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Stability and Inertia Theorems for Generalized Lyapunov Equations

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Abstract

We study generalized Lyapunov equations and present generalizations of Lyapunov stability theorems and some matrix inertia theorems for matrix pencils. We discuss applications of generalized Lyapunov equations with special right-hand sides in stability theory and control problems for descriptor systems.

Key words. generalized Lyapunov equations, descriptor systems, inertia, controllability, observability.

AMS subject classification. 15A22, 15A24, 93B

1 Introduction

Generalized continuous-time Lyapunov equations

$$E^*XA + A^*XE = -G \tag{1.1}$$

and generalized discrete-time Lyapunov equations

$$A^*XA - E^*XE = -G \tag{1.2}$$

with given matrices E, A, G and unknown matrix X arise naturally in control problems [2, 11], stability theory for the differential and difference equations [12, 13, 24] and problems of spectral dichotomy [17, 18].

Equations (1.1) and (1.2) with $E = I$ are the standard continuous-time and discrete-time Lyapunov equations (the latter is also known as the Stein equation). The theoretical analysis and numerical solution for these equations has been the topic of numerous publications, see [1, 12, 14, 15] and the references therein. The case of nonsingular E has been considered in [3, 21]. However, many applications in singular systems or descriptor systems [8] lead to generalized Lyapunov equations with a singular matrix E, see [2, 18, 24, 25].

The solvability of the generalized Lyapunov equations (1.1) and (1.2) can be described in terms of the generalized eigenstructure of the matrix pencil $\lambda E - A$. The pencil $\lambda E - A$ is

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called *regular* if E and A are square and $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. Otherwise, $\lambda E - A$ is called *singular*. A complex number $\lambda \neq \infty$ is said to be *generalized finite eigenvalue* of the regular pencil $\lambda E - A$ if $\det(\lambda E - A) = 0$. The pencil $\lambda E - A$ has an *infinite eigenvalue* if and only if the matrix E is singular.

A regular matrix pencil $\lambda E - A$ with singular E can be reduced to the *Weierstrass (Kronecker) canonical form* [23]. There exist nonsingular matrices W and T such that

$$E = W \begin{pmatrix} I_m & 0 \\ 0 & N \end{pmatrix} T \qquad \text{and} \qquad A = W \begin{pmatrix} J & 0 \\ 0 & I_{n-m} \end{pmatrix} T, \tag{1.3}$$

where I_m is the identity matrix of order m and N is nilpotent. The block J corresponds to the finite eigenvalues of the pencil $\lambda E - A$, the block N corresponds to the infinite eigenvalues. The index of nilpotency of N is called *index* of the pencil $\lambda E - A$. The spaces spanned by the first m columns of W and T^{-1} are, respectively, the *left* and *right deflating subspaces* of $\lambda E - A$ corresponding to the finite eigenvalues, whereas the spans of the last $n - m$ columns of W and T^{-1} form the *left* and *right deflating subspaces* corresponding to the infinite eigenvalues, respectively. For simplicity, the deflating subspaces of $\lambda E - A$ corresponding to the finite (infinite) eigenvalues we will call the *finite (infinite) deflating subspaces*. The matrices

$$P_l = W \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \qquad P_r = T^{-1} \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} T \tag{1.4}$$

are the *spectral projections* onto the left and right finite deflating subspaces of the pencil $\lambda E - A$ along the left and right infinite deflating subspaces, respectively.

The classical stability and inertia theorems [4, 6, 7, 9, 16, 20, 26, 27] relay the signatures of solutions of the standard Lyapunov equations and the numbers of eigenvalues of a matrix in the left and right open half-planes and on the imaginary axis in the continuous-time case and inside, outside and on the unit circle in the discrete-time case. A brief survey of matrix inertia theorems and their applications has been presented in [10]. In this paper we establish an analogous connection between the signatures of solutions of the generalized continuous-time Lyapunov equation

$$E^*XA + A^*XE = -P_r^* G P_r, \tag{1.5}$$

and the generalized discrete-time Lyapunov equation

$$A^*XA - E^*XE = -P_r^* G P_r - (I - P_r)^* G (I - P_r) \tag{1.6}$$

and the generalized eigenvalues of a matrix pencil $\lambda E - A$. Under some assumptions on the finite spectrum of $\lambda E - A$, equations (1.5) and (1.6) have solutions that are, in general, not unique. We are interested in the solution X of (1.5) satisfying $X = XP_l$ and the solution X of (1.6) satisfying $P_l^* X = XP_l$. Such solutions are uniquely defined and can be used to study the distribution of the generalized eigenvalues of a pencil in the complex plane with respect to the imaginary axis (Section 2) and the unit circle (Section 3).

Throughout the paper we will denote by $\mathbb{F}$ the field of real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers, $\mathbb{F}^{n,m}$ is the space of $n \times m$-matrices over $\mathbb{F}$. The matrix $A^* = A^T$ denotes the transpose of a real matrix A, $A^* = A^H$ denotes the complex conjugate transpose of complex A and $A^{-*} = (A^{-1})^*$. The matrix A is *Hermitian* if $A = A^*$. The matrix A is *positive definite* (*positive semidefinite*) if $x^*Ax > 0$ ($x^*Ax \geq 0$) for all nonzero $x \in \mathbb{F}^n$, and A is *positive definite on a subspace* $\mathcal{X} \subset \mathbb{F}^n$ if $x^*Ax > 0$ for all nonzero $x \in \mathcal{X}$. We will denote by $\|\cdot\|$

the Euclidean vector norm. A pencil $\lambda E - A$ is called *c-stable* if it is regular and all finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. A pencil $\lambda E - A$ is called *d-stable* if it is regular and all finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

2 Inertia with respect to the imaginary axis

First we recall the definition of the inertia with respect to the imaginary axis for matrices.

