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Balanced truncation model reduction for descriptor systems

We present a generalization of a balanced truncation model reduction method for descriptor systems. This method is closely related to the proper and improper controllability and observability Gramians and Hankel singular values that can be computed by solving projected generalized Lyapunov equations. We demonstrate the application of balanced truncation model reduction to the semidiscretized Stokes equation.

1. Introduction

Consider a linear time-invariant continuous-time descriptor system

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \qquad y(t) = C\,x(t), \tag{1}$$

where $E,\ A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the output. We will assume that system (1) is *asymptotically stable*, that is, the pencil $\lambda E - A$ is regular ($\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$) and all the finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. Descriptor systems arise naturally in many applications such as electrical circuit simulation and semidiscretization of partial differential equations. The order n of such systems is typically very large, while the number m of inputs and the number p of outputs are small compared to n.

The model order reduction problem consists in an approximation of system (1) by a reduced order system

$$\widetilde{E}\,\dot{\widetilde{x}}(t) = \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \qquad \widetilde{y}(t) = \widetilde{C}\,\widetilde{x}(t), \tag{2}$$

where $\widetilde{E},\ \widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$ and $\widetilde{C} \in \mathbb{R}^{p,\ell}$. The order ℓ of this system is much smaller than the order n of (1). It is desired that the approximate system (2) is also asymptotically stable and the approximation error is small in some sense. As a measure of the accuracy of the approximation we can use the $\mathcal{H}_\infty$-norm of an error system given by $\|\mathbf{G} - \widetilde{\mathbf{G}}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)\|$, where $\mathbf{G}(s) = C(sE - A)^{-1}B$ and $\widetilde{\mathbf{G}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B}$ are the *transfer functions* of systems (1) and (2), respectively, and $\|\cdot\|$ denotes the spectral matrix norm.

There exist various model reduction approaches for standard state space systems ($E = I$) such as balanced truncation, Hankel norm approximation and moment matching approximation, e.g., [1, 2]. In this paper, developing the ideas from [7, 8], we generalize a balanced truncation model reduction method for descriptor systems. Important properties of this method are that the asymptotic stability is preserved in the reduced order system and there is a priori bound on the approximation error.

2. Gramians and Hankel singular values

It is well known that balanced truncation model reduction for standard state space systems is closely related to the controllability and observability Gramians that satisfy standard Lyapunov equations [2]. These Gramians can be generalized for descriptor systems as follows. The *proper controllability Gramian* $\mathcal{G}_{pc}$ and the *proper observability Gramian* $\mathcal{G}_{po}$ of the continuous-time descriptor system (1) are defined as unique symmetric, positive semidefinite solutions of the *projected generalized continuous-time Lyapunov equations*

$$E\mathcal{G}_{pc}A^T + A\mathcal{G}_{pc}E^T = -P_lBB^TP_l^T, \qquad \mathcal{G}_{pc} = P_r\mathcal{G}_{pc}P_r^T, \tag{3}$$

$$E^T\mathcal{G}_{po}A + A^T\mathcal{G}_{po}E = -P_r^TC^TCP_r, \qquad \mathcal{G}_{po} = P_l^T\mathcal{G}_{po}P_l, \tag{4}$$

where P_r and P_l denote spectral projections onto the right and left deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues, see [8]. Unlike standard state space systems, the descriptor system (1) has also the *improper controllability Gramian* $\mathcal{G}_{ic}$ and the *improper observability Gramian* $\mathcal{G}_{io}$ that are defined as unique symmetric, positive semidefinite solutions of the *projected generalized discrete-time Lyapunov equations*

$$A\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T = (I - P_l)BB^T(I - P_l)^T, \qquad P_r\mathcal{G}_{ic}P_r^T = 0, \tag{5}$$

$$A^T\mathcal{G}_{io}A - E^T\mathcal{G}_{io}E = (I - P_r)^TC^TC(I - P_r), \qquad P_l^T\mathcal{G}_{io}P_l = 0. \tag{6}$$

The matrices $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ play the same role for descriptor systems as the product of the controllability and observability Gramians for standard state space systems [2]. It has been shown in [8] that all the eigenvalues of $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$ and $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$ are real and non-negative.

Let n_f and n_∞ be the dimensions of the deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite and infinite eigenvalues, respectively. The square roots of the n_f largest eigenvalues of the matrix $\mathcal{G}_{pc}E^T\mathcal{G}_{po}E$, denoted by ς_j, are called the *proper Hankel singular values* of system (1). The square roots of the n_∞ largest eigenvalues of the matrix $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$, denoted by θ_j, are called the *improper Hankel singular values* of (1).

