Solving projected generalized Lyapunov equations using SLICOT

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***Abstract*— We discuss the numerical solution of projected generalized Lyapunov equations. Such equations arise in many control problems for linear time-invariant descriptor systems including stability analysis, balancing and model order reduction. We present solvers for projected generalized Lyapunov equations based on matrix equations subroutines that are available in the Subroutine Library In COntrol Theory (SLICOT).**

I. INTRODUCTION

Consider the projected generalized continuous-time algebraic Lyapunov equation (GCALE)

$$\begin{aligned} E^T XA + A^T XE + P_r^T GP_r = 0,\\ X - P_l^T XP_l = 0, \end{aligned} \tag{1}$$

and the projected generalized discrete-time algebraic Lyapunov equation (GDALE)

$$\begin{aligned} A^T XA - E^T XE + s_1 P_r^T GP_r - s_2 Q_r^T GQ_r = 0,\\ X - P_l^T XP_l - Q_l^T XQ_l = 0, \end{aligned} \tag{2}$$

where E, A, $G \in \mathbb{R}^{n,n}$ are given matrices, $X \in \mathbb{R}^{n,n}$ is an unknown matrix, P_l and P_r are the spectral projectors onto the left and right deflating subspaces of the regular pencil $\lambda E - A$ corresponding to the finite eigenvalues, $Q_l = I - P_l$, $Q_r = I - P_r$ and s_1, s_2 are 0 or 1 with $s_1^2 + s_2^2 \neq 0$. Such equations arise in many control problems for linear time-invariant descriptor systems

$$\begin{aligned} E(\mathcal{D}x(t)) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t), \end{aligned} \tag{3}$$

where $\mathcal{D}x(t) = \dot{x}(t)$, $t \in \mathbb{R}$, in the continuous-time case and $\mathcal{D}x(t) = x_{t+1}$, $t \in \mathbb{Z}$, in the discrete-time case. In particular, the asymptotic stability as well as controllability and observability properties of system (3) can be characterized in terms of solutions of equations (1) and (2), see [3], [23], [31]. Furthermore, these equations can be used to compute the $\mathbb{H}_2$, Hilbert-Schmidt and Hankel norms of (3), see [25]. Finally, the projected Lyapunov equations play a fundamental role in balanced truncation model reduction of descriptor systems [24]. Note that in this problem it is also required to solve the projected GCALE

$$\begin{aligned} EXA^T + AXE^T + P_l GP_l^T = 0,\\ X - P_r XP_r^T = 0 \end{aligned} \tag{4}$$

and the projected GDALE

$$\begin{aligned} AXA^T - EXE^T + s_1 P_l GP_l^T - s_2 Q_l GQ_l^T = 0,\\ X - P_r XP_r^T - Q_r XQ_r^T = 0, \end{aligned} \tag{5}$$

that are dual to (1) and (2), respectively.

In the literature also other types of generalized Lyapunov equations have been considered that are useful in stability analysis and optimal regulator problem for descriptor systems [11], [16], [26], [27]. However, the application of such equations is usually limited to index one problems, whereas the existence and uniqueness results for projected Lyapunov equations can be stated independently of the index of the pencil $\lambda E - A$, see [21], [23].

In this paper, we discuss several solvers for projected Lyapunov equations that are based on the generalized Schur-Bartels-Stewart method and the generalized Schur-Hammarling method presented in [22]. These solvers are implemented using efficient matrix equations subroutines available in the SLICOT Library [4], [28]. In general, SLICOT includes Fortran implementations of numerical algorithms for solving different system and control problems together with standardized interfaces for MATLAB [17] and Scilab [9], see also the SLICOT webpage `http://www.slicot.de/`. Note that SLICOT routines often provide not only the solution of the problem but also condition number estimates and forward error bounds that allow the user to evaluate the accuracy of the computed solution.

II. PROJECTED LYAPUNOV EQUATIONS

In this section, we briefly describe the Schur-Bartels-Stewart and Schur-Hammarling methods for projected Lyapunov equations, see [21], [22] for details.

