reduced order system (3.9) is minimal and the pencil $\lambda\widetilde{E} - \widetilde{A}$ is c-stable.

The described decoupling of system matrices is equivalent to the additive decomposition of the transfer function as $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where $\mathbf{G}_{sp}(s) = C_f(sE_f - A_f)^{-1}B_f$ is the strictly proper part and $\mathbf{P}(s) = C_\infty(sE_\infty - A_\infty)^{-1}B_\infty$ is the polynomial part of $\mathbf{G}(s)$. The transfer function of the reduced system has the form $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_{sp}(s) + \widetilde{\mathbf{P}}(s)$, where $\widetilde{\mathbf{G}}_{sp}(s) = \widetilde{C}_f(s\widetilde{E}_f - \widetilde{A}_f)^{-1}\widetilde{B}_f$ and $\widetilde{\mathbf{P}}(s) = \widetilde{C}_\infty(s\widetilde{E}_\infty - \widetilde{A}_\infty)^{-1}\widetilde{B}_\infty$ are the reduced order subsystems. Note that $\widetilde{\mathbf{P}}(s) = \mathbf{P}(s)$, and, hence, the difference $\mathbf{G}(s) - \widetilde{\mathbf{G}}(s) = \mathbf{G}_{sp}(s) - \widetilde{\mathbf{G}}_{sp}(s)$

is a strictly proper rational function. Thus, we have the following upper bound for the $\mathbb{H}_\infty$-norm of the error system

$$\|\mathbf{G}(s) - \widetilde{\mathbf{G}}(s)\|_{\mathbb{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)\| \leq 2(\varsigma_{\ell_f+1} + \ldots + \varsigma_{n_f})$$

that can be derived as in [15]. Here $\|\cdot\|$ denotes the spectral matrix norm.

3.3 Algorithms

To reduce the order of the descriptor system (1.1) we have to compute the Cholesky factors of the proper and improper controllability and observability Gramians that satisfy the projected generalized Lyapunov equations (2.11), (2.12), (2.14) and (2.15). These factors can be computed using the *generalized Schur-Hammarling method* [34, 35].

Let the pencil $\lambda E - A$ be in generalized real Schur form

$$E = V \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} U^T \qquad \text{and} \qquad A = V \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} U^T, \tag{3.11}$$

where U and V are orthogonal, E_f is upper triangular nonsingular, E_∞ is upper triangular nilpotent, A_f is upper quasi-triangular and A_∞ is upper triangular nonsingular, and let the matrices

$$V^T B = \begin{bmatrix} B_u \\ B_\infty \end{bmatrix} \qquad \text{and} \qquad CU = [\, C_f, \;\; C_u \,] \tag{3.12}$$

be partitioned in blocks conformally to E and A. Then one can show [34, 35] that the Cholesky factors of the Gramians of system (1.1) have the form

$$\begin{array}{ll} R_p = U \begin{bmatrix} R_f \\ 0 \end{bmatrix}, & R_i = U \begin{bmatrix} Y R_\infty \\ R_\infty \end{bmatrix}, \\ L_p = [\, L_f, \;\; -L_f Z \,] V^T, & L_i = [\, 0, \;\; L_\infty \,] V^T, \end{array} \tag{3.13}$$

where (Y, Z) is the solution of the generalized Sylvester equation

$$\begin{array}{rcl} E_f Y - Z E_\infty & = & -E_u, \\ A_f Y - Z A_\infty & = & -A_u, \end{array} \tag{3.14}$$

the matrices R_f, L_f are the Cholesky factors of the solutions $X_{pc} = R_f R_f^T$, $X_{po} = L_f^T L_f$ of the generalized continuous-time Lyapunov equations

$$E_f X_{pc} A_f^T + A_f X_{pc} E_f^T = -(B_u - Z B_\infty)(B_u - Z B_\infty)^T, \tag{3.15}$$

$$E_f^T X_{po} A_f + A_f^T X_{po} E_f = -C_f^T C_f, \tag{3.16}$$

while R_∞ and L_∞ are the Cholesky factors of the solutions $X_{ic} = R_\infty R_\infty^T$ and $X_{io} = L_\infty^T L_\infty$ of the generalized discrete-time Lyapunov equations

$$A_\infty X_{ic} A_\infty^T - E_\infty X_{ic} E_\infty^T = B_\infty B_\infty^T, \tag{3.17}$$

$$A_\infty^T X_{io} A_\infty - E_\infty^T X_{io} E_\infty = (C_f Y + C_u)^T (C_f Y + C_u). \tag{3.18}$$