Since the proper and improper controllability and observability Gramians are symmetric and positive semidefinite, there exist full rank factorizations $\mathcal{G}_{pc} = R_pR_p^T$, $\mathcal{G}_{po} = L_p^TL_p$, $\mathcal{G}_{ic} = R_iR_i^T$ and $\mathcal{G}_{io} = L_i^TL_i$, where R_p, L_p, R_i and L_i are full rank Cholesky factors. One can show that the non-zero proper Hankel singular values of the descriptor system (1) coincide with the non-zero singular values of the matrix L_pER_p, while the non-zero improper Hankel singular values of (1) are the non-zero singular values of the matrix L_iAR_i, see [8].

3. Balanced truncation

A descriptor system (1) is called *balanced* if the Gramians of (1) satisfy

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix},$$

where $\Sigma = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{n_f})$ and $\Theta = \operatorname{diag}(\theta_1, \ldots, \theta_{n_\infty})$. It is well known that every completely controllable, completely observable and asymptotically stable descriptor system can be transformed to a balanced form [8]. If the descriptor system (1) has uncontrollable or/and unobservable states that, in fact, correspond to the zero proper and improper Hankel singular values, then these states can be truncated without changing the input-output relation in the system. Note that the number of non-zero improper Hankel singular values of (1) is the same as $\operatorname{rank}(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A)$ which is estimated as $\operatorname{rank}(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A) \le \min(\nu m, \nu p, n_\infty)$, where ν is the index of the pencil $\lambda E - A$. This estimate shows that if ν times the number m of inputs or the number p of outputs is much smaller than the dimension n_∞ of the deflating subspace of $\lambda E - A$ corresponding to the infinite eigenvalue, then the order of the descriptor system (1) can be reduced significantly. Moreover, taking into account the input-output energy characterization via the proper controllability and observability Gramians, see [8], we can conclude that the truncation of the states of the balanced descriptor system related to the small proper Hankel singular values does not change the system properties essentially. Note that this does not hold for the improper Hankel singular values. The truncation of the states that correspond to the small non-zero improper Hankel singular values may lead to an inaccurate approximation.

In summary, we have the following algorithm which is a generalization of the *square root balanced truncation method*, e.g., [4] for the descriptor system (1).

Algorithm. *Generalized Square Root Balanced Truncation (GSRBT) method.*

Input: $[\,E,\ A,\ B,\ C\,]$ *such that all the finite eigenvalues of* $\lambda E - A$ *have negative real part.*

Output: *A reduced order system* $[\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,]$.

1. *Compute the full rank Cholesky factors* R_p *and* L_p *of the proper controllability and observability Gramians* $\mathcal{G}_{pc} = R_pR_p^T$ *and* $\mathcal{G}_{po} = L_p^TL_p$ *that satisfy equations* (3) *and* (4), *respectively.*
2. *Compute the full rank Cholesky factors* R_i *and* L_i *of the improper controllability and observability Gramians* $\mathcal{G}_{ic} = R_iR_i^T$ *and* $\mathcal{G}_{io} = L_i^TL_i$ *that satisfy equations* (5) *and* (6), *respectively.*
3. *Compute the 'thin' singular value decomposition* $L_pER_p = [\,U_1,\ U_2\,]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}[\,V_1,\ V_2\,]^T$, *where* $[\,U_1,\ U_2\,]$ *and* $[\,V_1,\ V_2\,]$ *have orthonormal columns,* $\Sigma_1 = \operatorname{diag}(\varsigma_1, \ldots, \varsigma_{\ell_f})$ *and* $\Sigma_2 = \operatorname{diag}(\varsigma_{\ell_f+1}, \ldots, \varsigma_{r_p})$ *with* $r_p = \operatorname{rank}(L_pER_p)$ *and* $\varsigma_1 \ge \ldots \ge \varsigma_{\ell_f} > \varsigma_{\ell_f+1} \ge \ldots \ge \varsigma_{r_p}$.
4. *Compute the 'thin' singular value decomposition* $L_iAR_i = U_3\Theta_3V_3^T$, *where* U_3 *and* V_3 *have orthonormal columns and* $\Theta_3 = \operatorname{diag}(\theta_1, \ldots, \theta_{\ell_\infty})$ *with* $\ell_\infty = \operatorname{rank}(L_iAR_i)$.
5. *Compute the matrices* $W_\ell = [\,L_p^TU_1\Sigma_1^{-1/2},\ \ L_i^TU_3\Theta_3^{-1/2}\,]$ *and* $T_\ell = [\,R_pV_1\Sigma_1^{-1/2},\ \ R_iV_3\Theta_3^{-1/2}\,]$.
6. *Compute the reduced order system* $[\,\widetilde{E},\ \widetilde{A},\ \widetilde{B},\ \widetilde{C}\,] = [\,W_\ell^TET_\ell,\ W_\ell^TAT_\ell,\ W_\ell^TB,\ CT_\ell\,]$.