Definition 2.1 The *inertia of a matrix A with respect to the imaginary axis* (*c-inertia*) is defined by the triplet of integers

$$\mathrm{In}_c(A) = \{\ \pi_-(A),\ \pi_+(A),\ \pi_0(A)\ \},$$

where $\pi_-(A)$, $\pi_+(A)$ and $\pi_0(A)$ denote the numbers of eigenvalues of A with negative, positive and zero real part, respectively, counting multiplicities.

Taking into account that a matrix pencil may have finite as well as infinite eigenvalues, the c-inertia for matrices can be generalized for regular pencils as follows.

Definition 2.2 The *c-inertia of a regular pencil $\lambda E - A$* is defined by the quadruple of integers

$$\mathrm{In}_c(E, A) = \{\ \pi_-(E, A),\ \pi_+(E, A),\ \pi_0(E, A),\ \pi_\infty(E, A)\ \},$$

where $\pi_-(E, A)$, $\pi_+(E, A)$ and $\pi_0(E, A)$ denote the numbers of the finite eigenvalues of $\lambda E - A$ counted with their algebraic multiplicities with negative, positive and zero real part, respectively, and $\pi_\infty(E, A)$ denotes the number of infinite eigenvalues of $\lambda E - A$.

Clearly, $\pi_-(E, A) + \pi_+(E, A) + \pi_0(E, A) + \pi_\infty(E, A) = n$ is the size of E. A c-stable pencil $\lambda E - A$ has the c-inertia $\mathrm{In}_c(E, A) = \{\, m, 0, 0, n - m\,\}$, where m is the number of the finite eigenvalues of $\lambda E - A$ counting their multiplicities. If the matrix E is nonsingular, then $\pi_\infty(E, A) = 0$ and

$$\pi_\alpha(E, A) = \pi_\alpha(AE^{-1}) = \pi_\alpha(E^{-1}A),$$

where α is $-$, $+$ and 0. Thus, the classical stability and matrix inertia theorems [4, 6, 7, 12, 16, 20, 27] can be extended to the GCALE (1.1) with nonsingular E. Here we formulate only a generalization of the Lyapunov stability theorem [12].

Theorem 2.3 *Let $\lambda E - A$ be a regular matrix pencil. If all eigenvalues of $\lambda E - A$ are finite and lie in the open left half-plane, then for every Hermitian, positive (semi)definite matrix G, the GCALE (1.1) has a unique Hermitian, positive (semi)definite solution X. Conversely, if there exist Hermitian, positive definite matrices X and G satisfying (1.1), then all eigenvalues of the pencil $\lambda E - A$ are finite and lie in the open left half-plane.*

If the pencil $\lambda E - A$ has an infinite eigenvalue or, equivalently, if E is singular, then the GCALE (1.1) may have no solutions even if all finite eigenvalues of $\lambda E - A$ have negative real part.

Example 2.4 The GCALE (1.1) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = -I_2, \qquad G = I_2$$

has no solutions.

Consider the GCALE with a special right-hand side

$$E^* X A + A^* X E = -P_r^* G P_r, \tag{2.1}$$

where P_r is the spectral projection onto the right finite deflating subspace of $\lambda E - A$. The following theorem gives a connection between the c-inertia of a pencil $\lambda E - A$ and the c-inertia of an Hermitian solution X of this equation.

Theorem 2.5 *Let P_r and P_l be the spectral projection onto the right and left finite deflating subspaces of a regular pencil $\lambda E - A$ and let G be an Hermitian, positive definite matrix. If there exists an Hermitian matrix X which satisfies the GCALE (2.1) together with $X = XP_l$, then*

$$\begin{array}{ll} \pi_-(E, A) = \pi_+(X), & \pi_+(E, A) = \pi_-(X), \\ \pi_0(E, A) = 0, & \pi_\infty(E, A) = \pi_0(X). \end{array} \tag{2.2}$$

Conversely, if $\pi_0(E, A) = 0$, then there exists an Hermitian matrix X and an Hermitian, positive definite matrix G such that the GCALE (2.1) is fulfilled and the c-inertia identities (2.2) hold.

Proof. Assume that an Hermitian matrix X satisfies the GCALE (2.1) together with $X = XP_l$. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the Hermitian matrices

$$T^{-*} G T^{-1} = \begin{pmatrix} G_{11} & G_{12} \\ G_{12}^* & G_{22} \end{pmatrix} \quad \text{and} \quad W^* X W = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \tag{2.3}$$

be partitioned conformally to E and A. Then we obtain from (2.1) the system of matrix equations

$$X_{11} J + J^* X_{11} = -G_{11}, \tag{2.4}$$

$$X_{12} + J^* X_{12} N = 0, \tag{2.5}$$

$$N^* X_{22} + X_{22} N = 0. \tag{2.6}$$

Since N is nilpotent, equation (2.5) has the unique solution $X_{12} = 0$, whereas equation (2.6) is not uniquely solvable. It follows from $X = XP_l$ that

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1} = XP_l = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} W^{-1},$$

i.e., $X_{22} = 0$.