From (3.11) and (3.13) we obtain that $L_p E R_p = L_f E_f R_f$ and $L_i A R_i = L_\infty A_\infty R_\infty$. Thus, the proper and improper Hankel singular values of (1.1) can be computed from the singular value decompositions of the matrices $L_f E_f R_f$ and $L_\infty A_\infty R_\infty$. Furthermore, it follows from (3.10) and (3.13) that the projection matrices W_ℓ and T_ℓ have the form

$$W_\ell = V \begin{bmatrix} W_f & 0 \\ -Z^T W_f & W_\infty \end{bmatrix}, \qquad T_\ell = U \begin{bmatrix} T_f & Y T_\infty \\ 0 & T_\infty \end{bmatrix} \tag{3.19}$$

with $W_f = L_f^T U_1 \Sigma_1^{-1/2}$, $W_\infty = L_\infty^T U_3 \Theta_3^{-1/2}$, $T_f = R_f V_1 \Sigma_1^{-1/2}$ and $T_\infty = R_\infty V_3 \Theta_3^{-1/2}$. In this case the matrix coefficients of the reduced order system (3.9) are given by

$$\begin{aligned} \widetilde{E} &= \begin{bmatrix} I_{\ell_f} & 0 \\ 0 & W_\infty^T E_\infty T_\infty \end{bmatrix}, & \widetilde{A} &= \begin{bmatrix} W_f^T A_f T_f & 0 \\ 0 & I_{\ell_\infty} \end{bmatrix}, \\ \widetilde{B} &= \begin{bmatrix} W_f^T (B_u - Z B_\infty) \\ W_\infty^T B_\infty \end{bmatrix}, & \widetilde{C} &= [\, C_f T_f, \quad (C_f Y + C_u) T_\infty \,]. \end{aligned} \tag{3.20}$$

In summary, we have the following algorithm that is a generalization of the *square root balanced truncation method* [24, 37] for the continuous-time descriptor system (1.1).

Algorithm 3.1. *Generalized Square Root (GSR) method.*
Input: $[\,E,\, A,\, B,\, C\,]$ *such that* $\lambda E - A$ *is c-stable.*
Output: *A reduced order system* $[\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C}\,]$.

1. *Compute the generalized Schur form* (3.11).
2. *Compute the matrices* (3.12).
3. *Solve the generalized Sylvester equation* (3.14).
4. *Compute the Cholesky factors* R_f, L_f *of the solutions* $X_{pc} = R_f R_f^T$ *and* $X_{po} = L_f^T L_f$ *of equations* (3.15) *and* (3.16), *respectively.*
5. *Compute the Cholesky factors* R_∞, L_∞ *of the solutions* $X_{ic} = R_\infty R_\infty^T$ *and* $X_{io} = L_\infty^T L_\infty$ *of equations* (3.17) *and* (3.18), *respectively.*
6. *Compute the 'thin' singular value decomposition*

$$L_f E_f R_f = [\, U_1,\; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} [\, V_1,\; V_2 \,]^T,$$

   where $[\,U_1,\; U_2\,]$ *and* $[\,V_1,\; V_2\,]$ *have orthonormal columns,* $\Sigma_1 = \mathrm{diag}(\varsigma_1, \ldots, \varsigma_{\ell_f})$ *and* $\Sigma_2 = \mathrm{diag}(\varsigma_{\ell_f+1}, \ldots, \varsigma_r)$ *with* $r = \mathrm{rank}(L_f E_f R_f)$ *and* $\varsigma_1 \geq \ldots \geq \varsigma_{\ell_f} > \varsigma_{\ell_f+1} \geq \ldots \geq \varsigma_r$.
7. *Compute the 'thin' singular value decomposition* $L_\infty A_\infty R_\infty = U_3 \Theta_3 V_3^T$, *where* U_3 *and* V_3 *have orthonormal columns and* $\Theta_3 = \mathrm{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ *with* $\ell_\infty = \mathrm{rank}(L_\infty A_\infty R_\infty)$.
8. *Compute* $W_f = L_f^T U_1 \Sigma_1^{-1/2}$, $W_\infty = L_\infty^T U_3 \Theta_3^{-1/2}$, $T_f = R_f V_1 \Sigma_1^{-1/2}$, $T_\infty = R_\infty V_3 \Theta_3^{-1/2}$.
9. *Compute the reduced order system* $[\,\widetilde{E},\, \widetilde{A},\, \widetilde{B},\, \widetilde{C}\,]$ *as in* (3.20).