The computation of the reduced order descriptor system via balanced truncation can be interpreted as the additive decomposition of the transfer function as $\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s)$, where $\mathbf{G}_{sp}(s) = C_f(sE_f - A_f)^{-1}B_f$ is the strictly proper part and $\mathbf{P}(s) = C_\infty(sE_\infty - A_\infty)^{-1}B_\infty$ is the polynomial part of $\mathbf{G}(s)$, and then applying the classical continuous-time and discrete-time balanced truncation model reduction to the subsystems $[\,E_f,\ A_f,\ B_f,\ C_f\,]$ and $[\,A_\infty,\ E_\infty,\ B_\infty,\ C_\infty\,]$, respectively, where E_f and A_∞ are nonsingular. The reduced order system (2) has the transfer function $\widetilde{\mathbf{G}}(s) = \widetilde{\mathbf{G}}_{sp}(s) + \widetilde{\mathbf{P}}(s)$, where $\widetilde{\mathbf{G}}_{sp}(s) = \widetilde{C}_f(s\widetilde{E}_f - \widetilde{A}_f)^{-1}\widetilde{B}_f$ and $\widetilde{\mathbf{P}}(s) = \widetilde{C}_\infty(s\widetilde{E}_\infty - \widetilde{A}_\infty)^{-1}\widetilde{B}_\infty = \mathbf{P}(s)$. In this case the error system $\mathbf{G}(s) - \widetilde{\mathbf{G}}(s) = \mathbf{G}_{sp}(s) - \widetilde{\mathbf{G}}_{sp}(s)$ is strictly proper, and we have the following $\mathcal{H}_\infty$-norm error bound $\|\mathbf{G} - \widetilde{\mathbf{G}}\|_{\mathcal{H}_\infty} \le 2(\varsigma_{\ell_f+1} + \ldots + \varsigma_{n_f})$, see [2]. Moreover, we can show that the reduced order system computed by the GSRBT method is completely controllable, completely observable, asymptotically stable and balanced.

4. Semidiscretized Stokes equation

In this section we apply the balanced truncation model reduction to the descriptor system obtained by spatial discretization of the instationary Stokes equation that describes the flow of an incompressible fluid. Such a system has the form (1) with

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [\, C_1,\ C_2 \,], \tag{7}$$

where I_{n_1} is an identity matrix of order n_1, $A_{11} \in \mathbb{R}^{n_1,n_1}$ is symmetric, negative definite and $A_{12} \in \mathbb{R}^{n_1,n_2}$ has full column rank, see [9] for details. Using the block structure of the matrices E and A, we obtain that

$$P_r = \begin{bmatrix} \Pi & 0 \\ -(A_{12}^T A_{12})^{-1} A_{12}^T A_{11} \Pi & 0 \end{bmatrix} = P_l, \tag{8}$$

where $\Pi = I - A_{12}(A_{12}^T A_{12})^{-1} A_{12}^T$ is the orthogonal projection onto $\mathrm{Ker} A_{12}^T$ along $\mathrm{Im} A_{12}$.

Substituting (7) and (8) in the projected generalized continuous-time Lyapunov equations (3) and (4), we find that the proper controllability and observability Gramians of the semidiscretized Stokes equation (1), (7) can be computed in factored form $\mathcal{G}_{pc} = R_p R_p^T$ and $\mathcal{G}_{po} = L_p^T L_p$, where

$$R_p^T = [\, R_1^T,\ -R_1^T A_{11} A_{12} (A_{12}^T A_{12})^{-1} \,], \qquad L_p = [\, L_1,\ -L_1 A_{11} A_{12} (A_{12}^T A_{12})^{-1} \,]. \tag{9}$$