Consider now equation (2.4), where the matrix G_{11} is Hermitian and positive definite. By the Sylvester law of inertia [6] and the main inertia theorem [20, Theorem 1] we obtain that

$$\begin{array}{l} \pi_-(E, A) = \pi_-(J) = \pi_+(X_{11}) = \pi_+(X), \\ \pi_+(E, A) = \pi_+(J) = \pi_-(X_{11}) = \pi_-(X), \\ \pi_0(E, A) = \pi_0(J) = \pi_0(X_{11}) = 0 \end{array}$$

and, hence, $\pi_0(X) = \pi_0(X_{11}) + \pi_\infty(E, A) = \pi_\infty(E, A)$.

Assume now that $\pi_0(E, A) = 0$. Then $\pi_0(J) = 0$ and by the main inertia theorem [20, Theorem 1] there exists an Hermitian matrix X_{11} such that $G_{11} := -(X_{11}J + J^* X_{11})$ is Hermitian, positive definite and

$$\pi_-(J) = \pi_+(X_{11}), \qquad \pi_+(J) = \pi_-(X_{11}), \qquad \pi_0(J) = \pi_0(X_{11}) = 0.$$

In this case the matrices

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} W^{-1} \quad \text{and} \quad G = T^* \begin{pmatrix} G_{11} & 0 \\ 0 & I \end{pmatrix} T$$

satisfy the GCALE (2.1). Moreover, G is Hermitian, positive definite, X is Hermitian and the c-inertia identities (2.2) hold. ■

The following corollary gives necessary and sufficient conditions for the pencil $\lambda E - A$ to be c-stable.

Corollary 2.6 *Let $\lambda E - A$ be a regular pencil and let P_r and P_l be the spectral projections onto the right and left finite deflating subspaces of $\lambda E - A$.*

1. *If there exist an Hermitian, positive definite matrix G and an Hermitian, positive semidefinite matrix X satisfying the GCALE (2.1), then the pencil $\lambda E - A$ is c-stable.*
2. *If the pencil $\lambda E - A$ is c-stable, then the GCALE (2.1) has a solution for every matrix G. Moreover, if a solution X of (2.1) satisfies $X = XP_l$, then it is unique and given by*

$$X = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi E - A)^{-*} P_r^* G P_r (i\xi E - A)^{-1} d\xi.$$

   If G is Hermitian, then this solution X is Hermitian. If G is positive definite or positive semidefinite, then X is positive semidefinite.

Proof. Part 1 immediately follows from Theorem 2.5. The proof of part 2 can be found in [24]. ■

Corollary 2.6 is a generalization of the classical Lyapunov stability theorem [12] for the GCALE (2.1). We see that if the GCALE (2.1) has an Hermitian, positive semidefinite solution for some Hermitian, positive definite matrix G, then (2.1) has (nonunique) solution for every G. Constraining the solution of (2.1) to satisfy $X = XP_l$, we choose the nonunique part X_{22} to be zero. A system of matrix equations

$$\begin{aligned} &E^* X A + A^* X E = -P_r^* G P_r, \\ &X = XP_l \end{aligned} \tag{2.7}$$

is called *projected generalized continuous-time algebraic Lyapunov equation.* From the proof of Theorem 2.5 it follows that the solution of the projected GCALE (2.7) has the form

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \tag{2.8}$$

where X_{11} satisfies the standard Lyapunov equation (2.4). Thus, the matrix inertia theorems can be generalized for regular pencils by using the Weierstrass canonical form (1.3) and applying these theorems to equation (2.4).

Theorem 2.7 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian solution of the projected GCALE (2.7) with an Hermitian, positive semidefinite matrix G.*

1. If $\pi_0(E,A) = 0$, then $\pi_-(X) \leq \pi_+(E,A)$ and $\pi_+(X) \leq \pi_-(E,A)$.

2. If $\pi_0(X) = \pi_\infty(E,A)$, then $\pi_+(E,A) \leq \pi_-(X)$, $\pi_-(E,A) \leq \pi_+(X)$.

PROOF. The result immediately follows if we apply the matrix inertia theorems [7, Lemma 1 and Lemma 2] to equation (2.4). ■

As an immediate consequence of Theorem 2.7 we obtain a generalization of Theorem 2.5 for the case that G is Hermitian, positive semidefinite.

Corollary 2.8 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian solution of the projected GCALE (2.7) with an Hermitian, positive semidefinite matrix G. If $\pi_0(E,A) = 0$ and $\pi_\infty(E,A) = \pi_0(X)$, then the c-inertia identities (2.2) hold.*

Similar to the matrix case [15, 16, 27], the c-inertia identities (2.2) can be also derived using controllability and observability conditions.

Consider the linear continuous-time descriptor system

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \quad x(t) = x_0, \\ y(t) & = & Cx(t), \end{array} \tag{2.9}$$

where $E,\ A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,q}$, $C \in \mathbb{F}^{p,n}$, $x(t) \in \mathbb{F}^n$ is the state, $u(t) \in \mathbb{F}^q$ is the control input and $y(t) \in \mathbb{F}^p$ is the output. For descriptor systems there are various concepts of controllability and observability [5, 8, 28].