If the original system (1.1) is highly unbalanced or if the deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues are close, then the projection matrices W_ℓ and T_ℓ as in (3.19) are ill-conditioned. To avoid accuracy loss in the reduced order system, a *square root balancing free method* has been proposed in [38] for standard state space systems. This method can be generalized for descriptor systems as follows.

Algorithm 3.2. *Generalized Square Root Balancing Free (GSRBF) method.*
Input: $[E, A, B, C]$ *such that* $\lambda E - A$ *is c-stable.*
Output: *A reduced order system* $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$.

1.-7. as in Algorithm 3.1.

8. Compute the 'economy size' QR decompositions

$$\begin{bmatrix} R_f V_1 & Y R_\infty V_3 \\ 0 & R_\infty V_3 \end{bmatrix} = Q_R R, \qquad \begin{bmatrix} L_f^T U_1 & 0 \\ -Z^T L_f^T U_1 & L_\infty^T U_3 \end{bmatrix} = Q_L L,$$

where $Q_R,\ Q_L \in \mathbb{R}^{n,\ell}$ *have orthonormal columns and* $R,\ L \in \mathbb{R}^{\ell,\ell}$ *are nonsingular.*

9. Compute the reduced order system $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ *with*

$$\begin{aligned} \widetilde{E} &= Q_L^T \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} Q_R, & \widetilde{A} &= Q_L^T \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} Q_R, \\ \widetilde{B} &= Q_L^T \begin{bmatrix} B_u \\ B_\infty \end{bmatrix}, & \widetilde{C} &= [\,C_f,\ C_u\,] Q_R. \end{aligned} \tag{3.21}$$

The GSR method and the GSRBF method are mathematically equivalent in the sense that in exact arithmetic they return reduced systems with the same transfer function. It should be noted that the reduced order realization $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ as in (3.21) is, in general, not balanced and the corresponding pencil $\lambda\widetilde{E} - \widetilde{A}$ is not in Weierstrass-like canonical form.

We will now discuss the numerical aspects of Algorithms 3.1 and 3.2. To compute the generalized Schur form (3.11) we can use the QZ algorithm [16, 40] the GUPTRI algorithm [10, 11], or algorithms proposed in [2, 39].

To solve the generalized Sylvester equation (3.14) we can use the generalized Schur method [23] or its recursive blocked modification [21] that is more suitable for large problems. The upper triangular Cholesky factors R_f, L_f, R_∞ and L_∞ of the solutions of the generalized Lyapunov equations (3.15)-(3.18) can be determined without computing the solutions themself using the generalized Hammarling method [20, 28]. In the general case the generalized Schur and Hammarling methods are based on the preliminary reduction of the corresponding matrix pencils to the generalized Schur form, calculation of the solution of a reduced system and back transformation. Note that the pencils $\lambda E_f - A_f$ and $\lambda E_\infty - A_\infty$ in equations (3.14) – (3.18) are already in generalized Schur form. Thus, we need only to solve the upper (quasi-)triangular matrix equations.

Finally, the singular value decomposition of the matrices $L_f E_f R_f$ and $L_\infty A_\infty R_\infty$, where all three factors are upper triangular, can be computed without forming these products explicitly, see [5, 12, 17] and references therein.

Since the GSR method and the GSRBF method are based on computing the generalized Schur form, they cost $O(n^3)$ flops and have the memory complexity $O(n^2)$. Thus, these methods can be used for problems of small and medium size only. Moreover, they do not take into account the sparsity or any structure of the system and are not attractive for parallelization. Recently, iterative methods related to the ADI method and the Smith method have been proposed to compute low rank approximations of the solutions of standard large-scale sparse Lyapunov equations [25, 29]. It is important to extend these methods for projected generalized Lyapunov equations. This topic is currently under investigations.