Let the pencil $\lambda E - A$ be in generalized real Schur form

$$E = V \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} U^T, \quad A = V \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} U^T, \tag{6}$$

where U and V are orthogonal, E_f is upper triangular nonsingular, E_∞ is upper triangular nilpotent, A_f is upper quasi-triangular and A_∞ is upper triangular nonsingular. In this case the projectors P_l and P_r can be represented as

$$P_l = V \begin{bmatrix} I & -Z \\ 0 & 0 \end{bmatrix} V^T, \quad P_r = U \begin{bmatrix} I & -Y \\ 0 & 0 \end{bmatrix} U^T, \tag{7}$$

where Y and Z satisfy the generalized Sylvester equation

$$\begin{aligned} E_f Y - ZE_\infty &= -E_u,\\ A_f Y - ZA_\infty &= -A_u. \end{aligned} \tag{8}$$

Let the matrices

$$U^T GU = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \quad V^T GV = \begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix} \tag{9}$$

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be partitioned in blocks conformally with E and A in (6) and let

$$\begin{aligned}\widetilde{G}_{11} &= \hat{G}_{11} - Z\hat{G}_{21} - \hat{G}_{12}Z^T + Z\hat{G}_{22}Z^T,\\ \widetilde{G}_{22} &= Y^T G_{11} Y + Y^T G_{12} + G_{21} Y + G_{22}.\end{aligned} \tag{10}$$

Using (6)-(10), one can show that the solutions of the projected Lyapunov equations have the following representations:

- the projected GCALE (1) has the solution

$$X = V \begin{bmatrix} X_{11} & -X_{11}Z \\ -Z^T X_{11} & Z^T X_{11} Z \end{bmatrix} V^T, \tag{11}$$

  where X_{11} satisfies the GCALE

$$E_f^T X_{11} A_f + A_f^T X_{11} E_f = -G_{11}. \tag{12}$$

- the projected GCALE (4) has the solution

$$X = U \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} U^T,$$

  where X_{11} solves the GCALE

$$E_f X_{11} A_f^T + A_f X_{11} E_f^T = -\widetilde{G}_{11}. \tag{13}$$

- the projected GDALE (2) has the solution

$$X = V \begin{bmatrix} X_{11} & -X_{11}Z \\ -Z^T X_{11} & X_{22} + Z^T X_{11} Z \end{bmatrix} V^T, \tag{14}$$

  where X_{11} and X_{22} satisfy the GDALEs

$$\begin{aligned} A_f^T X_{11} A_f - E_f^T X_{11} E_f &= -s_1 G_{11},\\ A_\infty^T X_{22} A_\infty - E_\infty^T X_{22} E_\infty &= s_2 \widetilde{G}_{22}.\end{aligned} \tag{15}$$

- the projected GDALE (5) has the solution

$$X = U \begin{bmatrix} X_{11} + Y X_{22} Y^T & Y X_{22} \\ X_{22} Y^T & X_{22} \end{bmatrix} U^T,$$

  where X_{11} and X_{22} solve the GDALEs

$$\begin{aligned} A_f X_{11} A_f^T - E_f X_{11} E_f^T &= -s_1 \widetilde{G}_{11},\\ A_\infty X_{22} A_\infty^T - E_\infty X_{22} E_\infty^T &= s_2 \hat{G}_{22}.\end{aligned} \tag{16}$$

Thus, to compute the solution of the projected Lyapunov equation, we need to reduce the pencil to the generalized Schur form (6) and to solve the generalized Sylvester equation (8) as well as the corresponding generalized Lyapunov equations. To compute the generalized Schur form (6) we can use the QZ method [8], [30] or algorithms proposed in [2], [6], [7], [29] based on row/column compression. For solving the generalized Sylvester equation (8) one can use the generalized Schur method [15] or its recursive blocked modification [12] that is more suitable for large problems. The solutions of the generalized Lyapunov equations (12), (13), (15) and (16) can be computed using the generalized Bartels-Stewart method [1], [18].