Definition 2.9 System (2.9) and the triplet (E,A,C) are called *R-observable* if

$$\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \text{for all finite } \lambda \in \mathbb{C}\,. \tag{2.10}$$

System (2.9) and the triplet (E,A,C) are called *I-observable* if

$$\operatorname{rank}\begin{bmatrix} E \\ K_{E^*}^* A \\ C \end{bmatrix} = n, \tag{2.11}$$

where the columns of K_{E^*} span the nullspace of E^*.
System (2.9) and the triplet (E,A,C) are called *S-observable* if (2.10) and (2.11) are satisfied.
System (2.9) and the triplet (E,A,C) are called *C-observable* if (2.10) holds and

$$\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{2.12}$$

Note that condition (2.11) is weaker than (2.12) and, hence, the C-observability implies the S-observability.

Controllability is a dual property of observability. System (2.9) and the triplet (E,A,B) are R (I, S, C)-controllable, if the triplet (E^*,A^*,B^*) is R (I, S, C)-observable.

The following corollary shows that in the case of an Hermitian, positive semidefinite matrix $G = C^*C$, the conditions $\pi_0(X) = \pi_\infty(E,A)$ and $\pi_0(E,A) = 0$ in Corollary 2.8 may be replaced by the assumption that the triplet (E,A,C) is R-observable.

Corollary 2.10 *Consider system (2.9) with a regular pencil $\lambda E - A$. If the triplet (E, A, C) is R-observable and if there exists an Hermitian matrix X satisfying the projected GCALE*

$$E^*XA + A^*XE = -P_r^*C^*CP_r, \qquad X = XP_l, \tag{2.13}$$

then the c-inertia identities (2.2) hold.

Proof. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the matrix $CT^{-1} = [\,C_1,\ C_2\,]$ be partitioned in blocks conformally to E and A. Then the Hermitian solution of the projected GCALE (2.13) has the form (2.8), where X_{11} satisfies the standard Lyapunov equation

$$X_{11}J + J^*X_{11} = -C_1^*C_1. \tag{2.14}$$

From the R-observability condition (2.10) we have that the matrix $\begin{bmatrix} \lambda I - J \\ C_1 \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$, see [8]. In this case the solution X_{11} of (2.14) is nonsingular and the matrix J has no eigenvalues on the imaginary axis, e.g., [15, Theorem 13.1.4]. Hence, $\pi_0(E, A) = 0$ and $\pi_0(X) = \pi_\infty(E, A)$. The remaining relations in (2.2) immediately follow from Corollary 2.8. ■

The following corollary gives connections between c-stability of the pencil $\lambda E - A$, the R-observability of the triplet (E, A, C) and the existence of an Hermitian solution of the projected GCALE (2.13).

Corollary 2.11 *Consider the statements:*

1. *the pencil $\lambda E - A$ is c-stable,*
2. *the triplet (E, A, C) is R-observable,*
3. *the projected GCALE (2.13) has a unique solution X which is Hermitian, positive definite on the subspace $\operatorname{im} P_l$.*

Any two of these statements together imply the third.

Proof. '1 and 2 $\Rightarrow$ 3' and '2 and 3 $\Rightarrow$ 1' can be obtained from Corollaries 2.6 and 2.10.

'1 and 3 $\Rightarrow$ 2'. Suppose that (E, A, C) is not R-observable. Then there exists $\lambda_0 \in \mathbb{C}$ and a vector $z \neq 0$ such that

$$\begin{bmatrix} \lambda_0 E - A \\ C \end{bmatrix} z = 0.$$

We obtain that z is the eigenvector of the pencil $\lambda E - A$ corresponding to the finite eigenvalue λ_0. Hence $\Re e\, \lambda_0 > 0$ and $z \in \operatorname{im} P_r$. Moreover, we have $Cz = 0$. On the other hand, it follows from the Lyapunov equation in (2.13) that

$$-\|Cz\|^2 = z^*(E^*XA + A^*XE)z = 2(\Re e\, \lambda_0) z^*E^*XEz.$$

and, hence, $Cz \neq 0$. Thus, the triplet (E, A, C) is R-observable. ■

Corollary 2.11 generalizes the stability result (see Corollary 2.6) to the case that $G = C^*C$ is Hermitian, positive semidefinite. We see, that weakening the assumption for G to be positive semidefinite requires the additional R-observability condition. Moreover, Corollary 2.11 gives necessary and sufficient conditions for the triplet (E, A, C) to be R-observable.

It is natural to ask what happens if the triplet (E, A, C) is not R-observable. Consider a *proper observability matrix*

$$\mathbf{O}_p = \begin{bmatrix} CF_0 \\ CF_1 \\ \vdots \\ CF_{n-1} \end{bmatrix}, \tag{2.15}$$

where the matrices F_k have the form

$$F_k = T^{-1} \begin{pmatrix} J^k & 0 \\ 0 & 0 \end{pmatrix} W^{-1}, \qquad k = 0, 1, \ldots.$$

Here T, W and J are as in (1.3). If $E = I$, then $\mathbf{O}_p$ is an usual observability matrix. The property of the triplet (E, A, C) to be R-observable is equivalent to the condition $\operatorname{rank} \mathbf{O}_p = n - \pi_\infty(E, A)$, see [2]. The nullspace of $\mathbf{O}_p$ is the *proper unobservable subspace* for the descriptor system (2.9). Using the Weierstrass canonical form (1.3) and the matrix inertia theorems [16] we obtain the following c-inertia inequalities.