Here R_1 and L_1 are full rank Cholesky factors of the solutions $X_{11} = R_1 R_1^T$ and $Y_{11} = L_1^T L_1$ of the projected continuous-time Lyapunov equations

$$\Pi A_{11} \Pi X_{11} + X_{11} \Pi A_{11} \Pi = -\Pi B_{12} B_{12}^T \Pi, \tag{10}$$

$$\Pi A_{11} \Pi Y_{11} + Y_{11} \Pi A_{11} \Pi = -\Pi C_{12}^T C_{12} \Pi, \tag{11}$$

where $B_{12} = B_1 - A_{11} A_{12} (A_{12}^T A_{12})^{-1} B_2$, $C_{12} = C_1 - C_2 (A_{12}^T A_{12})^{-1} A_{12}^T A_{11}$. Using (7) and (9), we have $L_p E R_p = L_1 R_1$. Thus, the proper Hankel singular values of the semidiscretized Stokes equation (1), (7) can be computed from the singular value decomposition of the matrix $L_1 R_1$.

Analogously, we obtain from the projected generalized discrete-time Lyapunov equations (5), (6) that the improper controllability and observability Gramians of system (1), (7) have the form $\mathcal{G}_{ic} = R_i R_i^T$, $\mathcal{G}_{io} = L_i^T L_i$, where

$$R_i = \begin{bmatrix} A_{12}(A_{12}^T A_{12})^{-1} B_2 & 0 \\ (A_{12}^T A_{12})^{-1} A_{12}^T B_{12} & (A_{12}^T A_{12})^{-1} B_2 \end{bmatrix}, \qquad L_i = \begin{bmatrix} C_2 (A_{12}^T A_{12})^{-1} A_{12}^T & C_{12} A_{12} (A_{12}^T A_{12})^{-1} \\ 0 & C_2 (A_{12}^T A_{12})^{-1} \end{bmatrix}. \tag{12}$$

It follows from (7) and (12) that

$$L_i A R_i = \begin{bmatrix} C_{12} A_{12} (A_{12}^T A_{12})^{-1} B_2 + C_2 (A_{12}^T A_{12})^{-1} A_{12}^T B_1 & C_2 (A_{12}^T A_{12})^{-1} B_2 \\ C_2 (A_{12}^T A_{12})^{-1} B_2 & 0 \end{bmatrix} \in \mathbb{R}^{2p,2m}. \tag{13}$$

Hence, to determine the improper Hankel singular values of (1), (7) we have to compute the singular value decomposition of the matrix $L_i A R_i$ as in (13) that has only a few columns and rows if m and p are small.

Example. *The spatial discretization of the Stokes equation on a square domain* $[0,1] \times [0,1]$ *by the finite volume method on a uniform staggered* 22×22 *grid leads to a problem of dimension* $n = n_1 + n_2 = 1540$ *with* $n_1 = 1012$ *and* $n_2 = 528$. *The dimensions of the deflating subspaces of the pencil* $\lambda E - A$ *corresponding to the finite and infinite eigenvalues are* $n_f = 484$ *and* $n_\infty = 1056$, *respectively. In our experiments* $B \in \mathbb{R}^{n,1}$ *and* $C \in \mathbb{R}^{1,n}$ *are chosen at random. The computations were done on a SUN OS 5.8 workstation with machine precision* $\varepsilon = 2.22 \times 10^{16}$ *using MATLAB.*

Figure 1 *shows the* 25 *largest proper Hankel singular values and eigenvalues of the solutions* X_{11} *and* Y_{11} *of equations* (10) *and* (11), *respectively, computed by the Hammarling method* [3]. *One can see that the eigenvalues decay very fast and, hence, the matrices* X_{11} *and* Y_{11} *can be well approximated by matrices of low rank. Using the low rank alternating direction implicit (LRADI) method* [5, 6], *we have computed the low rank Cholesky factors* X,

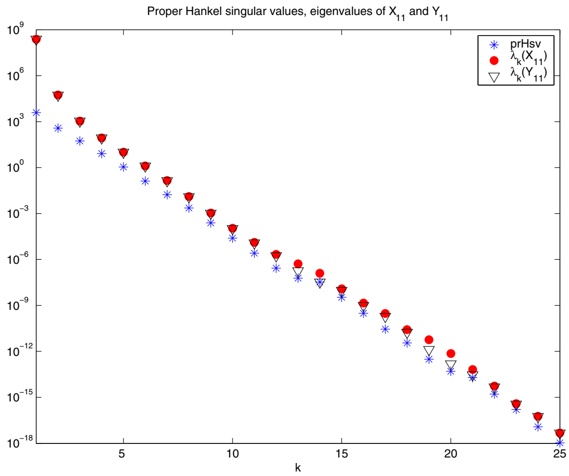

Figure 1: Proper Hankel singular values and eigenvalues of the matrices X_{11} and Y_{11}.