In some applications, such as model order reduction, it is required to compute the Cholesky factor of the solution of stable projected Lyapunov equations rather than the solution itself. The term *stable* means here that all the finite eigenvalues of the pencil $\lambda E - A$ have negative real part in the continuous-time case or moduli less than one in the discrete-time case, and the matrix $G = C^T C$ is symmetric, positive semidefinite. The solution of such projected Lyapunov equations can be computed in the factored form $X = R^T R$, where R is a full row rank Cholesky factor of X. This factor can be determined directly without forming the product $C^T C$ and the solution X itself if we apply the generalized Hammarling method [10], [18] for computing the Cholesky factors R_{11} and R_{22} of the solutions $X_{11} = R_{11}^T R_{11}$ and $X_{22} = R_{22}^T R_{22}$ of the corresponding generalized Lyapunov equations. Then, for example, the Cholesky factor of the solution $X = \hat{R}^T \hat{R}$ of the projected GDALE (2) is given by

$$\hat{R} = \begin{bmatrix} R_{11} & -R_{11} Z \\ 0 & R_{22} \end{bmatrix} V^T.$$

The full row rank Cholesky factor R of $X = R^T R$ can then be computed from the QR decomposition $\hat{R} = QR$, where Q has orthonormal columns and R is of full row rank. The full rank Cholesky factors of the solutions of the projected Lyapunov equations (1), (4) and (5) can be determined similarly.

In solving matrix equations it is very important to study the sensitivity of the problem to perturbations in the input data and to bound errors in the computed solution. The solution of projected Lyapunov equations is determined essentially in two steps that include first a computation of the deflating subspaces of a pencil corresponding to the finite and infinite eigenvalues via a reduction to the generalized Schur form (6) and solving the generalized Sylvester equation (8) and then a calculation of the solution of the corresponding generalized Lyapunov equations as in (12)-(16). In this case it may happen that although the projected Lyapunov equation is well-conditioned, one of the intermediate problems may be ill-conditioned. This may lead to large inaccuracy in the computed solution of the original problem. Therefore, along with the conditioning of the projected Lyapunov equation we should also consider the condition numbers for the deflating subspaces.

An important quantity that measures the sensitivity of the right and left deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues to perturbations in E and A is a separation $\mathrm{Dif} = \mathrm{Dif}(E_f, A_f; E_\infty, A_\infty)$ of the pencils $\lambda E_f - A_f$ and $\lambda E_\infty - A_\infty$ defined by

$$\begin{aligned}\mathrm{Dif} &= \inf_{\|[Y,Z]\|_F = 1} \big\| [E_f Y - Z E_\infty, A_f Y - Z A_\infty] \big\|_F \\ &= \sigma_{\min}(S),\end{aligned} \tag{17}$$

see [5], [14], [20] for details. Here $\|\cdot\|_F$ denotes the Frobenius matrix norm, the matrix S has the form

$$S = \begin{bmatrix} I \otimes E_f & -E_\infty^T \otimes I \\ I \otimes A_f & -A_\infty^T \otimes I \end{bmatrix},$$

where the symbol $\otimes$ stands for the Kronecker product of two matrices, and $\sigma_{\min}(S)$ is the smallest singular value of S. The reciprocal of the separation $\mathrm{Dif}(E_f, A_f; E_\infty, A_\infty)$ can also be used as a condition number of the generalized Sylvester equation (8) that measures the sensitivity of the solution of this equation to perturbations in the data [13],

[15]. The conditioning of the deflating subspaces of $\lambda E - A$ can also be characterized by the spectral norms of the projectors P_l and P_r given by

$$\|P_r\|_2 = \sqrt{1+\|Y\|_2^2}, \quad \|P_l\|_2 = \sqrt{1+\|Z\|_2^2}.$$

The *condition numbers* for the projected GCALE (1) and the projected GDALE (2) are defined by

$$\begin{aligned} \kappa_c(E,A) &= 2\|E\|_2\|A\|_2\|H_c\|_2, \\ \kappa_d(E,A) &= \big(\|E\|_2^2 + \|A\|_2^2\big)\|H_d\|_2, \end{aligned} \tag{18}$$

respectively, where H_c and H_d are the solutions of (1) and (2), respectively, with $G = I$. These condition numbers measure the sensitivity of the solutions of the projected Lyapunov equations (1) and (2) to perturbations in E, A and G, see [21], [22]. The condition numbers for the projected Lyapunov equations (4) and (5) can be defined similarly.