Theorem 2.12 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian solution of the projected GCALE (2.13). Assume that* $\operatorname{rank} \mathbf{O}_p < n - \pi_\infty(E, A)$. *Then*

$$\begin{aligned} |\pi_-(E, A) - \pi_+(X)| &\le n - \pi_\infty(E, A) - \operatorname{rank} \mathbf{O}_p, \\ |\pi_+(E, A) - \pi_-(X)| &\le n - \pi_\infty(E, A) - \operatorname{rank} \mathbf{O}_p. \end{aligned} \tag{2.16}$$

Other matrix inertia theorems concerning the matrix c-inertia and the rank of the observability matrix [4, 22] can be generalized for matrix pencils in the same way.

By duality of controllability and observability conditions analogies of Corollaries 2.10, 2.11 and Theorem 2.12 can be proved for the dual projected GCALE

$$EXA^* + AXE^* = -P_l BB^* P_l^*, \qquad X = P_r X.$$

3 Inertia with respect to the unit circle

We recall that the *inertia of a matrix A with respect to the unit circle* or *d-inertia* is defined by the triplet of integers

$$\operatorname{In}_d(A) = \{\ \pi_{<1}(A),\ \pi_{>1}(A),\ \pi_1(A)\ \},$$

where $\pi_{<1}(E, A)$, $\pi_{>1}(E, A)$ and $\pi_1(E, A)$ denote the numbers of the eigenvalues of A counted with their algebraic multiplicities inside, outside and on the unit circle, respectively.

Before extending the d-inertia for matrix pencils, it should be noted that in some problems it is necessary to distinguish the finite eigenvalues of a matrix pencil of modulus larger that 1 and the infinite eigenvalues although the latter also lie outside the unit circle. For example, the presence of infinite eigenvalues of $\lambda E - A$, in contrast to the finite eigenvalues outside the unit circle, does not affect the behavior at infinity of solutions of the discrete-time singular system, see [8].

Definition 3.1 The *d-inertia of a regular pencil* $\lambda E - A$ is defined by the quadruple of integers

$$\mathrm{In}_d(E, A) = \{\ \pi_{<1}(E, A),\ \pi_{>1}(E, A),\ \pi_1(E, A),\ \pi_\infty(E, A)\ \},$$

where $\pi_{<1}(E, A)$, $\pi_{>1}(E, A)$ and $\pi_1(E, A)$ denote the numbers of the finite eigenvalues of $\lambda E - A$ counted with their algebraic multiplicities inside, outside and on the unit circle, respectively, and $\pi_\infty(E, A)$ denotes the number of infinite eigenvalues of $\lambda E - A$.

For a d-stable pencil $\lambda E - A$ we have $\mathrm{In}_d(E, A) = \{\, m, 0, 0, n-m\,\}$, where m is the number of finite eigenvalues of $\lambda E - A$ counting their multiplicities.

It is well known that the standard continuous-time and discrete-time Lyapunov equations are related via a *Cayley transformation* defined by $\mathcal{C}(A) := (A - I)^{-1}(A + I) = \mathcal{A}$, see, e.g., [15]. A *generalized Cayley transformation* for matrix pencils is given by

$$\mathcal{C}(E, A) = \lambda(A - E) - (E + A) = \lambda\mathcal{E} - \mathcal{A}. \tag{3.1}$$

Under this transformation the finite eigenvalues of $\lambda E - A$ inside and outside the unit circle are mapped to eigenvalues in the open left and right half-planes, respectively; the finite eigenvalues on the unit circle except $\lambda = 1$ are mapped to eigenvalues on the imaginary axis, the eigenvalue $\lambda = 1$ is mapped to ∞; the infinite eigenvalues of $\lambda E - A$ are mapped to $\lambda = 1$ in the open right half-plane, see [19] for details. Thus, even if the pencil $\lambda E - A$ with singular E is d-stable, the Cayley-transformed pencil $\lambda\mathcal{E} - \mathcal{A}$ is not c-stable. Therefore, in the sequel the inertia theorems with respect to the unit circle will be established independently.

If one of the matrices E or A is nonsingular, then the GDALE (1.2) is equivalent to the standard discrete-time Lyapunov equations

$$(AE^{-1})^* X AE^{-1} - X = -E^{-*} G E^{-1} \tag{3.2}$$

or

$$X - (EA^{-1})^* X EA^{-1} = -A^{-*} G A^{-1}. \tag{3.3}$$

In this case the classical stability and inertia theorems [4, 15, 26] for (3.2) or (3.3) can be generalized to equation (1.2). The following stability theorem is a unit circle analogue of Theorem 2.3.

Theorem 3.2 *Let $\lambda E - A$ be a regular pencil. If all eigenvalues of $\lambda E - A$ are finite and lie inside the unit circle, then for every Hermitian, positive (semi)definite matrix G, the GDALE (1.2) has a unique Hermitian, positive (semi)definite solution X. Conversely, if there exist Hermitian, positive definite matrices X and G satisfying (1.2), then all eigenvalues of the pencil $\lambda E - A$ are finite and lie inside the unit circle.*

Unlike the GCALE (1.1), the GDALE (1.2) with singular E and positive definite G has a unique negative definite solution X if and only if the matrix A is nonsingular and all eigenvalues of the pencil $\lambda E - A$ lie outside the unit circle or, equivalently, the eigenvalues of the reciprocal pencil $E - \mu A$ are finite and lie inside the unit circle. However, if both the matrices E and A are singular, then the GDALE (1.2) may have no solutions although all finite eigenvalues of $\lambda E - A$ lie inside the unit circle.

