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Passivity-preserving model reduction of differential-algebraic equations in circuit simulation

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We present an extension of the positive real balanced truncation model reduction method for differential-algebraic equations that arise in circuit simulation. This method is based on balancing the solutions of the projected generalized algebraic Riccati equations. Important properties of this method are that passivity is preserved in the reduced-order model and that there exists an approximation error bound. Numerical solution of the projected Riccati equations using the special structure of circuit equations is also discussed.

1 Introduction

Consider a linear time-invariant system of differential-algebraic equations (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t), \tag{1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control input and $y(t) \in \mathbb{R}^p$ is an output. Such systems arise frequently in circuit simulation. A modified nodal analysis (MNA) for linear RLC circuits yields system (1) with

$$E = \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T, \quad D = 0. \tag{2}$$

Here $A_{\mathcal{C}}$, $A_{\mathcal{L}}$, $A_{\mathcal{R}}$, $A_{\mathcal{V}}$ and $A_{\mathcal{I}}$ are the incidence matrices describing the circuit topology, and $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$ are the symmetric, positive definite resistance, inductance and capacitance matrices, respectively. Linear RLC circuits are often used to model interconnects, transmission lines and pin packages in VLSI networks. The order n of the MNA system (1) is related to the number of circuit elements and it is usually very large. This makes the analysis and numerical simulation of circuit equations unacceptably time consuming. Therefore, model order reduction is of great importance.

A general idea of model order reduction is to approximate the large-scale system (1) by a reduced-order model

$$\widetilde{E}\,\dot{\widetilde{x}}(t) = \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \qquad \widetilde{y}(t) = \widetilde{C}\,\widetilde{x}(t) + \widetilde{D}\,u(t), \tag{3}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{p,\ell}$, $\widetilde{D} \in \mathbb{R}^{p,m}$ and $\ell \ll n$. It is required that the approximate model (3) captures the input-output behavior of the original system (1) to a required accuracy and preserves essential physical properties such as stability and passivity. These properties are important in circuit simulation and system design. Generally speaking, passivity means that system does not produce energy via the input-output channel. System (1) is *passive* if and only if its transfer function given by $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ is *positive real*, i.e., $\boldsymbol{G}$ is analytic in $\mathbb{C}_+ = \{\, s \in \mathbb{C} \,:\, \mathrm{Re}(s) > 0 \,\}$ and the matrix $\boldsymbol{G}(s) + \boldsymbol{G}^T(\bar{s})$ is positive semidefinite for all $s \in \mathbb{C}_+$, see [1].

2 Positive real balanced truncation for circuit equations

Balanced truncation model reduction is based on the transformation of the dynamical system into a balanced form whose controllability and observability Gramians are both equal to a diagonal matrix. Then a reduced-order model is determined by the truncation of the states corresponding to small diagonal elements of the balanced Gramians. Depending on system properties, different types of balancing may be defined. In the *positive real balanced truncation* method, the Gramians are defined as stabilizing solutions of the projected generalized algebraic Riccati equations

$$E\,X\hat{A}^T + \hat{A}XE^T + EXC^T\hat{D}^{-1}CXE^T + P_lB\hat{D}^{-1}B^TP_l^T = 0, \qquad X = P_rXP_r^T, \tag{4}$$

$$E^TY\hat{A} + \hat{A}^TYE + E^TYB\hat{D}^{-1}B^TYE + P_r^TC^T\hat{D}^{-1}CP_r = 0, \qquad Y = P_l^TYP_l, \tag{5}$$

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where $\hat{A} = A - P_l B \hat{D}^{-1} C P_r$, P_r and P_l are the spectral projectors onto the right and left deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues, $\hat{D} = M_0 + M_0^T$ is assumed to be nonsingular, and M_0 is a constant term in the Laurent series of the transfer function $\boldsymbol{G}$ at infinity $\boldsymbol{G}(s) = \sum_{j=-\infty}^{\infty} M_j s^{-j}$. For the DAE control system (1) that is not necessarily minimal but that has the proper transfer function $\boldsymbol{G}$, we have the following algorithm.

Algorithm 1 Positive real balanced truncation method for DAE systems.
Given a passive system $\boldsymbol{G} = [E,\ A,\ B,\ C,\ D]$ and projectors P_r, P_l, compute a reduced-order system $\widetilde{\boldsymbol{G}} = [\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}]$.

1. Compute the matrices $M_0 = C(I - P_r)(sE - A)^{-1}B + D$ and $\hat{D} = M_0 + M_0^T$.
2. Compute the Cholesky factors R and L of the positive real Gramians $X = RR^T$ and $Y = LL^T$ that are the stabilizing solutions of the projected generalized Riccati equations (4) and (5), respectively.
3. Compute the singular value decomposition $L^T E R = [\,U_1,\ U_2\,]\mathrm{diag}(\Pi_1, \Pi_2)[\,V_1,\ V_2\,]^T$, where $[\,U_1, U_2\,]$, $[\,V_1, V_2\,]$ have orthonormal columns, $\Pi_1 = \mathrm{diag}(\pi_1, \ldots, \pi_\ell)$, $\Pi_2 = \mathrm{diag}(\pi_{\ell+1}, \ldots, \pi_r)$ with $\pi_1 \geq \ldots \geq \pi_\ell > \pi_{\ell+1} \geq \ldots \geq \pi_r > 0$.
4. Compute $[\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D}\,] = [\,W^T E T,\ W^T A T,\ W^T B,\ C T,\ M_0\,]$ with $W = L U_1 \Pi_1^{-1/2}$ and $T = R V_1 \Pi_1^{-1/2}$.