III. SOLVERS

The following MATLAB functions have been implemented

```
[X, out] = pgcale(A, E, G, flag, trans),
[X, out] = pgdale(A, E, G, flag, trans,s)
```

that can be used for solving the projected GCALE (1) or (4) and the projected GDALE (2) or (5), respectively. The optional input parameter `flag` is the vector with two components characterizing the structure of the pencil $\lambda E - A$. Specifically, if `flag(1)` < 0, then $\lambda E - A$ is in general form; otherwise, $\lambda E - A$ is in the generalized Schur form (6) and `flag(1)` ≥ 0 is the number of the finite eigenvalues of the pencil $\lambda E - A$ counting their multiplicities. If `flag(2)` $= 0$, then solving the Sylvester equation (8) is required; otherwise, the solution Y and Z of (8) is known. In this case the input matrices E and A should have the form

$$E = \begin{bmatrix} E_f & Y \\ 0 & E_\infty \end{bmatrix}, \qquad A = V\begin{bmatrix} A_f & Z \\ 0 & A_\infty \end{bmatrix}.$$

Default value is `flag = [-1,0]`. The optional input parameter `trans` determines the type of the projected Lyapunov equation. In particular, `trans` $= 0$ if the projected Lyapunov equation (1) or (2) has to be solved, and `trans` $= 1$ if the solution of the projected Lyapunov equation (4) or (5) is required. Default value is `trans = 0`. The input parameter `s` in `pgdale` is the vector with two components that gives the values `s(1)` $= s_1$ and `s(2)` $= s_2$ for the projected GDALEs (2) and (5).

The optional output parameter

```
out = [dif, pl, pr, kappa]
```

contains the estimate `dif` on the separation $\mathrm{Dif} = \mathrm{Dif}(E_f, A_f; E_\infty, A_\infty)$ defined in (17), the spectral norms `pl` $= \|P_l\|_2$ and `pr` $= \|P_r\|_2$ and the condition number `kappa` which is equal to $\kappa_c(E,A)$ in the continuous-time case and $\kappa_d(E,A)$ in the discrete-time case as defined in (18).

We have also implemented the MATLAB functions

```
R = pgscle(A, E, C, flag, trans),
R = pgsdle(A, E, C, flag, trans, s)
```

that can be used to compute the full rank Cholesky factor R of the solution $X = \mathrm{op}(R)^T\mathrm{op}(R)$ of the projected GCALE (1) or (4) and the projected GDALE (2) or (5), respectively, with $G = \mathrm{op}(C)^T\mathrm{op}(C)$. Here $\mathrm{op}(C) = C$ for equations (1) and (2) and $\mathrm{op}(C) = C^T$ for equations (4) and (5).

In our implementations we have used the following MATLAB functions for solving Sylvester and Lyapunov equations that are available in SLICOT [19]:

`slgesg`	for the generalized Sylvester equation (8),
`slgely`	for the GCALEs (12) and (13),
`slgest`	for the GDALEs (15) and (16),
`slgsly`	for the stable GCALEs (12) and (13),
`slgsst`	for the stable GDALEs (15) and (16).

These functions call the MEX-file `genleq` based on the corresponding SLICOT Fortran routines for generalized Sylvester and Lyapunov equations.

IV. NUMERICAL EXAMPLES

In this section we present the results of some numerical experiments. Computations were carried out on IBM PC computer using MATLAB 7 (R14) with relative machine precision $\varepsilon \approx 2.22 \cdot 10^{-16}$.

Example 1: Consider the projected GCALE (1) with

$$E = V\begin{bmatrix} I_3 & D(N_3 - I_3) \\ 0 & N_3 \end{bmatrix}U^T, \quad A = V\begin{bmatrix} J & (I_3 - J)D \\ 0 & I_3 \end{bmatrix}U^T,$$

$$G = U\begin{bmatrix} G_{11} & -G_{11}D \\ -DG_{11} & DG_{11}D \end{bmatrix}U^T, \tag{19}$$

where N_3 is the nilpotent Jordan block of order 3,

$$\begin{aligned} G_{11} &= \mathrm{diag}(2,\,4,\,6), \\ J &= \mathrm{diag}(-10^{-k},\,-2,\,-3\times 10^{k}), \\ D &= \mathrm{diag}(10^{-k},\,1,\,10^{k}), \end{aligned}$$

with $k \geq 0$. The transformation matrices V and U are elementary reflections chosen as

$$\begin{aligned} V &= I_6 - \frac{1}{3}ee^T, \qquad e = (1,\,1,\,1,\,1,\,1,\,1)^T, \\ U &= I_6 - \frac{1}{3}ff^T, \qquad f = (1,-1,\,1,-1,\,1,-1)^T. \end{aligned} \tag{20}$$