Example 3.3 The GDALE (1.2) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is not solvable.

Consider the GDALE with a special right-hand side

$$A^*XA - E^*XE = -P_r^*GP_r - (I-P_r)^*G(I-P_r). \tag{3.4}$$

The following theorem generalizes the matrix inertia theorem [26] and gives a connection between the d-inertia of the pencil $\lambda E - A$ and the c-inertia of the Hermitian solution of the GDALE (3.4).

Theorem 3.4 *Let P_l and P_r be the spectral projections onto the left and right deflating subspaces of a regular pencil $\lambda E - A$ and let G be an Hermitian, positive definite matrix. If there exists an Hermitian matrix X that satisfies the GDALE (3.4) together with $P_l^*X = XP_l$, then*

$$\begin{aligned} &\pi_{<1}(E,A) = \pi_+(X), \\ &\pi_{>1}(E,A) + \pi_\infty(E,A) = \pi_-(X), \\ &\pi_1(E,A) = \pi_0(X) = 0. \end{aligned} \tag{3.5}$$

Conversely, if $\pi_1(E,A) = 0$, then there exist an Hermitian matrix X and an Hermitian, positive definite matrix G that satisfy the GDALE (3.4) and the inertia identities (3.5) hold.

Proof. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let Hermitian matrices G and X be as in (2.3). If X satisfies the GDALE (3.4), then the matrix equations

$$J^*X_{11}J - X_{11} = -G_{11}, \tag{3.6}$$
$$J^*X_{12} - X_{12}N = 0, \tag{3.7}$$
$$X_{22} - N^*X_{22}N = -G_{22} \tag{3.8}$$

are fulfilled. From $P_l^*X = XP_l$ we have that $X_{12} = 0$ and it satisfies equation (3.7). Since N is nilpotent, equation (3.8) has a unique Hermitian solution

$$X_{22} = -\sum_{j=0}^{\nu-1} (N^*)^j G_{22} N^j \tag{3.9}$$

that is negative definite if G_{22} is positive definite.

Consider now equation (3.6). It follows from the Sylvester law of inertia [6] and the matrix inertia theorem [26] that

$$\begin{aligned} &\pi_{<1}(E,A) = \pi_{<1}(J) = \pi_+(X_{11}) = \pi_+(X) - \pi_+(X_{22}) = \pi_+(X), \\ &\pi_{>1}(E,A) = \pi_{>1}(J) = \pi_-(X_{11}) = \pi_-(X) - \pi_-(X_{22}) = \pi_-(X) - \pi_\infty(E,A), \\ &\pi_1(E,A) = \pi_1(J) = \pi_0(X_{11}) = 0. \end{aligned}$$

Moreover, $\pi_0(X) = \pi_0(X_{11}) + \pi_0(X_{22}) = 0$.

Suppose that $\pi_1(E,A) = 0$. Then by the matrix inertia theorem [26] there exists an Hermitian matrix X_{11} such that $G_{11} = X_{11} - J^*X_{11}J$ is Hermitian, positive definite and

$$\pi_{<1}(J) = \pi_+(X_{11}), \qquad \pi_{>1}(J) = \pi_-(X_{11}), \qquad \pi_1(J) = \pi_0(X_{11}) = 0.$$

Furthermore, for every Hermitian positive definite matrix G_{22}, the matrix X_{22} as in (3.9) is Hermitian, negative definite and satisfies equation (3.8). Then $\pi_\infty(E,A) = \pi_-(X_{22})$ and $\pi_+(X_{22}) = \pi_0(X_{22}) = 0$. Thus, the Hermitian matrices

$$X = W^{-*}\begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1}, \qquad G = T^*\begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix} T$$

satisfy the GDALE (3.4), G is positive definite and the inertia identities (3.5) hold. ■

Remark 3.5 Note that if the GDALE (3.4) is solvable and if A is nonsingular, then the solution of (3.4) is unique. If both the matrices E and A are singular, then the nonuniqueness of the solution of (3.4) is resolved by requiring the extra condition for the nonunique part X_{12} to be zero. In terms of the original data this requirement can be expressed as $P_l^*X = XP_l$.

From Theorem 3.4 we obtain the following necessary and sufficient conditions for the pencil $\lambda E - A$ to be d-stable.

Corollary 3.6 *Let $\lambda E - A$ be a regular pencil and let P_l and P_r be spectral projection as in (1.4). For every Hermitian, positive definite matrix G, the GDALE (3.4) has an Hermitian solution X which is positive definite on $\operatorname{im} P_l$ if and only if the pencil $\lambda E - A$ is d-stable. Moreover, if E and A are singular and if a solution of (3.4) satisfies $P_l^*X = XP_l$, then it is unique and given by*

$$X = \frac{1}{2\pi}\int_0^{2\pi} (e^{i\varphi}E - A)^{-*}\Big(P_r^*GP_r - (I-P_r)^*G(I-P_r)\Big)(e^{i\varphi}E - A)^{-1}d\varphi.$$

A system of matrix equations

$$\begin{array}{l} A^*XA - E^*XE = -P_r^*GP_r - (I-P_r)^*G(I-P_r), \\ P_l^*X = XP_l \end{array} \tag{3.10}$$

is called *projected generalized discrete-time algebraic Lyapunov equation.*

There are unit circle analogies of Theorem 2.7 and Corollary 2.8 that can be established in the same way.