Remark 3.5. The GSR method and the GSRBF method can be used to reduce the order of unstable descriptor systems. Firstly, we compute the additive decomposition [22] of the transfer function $\mathbf{G}(s) = \mathbf{G}_-(s) + \mathbf{G}_+(s)$, where

$$\mathbf{G}_-(s) = C_-(sE_- - A_-)^{-1}B_-, \qquad \mathbf{G}_+(s) = C_+(sE_+ - A_+)^{-1}B_+.$$

Here the matrix pencil $\lambda E_- - A_-$ is c-stable and all the eigenvalues of the pencil $\lambda E_+ - A_+$ are finite and have non-negative real part. Then we determine the reduced order system $\widetilde{\mathbf{G}}_-(s) = \widetilde{C}_-(s\widetilde{E}_- - \widetilde{A}_-)^{-1}\widetilde{B}_-$ by applying the balanced truncation model reduction method to the subsystem $[\,E_-,\, A_-,\, B_-,\, C_-\,]$. Finally, the reduced order approximation of $\mathbf{G}_-(s)$ is given by $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_-(s) + \mathbf{G}_+(s)$, where $\mathbf{G}_+(s)$ is included unmodified.

Remark 3.6. The controllability and observability Gramians as well as Hankel singular values can also be generalized for discrete-time descriptor systems, see [36] for details. In this case an extension of balanced truncation model reduction methods for such systems is straightforward.

4 Numerical examples

In this section we consider numerical examples to illustrate the effectiveness of the proposed model reduction methods for descriptor systems.

All of the following results were obtained on a SunOS 5.8 workstation with relative machine precision $\epsilon = 2.22 \times 10^{-16}$ using the MATLAB mex-functions based on the GUPTRI routine[1] [10, 11] and the SLICOT library routines[2] [4].

Example 4.1. Consider the holonomically constrained planar model of a truck [32]. The linearized equation of motion has the form

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \mathbf{v}(t), \\ M\dot{\mathbf{v}}(t) &= K\mathbf{p}(t) + D\mathbf{v}(t) - G^T\boldsymbol{\lambda}(t) + B_2u(t), \\ 0 &= G\mathbf{p}(t), \end{aligned} \tag{4.1}$$

where $\mathbf{p}(t) \in \mathbb{R}^{11}$ is the position vector, $\mathbf{v}(t) \in \mathbb{R}^{11}$ is the velocity vector, $\boldsymbol{\lambda}(t) \in \mathbb{R}$ is the Lagrange multiplier, M is the positive definite mass matrix, K is the stiffness matrix, D is the damping matrix, G is the constraint matrix and B_2 is the input matrix. System (4.1) together with the output equation $y(t) = \mathbf{p}(t)$ forms a descriptor system of order $n = 23$ with $m = 1$ input and $p = 11$ outputs. The dimension of the deflating subspace corresponding to the finite eigenvalues is $n_f = 20$.

The proper Hankel singular values of the linearized truck model (4.1) are given in Table 1. All the improper Hankel singular values are zero and, hence, the transfer function of (4.1) is strictly proper.

We approximate system (4.1) by two models of order $\ell = \ell_f = 12$ computed by the GSR method and the GSRBF method. Figure 1 illustrates how accurate the reduced order models approximate the original system. We display the plots of the spectral norm of the frequency responses of the original system $\mathbf{G}(i\omega)$ and the reduced order systems $\widetilde{\mathbf{G}}(i\omega)$ for a frequency range $\omega \in [10^{-1}, 10^5]$. One can see that the approximate system delivered by the GSR method differs slightly from the original one for high frequencies only, while the plots of the full order

[1]Available from `http://www.cs.umu.se/research/nla/singular_pairs/guptri`.

[2]Available from `http://www.win.tue.nl/niconet/NIC2/slicot.html`.