The exact solution of the generalized Sylvester equation (8) is $Y = Z = D$ and the exact solution of the projected GCALE (1) is given in (11) with $X_{11} = \mathrm{diag}(10^k,\,1,\,10^{-k})$.

Figure 1 shows the values of $1/\mathrm{Dif}$ and $\kappa_c(E,A)$ as functions of k. One can see that the condition numbers of the generalized Sylvester equation (8) and the projected GCALE (1) increase as k grows, i.e., the problem tends to be ill-conditioned for increasing k. Figure 2 presents the relative error $\mathtt{RERR} = \|\hat{X} - X\|_2/\|X\|_2$ (top plot) and the relative residual (bottom plot)

$$\mathtt{RRESC} = \frac{\|E^T\hat{X}A + A^T\hat{X}E + \hat{P}_r^T G\hat{P}_r\|_2}{2\|E\|_2\|A\|_2\|X\|_2},$$

where $\hat{X}$ is the computed solution of (1) and $\hat{P}_r$ is the computed projector onto the right deflating subspace of the pencil $\lambda E - A$ corresponding to the finite eigenvalues. We

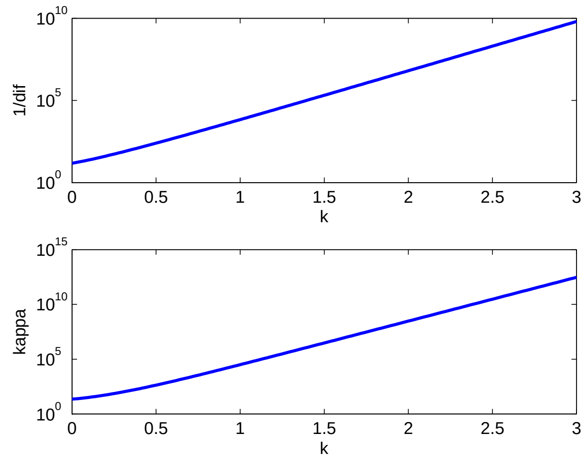

Fig. 1. Reciprocal of the separation `dif` $=$ Dif (top) and the condition number `kappa` $= \kappa_c(E,A)$ (bottom) for the projected continuous-time Lyapunov equation.

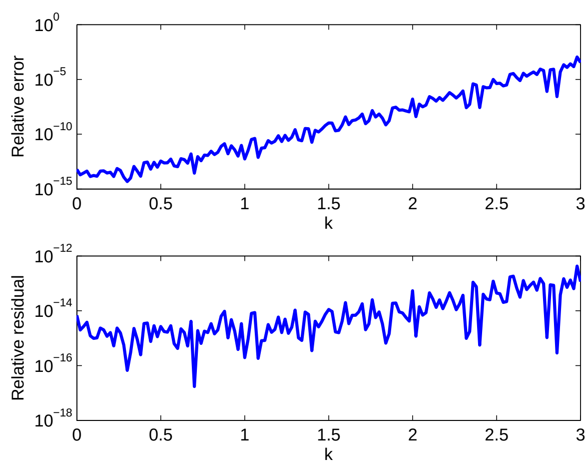

Fig. 2. Relative errors (top) and relative residuals (bottom) for the projected continuous-time Lyapunov equation.

see that the relative residuals are small even for the ill-conditioned problems. However, this does not imply that the relative error in the computed solution remains close to zero when the condition number $\kappa_c(E,A)$ is large. The relative error in $\hat{X}$ increases as $\kappa_c(E,A)$ grows.