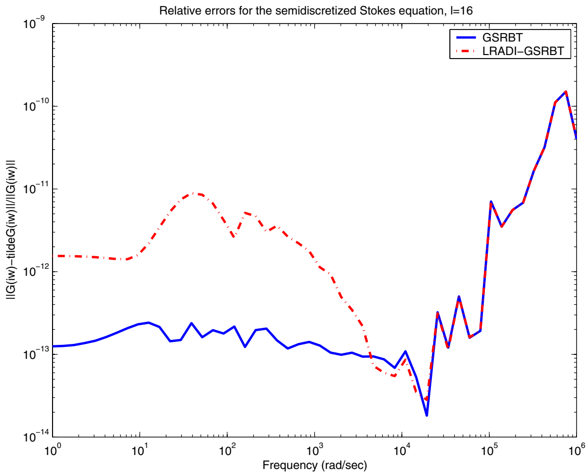

Figure 2: Relative error plots.

$Y \in \mathbb{R}^{1012,16}$ *of the solutions* $X_{11} \approx XX^T$ *and* $Y_{11} \approx Y^TY$ *of equations* (10) *and* (11)*, respectively. The dominant proper Hankel singular values have been approximated by the singular values of* YX*. The non-zero improper Hankel singular values are* $\theta_1 = 133780.8354$ *and* $\theta_2 = 0.4296$*. We approximate the semidiscretized Stokes equation by two models of order* $\ell = 16$ *(*$\ell_f = 14$*,* $\ell_\infty = 2$*) computed by the GSRBT method using the full rank factors* R_p *and* L_p *of the proper Gramians as in* (9) *and their low rank Cholesky factors given by* $\widetilde{R}_p = [\,X^T,\ -X^TA_{11}A_{12}(A_{12}^TA_{12})^{-1}\,]^T$ *and* $\widetilde{L}_p = [\,Y,\ -YA_{11}A_{12}(A_{12}^TA_{12})^{-1}\,]$*. The absolute values of the frequency responses of the full order and reduced order systems are not presented since they were impossible to distinguish. In Figure* 2 *we display the relative errors* $\|\mathbf{G}(i\omega) - \widetilde{\mathbf{G}}(i\omega)\|/\|\mathbf{G}(i\omega)\|$ *of the two different approximations. One can see that the both reduced order systems approximate the original one quite well. Note that the computational costs and memory requirements for the GSRBT method based on the LRADI iteration are much smaller than for standard implementations based on the direct Lyapunov equation solvers, see* [5, 6] *for details.*

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5. References

1 Antoulas, A.C.; Sorensen, D.C.: Approximation of large-scale dynamical systems: an overview. Int. J. Appl. Math. Comput. Sci. **11** (2001), 1093–1121.

2 Glover, K.: All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-errors bounds. Internat. J. Control **39** (1984), 1115–1193.

3 Hammarling, S.J.: Numerical solution of the stable non-negative definite Lyapunov equation. IMA J. Numer. Anal. **2** (1982), 303–323.

4 Laub, A.J.; Heath, M.T.; Paige, C.C.; Ward, R.C.: Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms. IEEE Trans. Automat. Control **32** (1987), 115–122.

5 Li, J.-R.: Model reduction of large linear systems via low rank system Gramians. Ph.D. thesis, Department of Mathematics, Massachusetts Institute of Technology, 2000.

6 Penzl, T.: A cyclic Low-Rank Smith method for large sparse Lyapunov equations. SIAM J. Sci. Comput. **21** (2000), 1401–1418.

7 Perev, K.; Shafai, B.: Balanced realization and model reduction of singular systems. Internat. J. Systems Sci. **25** (1994), 1039–1052.

8 Stykel, T.: Analysis and numerical solution of generalized Lyapunov equations. Ph.D. thesis, Institut für Mathematik, Technische Universität Berlin, 2002.

9 Weickert, J.: Applications of the theory of differential-algebraic equations to partial differential equations of fluid dynamics. Ph.D. thesis, Technische Universität Chemnitz, 1997.

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