Theorem 3.7 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian matrix that satisfy the projected GDALE (3.10) with an Hermitian, positive semidefinite matrix G.*
1. If $\pi_1(E,A) = 0$, then $\pi_-(X) \le \pi_{>1}(E,A) + \pi_\infty(E,A)$, $\pi_+(X) \le \pi_{<1}(E,A)$.
2. If $\pi_0(X) = 0$, then $\pi_-(X) \ge \pi_{>1}(E,A) + \pi_\infty(E,A)$, $\pi_+(X) \ge \pi_{<1}(E,A)$.

Corollary 3.8 *Let $\lambda E - A$ be a regular pencil and let G be an Hermitian, positive semidefinite matrix. Assume that $\pi_1(E,A) = 0$. If there exists a nonsingular Hermitian matrix X that satisfies the projected GDALE (3.10), then the inertia identities (3.5) hold.*

Like the continuous-time case, the inertia identities (3.5) for Hermitian, positive semidefinite G can be obtained from controllability and observability conditions for the linear discrete-time descriptor system

$$\begin{aligned} Ex_{k+1} &= Ax_k + Bu_k, \quad x_0 = x^0, \\ y_k &= Cx_k, \end{aligned} \tag{3.11}$$

where $E,\, A \in \mathbb{F}^{n,n}$, $B \in \mathbb{F}^{n,q}$, $C \in \mathbb{F}^{p,n}$, $x_k \in \mathbb{F}^n$ is the state, $u_k \in \mathbb{F}^q$ is the control input and $y_k \in \mathbb{F}^p$ is the output, see [8].

The discrete-time descriptor system (3.11) is R (I ,S ,C)-controllable if the triplet (E, A, B) is R (I ,S ,C)-controllable and (3.11) is R (I ,S ,C)-observable if the triplet (E, A, C) is R (I S C)-observable.

Consider the projected GDALE

$$\begin{aligned} &A^*XA - E^*XE = -P_r^*C^*CP_r - (I-P_r)^*C^*C(I-P_r), \\ &P_l^*X = XP_l. \end{aligned} \tag{3.12}$$

Note that, in contrast with the GCALE in (2.13), the GDALE in (3.12) has two terms in the right-hand side. This makes possible to characterize not only R-observability but also S-observability and C-observability properties of the discrete-time descriptor system (3.11). We will show that the condition for the pencil $\lambda E - A$ to have no eigenvalues of modulus 1 and the condition for the solution of (3.12) to be nonsingular together are equivalent to the property for (E, A, C) to be C-observable.

Theorem 3.9 *Consider system (3.11) with a regular pencil $\lambda E - A$. Let X be an Hermitian solution of the projected GDALE (3.12). The triplet (E, A, C) is C-observable if and only if $\pi_1(E, A) = 0$ and X is nonsingular.*

Proof. Let the pencil $\lambda E - A$ be in Weierstrass canonical form (1.3) and let the matrix $CT^{-1} = [\, C_1,\, C_2\,]$ be partitioned conformally to E and A. The solution of the projected GDALE (3.12) has the form

$$X = W^{-*} \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} W^{-1}, \tag{3.13}$$

where X_{11} satisfies the Lyapunov equation

$$J^*X_{11}J - X_{11} = -C_1^*C_1, \tag{3.14}$$

and X_{22} satisfies the Lyapunov equation

$$X_{22} - N^*X_{22}N = -C_2^*C_2. \tag{3.15}$$

Since the triplet (E, A, C) is C-observable, conditions (2.10) and (2.12) hold. From (2.10) we obtain that the solution X_{11} of (3.14) is nonsingular and J has no eigenvalues on the unit circle [15, Theorem 13.2.4].

From (1.3) and (2.12) we have that

$$n = \operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} I_m & 0 \\ 0 & N \\ C_1 & C_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} N \\ C_2 \end{bmatrix} + m.$$

and, hence, the matrix $\begin{bmatrix} \lambda I - N \\ C_2 \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Then the solution X_{22} of (3.15) is nonsingular, since equation (3.15) is a special case of (3.14). Thus, the solution X of the projected GDALE (3.12) is nonsingular and $\pi_1(E, A) = 0$.

Conversely, let z be a right eigenvector of $\lambda E - A$ corresponding to a finite eigenvalue λ with $|\lambda| \neq 1$. We have

$$-\|Cz\|^2 = -z^*C^*Cz = z^*(A^*XA - E^*XE)z = (|\lambda|^2 - 1)z^*E^*XEz.$$

Since X is nonsingular, $Ez \neq 0$ and $\pi_1(E, A) = 0$, then $Cz \neq 0$, i.e., (E, A, C) satisfies (2.10).

For $z \in \ker E$, we obtain that $\|Cz\|^2 = z^*C^*Cz = -z^*A^*XAz \neq 0$ and, hence, (2.12) holds. Thus, the triplet (E, A, C) is C-observable. ■

Remark 3.10 It follows from Theorem 3.9 that if $\pi_1(E, A) = 0$ and an Hermitian solution X of (3.12) is nonsingular, then the triplet (E, A, C) is S-observable. However, S-observability of (E, A, C) does not imply that the solution of (3.12) is nonsingular.