Example 2: Consider the projected GDALE (2) with

$$E = V\begin{bmatrix} I_3 & D(N_3 - I_3) \\ 0 & N_3 \end{bmatrix} U^T, \quad A = V\begin{bmatrix} J_1 & (J_2 - J_1)D \\ 0 & J_2 \end{bmatrix} U^T,$$

where U, V are given in (20) and

$$\begin{aligned} J_1 &= \mathrm{diag}(1 - 10^{-k},\, 1/2,\, 0), \\ J_2 &= \mathrm{diag}(10^{k},\, 1,\, 10^{-k}), \\ D &= \mathrm{diag}(10^{-3k/2},\, 1,\, 10^{3k/2}) \end{aligned}$$

with $k \geq 0$. The matrix G is as in (19) with

$$G_{11} = \mathrm{diag}(2 - 10^{-k},\, 3/4,\, 10^{-k})$$

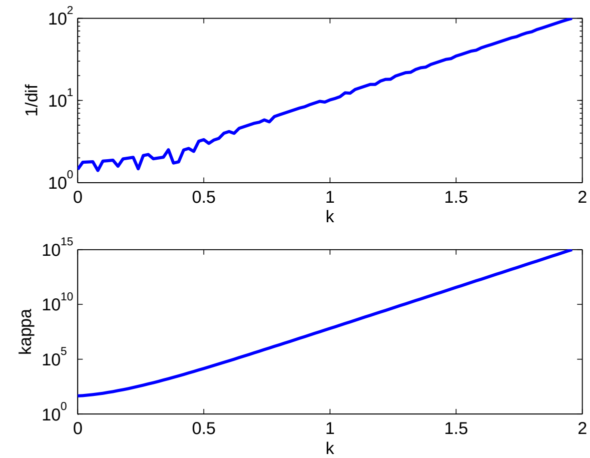

Fig. 3. Reciprocal of the separation `dif` $=$ Dif (top) and the condition number `kappa` $= \kappa_d(E,A)$ (bottom) for the projected discrete-time Lyapunov equation.

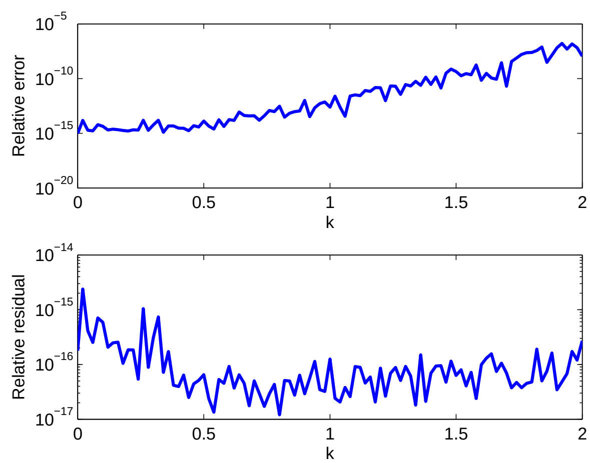

Fig. 4. Relative errors (top) and relative residuals (bottom) for the projected discrete-time Lyapunov equation.

and $s_1 = s_2 = 1$. The exact solution of the generalized Sylvester equation (8) is $Y = Z = D$. The exact solution of the projected GDALE (2) has the form (14) with $X_{22} = 0$ and $X_{11} = \mathrm{diag}(10^k,\, 1,\, 10^{-k})$.

Figure 3 shows the values of $1/\mathrm{Dif}$ and $\kappa_d(E,A)$ as functions of k. One can see that the generalized Sylvester equation (8) is well-conditioned for all $k \in [0,2]$, while the condition number of the projected GDALE (2) grows with k. The relative error $\mathtt{RERR} = \|\hat{X} - X\|_2 / \|X\|_2$ and the relative residual

$$\mathtt{RRESD} = \frac{\|A^T\hat{X}A - E^T\hat{X}E + \hat{P}_r^T G \hat{P}_r - \hat{Q}_r^T G \hat{Q}_r\|_2}{(\|E\|_2^2 + \|A\|_2^2)\|X\|_2}$$

are shown in Figure 4. We see that even though the relative residual remains small, the accuracy in $\hat{X}$ is getting worse for the large condition number $\kappa_d(E,A)$.

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