Similarly to Lyapunov-based balanced truncation [2], this method can also be extended to systems with the improper transfer function $\boldsymbol{G}$. In this case we have, in addition, to compute the improper controllability and observability Gramians by solving the projected generalized discrete-time Lyapunov equations.

Analogously to the standard state space case [3], one can show that the reduced-order model (3) computed by Algorithm 1 is passive, and we have the following $\mathbb{H}_\infty$-norm error bound

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \leq 2\|\hat{D}^{-1}\|\|\boldsymbol{G} + M_0^T\|_{\mathbb{H}_\infty}\|\widetilde{\boldsymbol{G}} + M_0^T\|_{\mathbb{H}_\infty}(\pi_{\ell+1} + \ldots + \pi_r),$$

where $\|\boldsymbol{G}\|_{\mathbb{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{G}(i\omega)\|$ denotes the $\mathbb{H}_\infty$-norm of $\boldsymbol{G}$ and $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}$.

Since the MNA matrices in (2) satisfy $E^T = SES$, $A^T = SAS$ and $B^T = S_1 C S$, where $S = \mathrm{diag}(I, -I, -I) = S^T$ and $S_1 = \mathrm{diag}(I, -I) = S_1^T$ are partitioned in accordance with A and C, respectively, we obtain that $P_l = S P_r^T S$ and $X = SYS^T = SLL^T S^T = RR^T$ with $R = SL$. Then the matrix $L^T E R = L^T E S L$ is symmetric. Thus, for circuit equations it is enough to solve only one Riccati equation, the solution of the second one is given for free. Furthermore, instead of the singular value decomposition (SVD) we can compute the eigenvalue decomposition $L^T E S L = U \Lambda U^T$ that is less expensive than the computation of the SVD.

To compute the solution of the projected Riccati equation (5) we can use a method based on the construction of a basis of deflating subspaces of an extended Hamiltonian pencil

$$\lambda\mathcal{M} - \mathcal{N} = \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & P_l B \\ 0 & -A^T & -P_r^T C^T \\ C P_r & B^T P_l^T & \hat{D} \end{bmatrix}.$$

Note that if λ is an eigenvalue of $\lambda\mathcal{M} - \mathcal{N}$, then $-\overline{\lambda}$ is also an eigenvalue of this pencil. This property is known as Hamiltonian symmetry of the pencil. One can show that if the columns of a matrix $\mathcal{Z} = [Z_1^T, Z_2^T, Z_3^T]^T$ partitioned in blocks in accordance with $\mathcal{N}$, span a right deflating subspace of the pencil $\lambda\mathcal{M} - \mathcal{N}$ corresponding to the eigenvalues with negative real part, i.e., $\mathcal{M}\mathcal{Z}\Lambda_- = \mathcal{N}\mathcal{Z}$, then the stabilizing solution of the projected Riccati equation (5) is given by $Y = -Z_2(EZ_1)^-$, provided Z_1 is of full column rank. Here $(EZ_1)^-$ is a pseudoinverse of EZ_1 satisfying $(EZ_1)^-(EZ_1) = I$. The deflating subspaces of $\lambda\mathcal{M} - \mathcal{N}$ can be computed using Krylov subspace methods like Lanczos or Arnoldi procedure.

The projected Riccati equation (5) can also be solved using Newton's method [4]. In each step of this method, the projected generalized Lyapunov equation of the form $E^T Z F + F^T Z E + P_r^T Q^T Q P_r = 0$, $Z = P_l^T Z P_l$ has to be solved for Z. For this propose we can use the generalized alternating direction implicit method [5].

Note that the solution of the large projected Riccati equation has often a low numerical rank. Such a solution can be well approximated by a matrix of low rank. Moreover, this low-rank approximation can be constructed directly in factored form $Y \approx \hat{L}\hat{L}^T$ with $\hat{L} \in \mathbb{R}^{n,k}$ that significantly reduces the memory requirements when $k \ll n$. In this case we have to compute the eigenvalue decomposition of the $k \times k$ matrix $\hat{L}^T E S \hat{L}$ only.

A major difficulty in the numerical solution of the projected Lyapunov and Riccati equations with large matrix coefficients is that the spectral projectors P_l and P_r are required. Fortunately, the matrix coefficients (2) in the MNA circuit equations have some special block structure. We can exploit this structure to construct the required projectors in explicit form using a matrix chain approach presented in [6].

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