$\varsigma_1 = 7.072 \times 10^{-4}$	$\varsigma_6 = 2.715 \times 10^{-6}$	$\varsigma_{11} = 1.134 \times 10^{-7}$	$\varsigma_{16} = 9.988 \times 10^{-9}$
$\varsigma_2 = 1.962 \times 10^{-4}$	$\varsigma_7 = 1.578 \times 10^{-6}$	$\varsigma_{12} = 1.013 \times 10^{-7}$	$\varsigma_{17} = 8.833 \times 10^{-9}$
$\varsigma_3 = 5.286 \times 10^{-6}$	$\varsigma_8 = 7.925 \times 10^{-7}$	$\varsigma_{13} = 5.392 \times 10^{-8}$	$\varsigma_{18} = 3.847 \times 10^{-9}$
$\varsigma_4 = 2.845 \times 10^{-6}$	$\varsigma_9 = 3.192 \times 10^{-7}$	$\varsigma_{14} = 2.596 \times 10^{-8}$	$\varsigma_{19} = 5.108 \times 10^{-10}$
$\varsigma_5 = 2.750 \times 10^{-6}$	$\varsigma_{10} = 2.535 \times 10^{-7}$	$\varsigma_{15} = 1.969 \times 10^{-8}$	$\varsigma_{20} = 1.163 \times 10^{-10}$

Table 1: Proper Hankel singular values of the linearized truck model.

system and the reduced order system computed by the GSRBF method coincide. Note that the projection matrices W_ℓ and T_ℓ in the GSR method (see (3.19)) have the condition numbers

$$\kappa(W_\ell) = \frac{\sigma_{\max}(W_\ell)}{\sigma_{\min}(W_\ell)} = 5.385 \times 10^3, \qquad \kappa(T_\ell) = \frac{\sigma_{\max}(T_\ell)}{\sigma_{\min}(T_\ell)} = 4.965 \times 10^5,$$

whereas the projection matrices in the GSRBF method given by $W_\ell = VQ_L$ and $T_\ell = UQ_R$ have orthonormal columns. In Figure 2 we compare the absolute approximation errors $\|\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)\|$ with the upper bound which is the twice sum of the truncated proper Hankel singular values $\varsigma_{13}, \ldots, \varsigma_{20}$. We see that the approximation error for the GSRBF method is considerable smaller than for the GSR method.

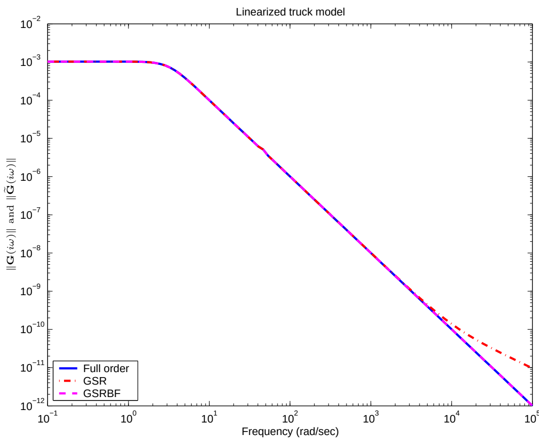

Figure 1: Frequency responses of the full order system and the reduced order systems.

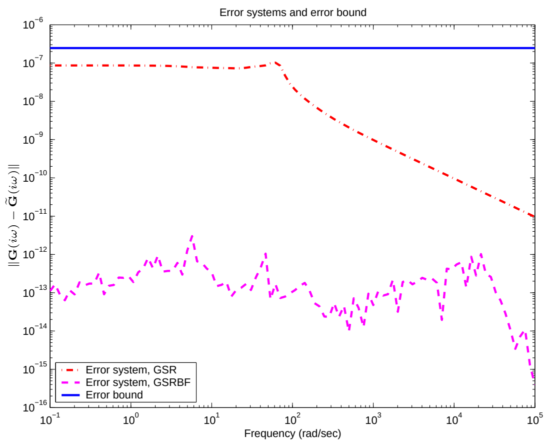

Figure 2: Error systems and error bound for the linearized truck model.