Example 3.11 The projected GDALE (3.12) with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = (1,\ \ 0)$$

has the unique solution

$$X = \begin{pmatrix} -1/3 & 0 \\ 0 & 0 \end{pmatrix}$$

which is singular although $\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = 2$ and $\operatorname{rank}\begin{bmatrix} E \\ K_{E^*}^* A \\ C \end{bmatrix} = 2$.

As immediate consequence of Corollary 3.8 and Theorem 3.9 we obtain the following result.

Corollary 3.12 *Consider system (3.11) with a regular pencil $\lambda E - A$. Let the triplet (E, A, C) be C-observable. If an Hermitian matrix X satisfies the projected GDALE (3.12), then the inertia identities (3.5) hold.*

Furthermore, from Theorem 3.9 and Corollary 3.12 we have the following connection between d-stability of the pencil $\lambda E - A$, the C-observability of the triplet (E, A, C) and the existence of an Hermitian solution of the projected GDALE (3.12).

Corollary 3.13 *Consider the statements:*

1. *the pencil $\lambda E - A$ is d-stable,*
2. *the triplet (E, A, C) is C-observable,*
3. *the projected GDALE (3.12) has a unique solution X which is Hermitian, positive definite on* $\operatorname{im} P_l$ *and negative definite on* $\ker P_l$.

Any two of these statements together imply the third.

Remark 3.14 Note that Corollary 3.13 still holds if we replace the C-observability condition by the weaker condition for (E, A, C) to be R-observable and if we require for the solution of (3.12) only to be positive definite on $\operatorname{im} P_l$.

If the triple (E, A, C) is not C-observable, then we can derive the inertia inequalities similar to (2.16). Consider a proper observability matrix $\mathbf{O}_p$ as in (2.15) and an *improper observability matrix*

$$\mathbf{O}_i = \begin{bmatrix} CF_{-1} \\ CF_{-2} \\ \vdots \\ CF_{-\nu} \end{bmatrix},$$

where ν is the index of the pencil $\lambda E - A$ and the matrices F_{-k} have the form

$$F_{-k} = T^{-1} \begin{pmatrix} 0 & 0 \\ 0 & N^{k-1} \end{pmatrix} W^{-1}, \qquad k = 1, 2, \ldots.$$

Here T, W and N are as in (1.3). Clearly, $F_{-k} = 0$ for $k > \nu$. The triplet (E, A, C) is C-observable if and only if $\operatorname{rank} \mathbf{O}_p = n - \pi_\infty(E, A)$ and $\operatorname{rank} \mathbf{O}_i = \pi_\infty(E, A)$, see [2]. The nullspaces of $\mathbf{O}_p$ and $\mathbf{O}_i$ are the *proper and improper unobservable subspaces*, respectively, for the descriptor system (3.11). Using the Weierstrass canonical form (1.3) and representation (3.13) for the solution X of the projected GDALE (3.12) we obtain the following inertia inequalities.

Theorem 3.15 *Let $\lambda E - A$ be a regular pencil and let X be an Hermitian solution of the projected GDALE (3.12). Then*

$$\begin{aligned} |\pi_{<1}(E, A) - \pi_+(X)| &\leq n - \pi_\infty(E, A) - \operatorname{rank} \mathbf{O}_p, \\ |\pi_{>1}(E, A) - \pi_-(X) + \operatorname{rank} \mathbf{O}_i| &\leq n - \pi_\infty(E, A) - \operatorname{rank} \mathbf{O}_p. \end{aligned}$$

Remark 3.16 All results of this section can be reformulated for the projected GDALE

$$\begin{aligned} &A^*XA - E^*XE = -P_r^*GP_r + s(I - P_r)^*G(I - P_r), \\ &P_l^*X = XP_l, \end{aligned}$$

where s is 0 or 1. For these equations we must consider instead of (3.5) the inertia identities

$$\begin{aligned} \pi_{<1}(E, A) &= \pi_+(X), \qquad & \pi_{>1}(E, A) &= \pi_-(X), \\ \pi_1(E, A) &= 0, \qquad & \pi_\infty(E, A) &= \pi_0(X) \end{aligned}$$

for the case $s = 0$, and

$$\begin{aligned} \pi_{<1}(E, A) + \pi_\infty(E, A) &= \pi_+(X), \qquad & \pi_{>1}(E, A) &= \pi_-(X), \\ \pi_1(E, A) &= \pi_0(X) = 0 \end{aligned}$$

for the case $s = 1$.

By duality of controllability and observability conditions analogies of Theorems 3.9, 3.15 and Corollaries 3.12, 3.13 can be obtained for the dual projected GDALE

$$\begin{aligned} &AXA^* - EXE^* = -P_lBB^*P_l^* + s(I - P_l)BB^*(I - P_l)^*, \\ &P_rX = XP_r^*. \end{aligned}$$

4 Conclusions

We have studied generalized continuous-time and discrete-time Lyapunov equations and presented generalizations of Lyapunov stability theorems and matrix inertia theorems for matrix pencils. We also have shown that the stability, controllability and observability properties of descriptor systems can be characterized in terms of solutions of generalized Lyapunov equations with special right-hand sides.

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