Example 4.2. Consider the flow of an incompressible fluid describing by the instationary Stokes equation on a square region $[0,1] \times [0,1]$. The spatial discretization of this equation by the finite volume method on a uniform staggered $k \times k$ grid leads to the descriptor system

$$\begin{aligned} \dot{\mathbf{v}}_h(t) &= L_h \mathbf{v}_h(t) - G_h \mathbf{p}_h(t) + B_1 u(t), \\ 0 &= G_h^T \mathbf{v}_h(t), \\ y(t) &= C_2 \mathbf{p}_h(t), \end{aligned} \tag{4.2}$$

where $\mathbf{v}_h(t) \in \mathbb{R}^{n_\mathbf{v}}$ and $\mathbf{p}_h(t) \in \mathbb{R}^{n_\mathbf{p}}$ are the semidiscretized vectors of velocities and pressures, respectively, $L_h \in \mathbb{R}^{n_\mathbf{v}, n_\mathbf{v}}$ is the discrete Laplace operator and $G_h \in \mathbb{R}^{n_\mathbf{v}, n_\mathbf{p}}$ is obtained from the discrete gradient operator by discarding the last column [41]. For simplicity, $B_1 \in \mathbb{R}^{n_\mathbf{v}, 2}$

is chosen at random and $C_2 = [1, 0, \ldots, 0] \in \mathbb{R}^{1,n_\mathbf{p}}$. For $k = 12$, we have $n = n_\mathbf{v} + n_\mathbf{p} = 480$ and the dimensions of the deflating subspaces of the pencil in (4.2) corresponding to the finite and infinite eigenvalues are $n_f = 144$ and $n_\infty = 336$.

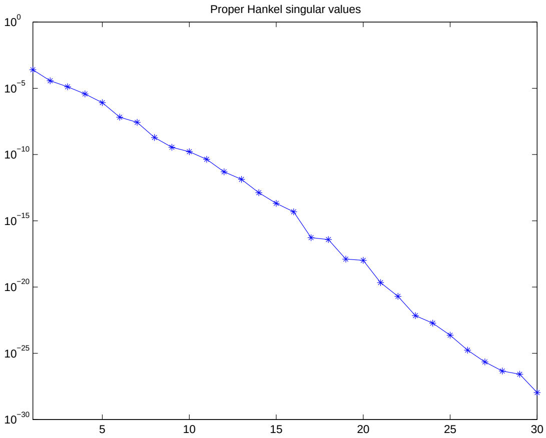

Figure 3: Proper Hankel singular values for the semidiscretized Stokes equation.

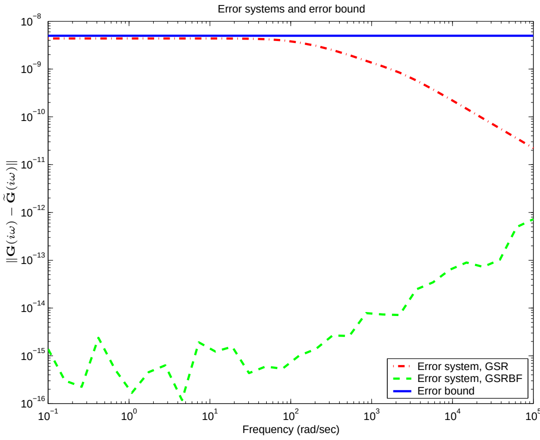

Figure 4: Error systems and error bound for the semidiscretized Stokes equation.

The 30 largest proper Hankel singular values of the semidiscretized Stokes equation (4.2) are given in Figure 3. One can see that they decay very fast, and, hence, system (4.2) can be well approximated by a model of low order. All the improper Hankel singular values are zero. We approximate system (4.2) by two models of order $\ell = \ell_f = 7$ computed by the GSR method and the GSRBF method. The spectral norm of the frequency responses of the full order and reduced order systems are not presented since they were impossible to distinguish. Figure 4 shows the plots of the error bound and the spectral norm of the error systems $\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)$ for a frequency rang $\omega \in [10^{-1}, 10^5]$. We see again that the approximate system delivered by the GSRBF method is better than the one computed by the GSR method.

5 Conclusion

We have generalized the controllability and observability Gramians as well as Hankel singular values for descriptor systems and studied their important features. Balanced truncation model reduction methods for descriptor systems have been presented. These methods are closely related to the Gramians and deliver reduced order systems that preserve the regularity and stability properties of the original system. Moreover, for these methods a priori bound on the approximation error is available.

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