Model Reduction of Periodic Descriptor Systems Using Balanced Truncation

Peter Benner, Mohammad-Sahadet Hossain and Tatjana Stykel

Abstract Linear periodic descriptor systems represent a broad class of time evolutionary processes in micro-electronics and circuit simulation. In this paper, we consider discrete-time linear periodic descriptor systems and study the concepts of periodic reachability and observability Gramians. We also discuss a lifted representation of periodic descriptor systems and propose a balanced truncation model reduction method for such systems. The behaviour of the suggested model reduction technique is illustrated using a numerical example.

Keywords: Periodic descriptor systems, lifted state space representation, periodic projected Lyapunov equations, balanced realization, model reduction.

1 Introduction

Linear discrete-time periodic descriptor systems have received a lot of attention over the last twenty years. They are suitable models for several natural as well as man-made phenomena, and have applications in modeling of periodic time-varying filters and networks [16, 21], multirate sampled-data systems [16, 18], circuit simulation [3, 8, 10, 18], micro-electronics [19, 20], aerospace realm [34, 35], control of industrial processes and communication systems [1, 20].

A linear discrete-time periodic descriptor system with time-varying dimensions has the form

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \qquad k \in \mathbb{Z}, \tag{1}$$

Peter Benner, Mohammad-Sahadet Hossain
Fakultät für Mathematik, TU Chemnitz, D-09107 Chemnitz, Germany,
e-mail: {benner@mathematik.tu-chemnitz.de, mohh@hrz.tu-chemnitz.de}

Tatjana Stykel
Institut für Mathematik, MA 4-5, TU Berlin, Straße des 17.Juni 136, D-10623 Berlin, Germany,
e-mail: stykel@math.tu-berlin.de

where $E_k \in \mathbb{R}^{\mu_{k+1}\times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1}\times m_k}$, $C_k \in \mathbb{R}^{p_k\times n_k}$ are periodic with a period $K \geq 1$ and $\sum_{k=0}^{K-1}\mu_k = \sum_{k=0}^{K-1} n_k = n$. The matrices E_k are allowed to be singular for all k. For system (1), a reduced-order model of dimension r would be a system of the form

$$\tilde{E}_k\tilde{x}_{k+1} = \tilde{A}_k\tilde{x}_k + \tilde{B}_k u_k, \quad \tilde{y}_k = \tilde{C}_k\tilde{x}_k, \qquad k \in \mathbb{Z}, \tag{2}$$

where $\tilde{E}_k \in \mathbb{R}^{\gamma_{k+1}\times r_{k+1}}$, $\tilde{A}_k \in \mathbb{R}^{\gamma_{k+1}\times r_k}$, $\tilde{B}_k \in \mathbb{R}^{\gamma_{k+1}\times m_k}$, $\tilde{C}_k \in \mathbb{R}^{p_k\times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1}\gamma_k = \sum_{k=0}^{K-1} r_k = r$ and $r \ll n$. Apart from having a much smaller state-space dimension, it is also important that the reduced-order model preserves physical properties of the original system such as regularity, stability and passivity, and that the approximation error is small.

The dynamics of the discrete-time periodic descriptor system (1) are often addressed by the regularity and the eigenstructure of the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$. If all E_k are nonsingular, the eigenvalues (also called characteristic multipliers) of system (1) are given by the eigenvalues of the matrix product

$$E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_0^{-1}A_0 \tag{3}$$

associated with the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$. This product only yields a well-defined matrix if all E_k are nonsingular. Even if they are, the formulation of that matrix should be avoided for reasons of numerical stability. Note that even for some E_k being singular, we use (3) in a formal way to denote a generalization of matrix pencils to this periodic case (see [2] for details of this formal matrix product calculus). We compute the eigenvalues of (3) via the generalized periodic Schur decomposition [11, 32].

There exist unitary matrices $P_k \in \mathbb{C}^{\mu_{k+1}\times\mu_{k+1}}$ and $Q_k \in \mathbb{C}^{n_k\times n_k}$, with $Q_{k+K} = Q_k$ such that the transformed matrices

$$N_k = P_k^* E_k Q_{k+1}, \quad M_k = P_k^* A_k Q_k, \quad k = 0, \ldots, K-1,$$

are all upper triangular, where for the ease of notation we allow complex arithmetic (in practice, however, computations can be performed in real arithmetic leading to quasi-triangular structure of one of the M_k).

Then the formal matrix product

$$N_{K-1}^{-1}M_{K-1}N_{K-2}^{-1}M_{K-2}\cdots N_0^{-1}M_0$$

has the same eigenvalues as (3). The blocks on the diagonals of the transformed matrices M_k and N_k are used to define the eigenvalues of the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$. A *finite eigenvalue* is given by

$$\lambda_l = \prod_{k=0}^{K-1} \frac{m_{ll}^{(k)}}{n_{ll}^{(k)}},$$

provided $n_{ll}^{(k)} \neq 0$ for $k = 0, \ldots, K-1$. Here $m_{ll}^{(k)} \in \Lambda(M_k)$ and $n_{ll}^{(k)} \in \Lambda(N_k)$, where Λ denotes the eigenspectrum of the corresponding matrix. An eigenvalue is called infinite if $\prod_{k=0}^{K-1} n_{ll}^{(k)} = 0$, but $\prod_{k=0}^{K-1} m_{ll}^{(k)} \neq 0$.

In this paper, we briefly review some basic concepts of discrete-time periodic descriptor systems (Section 2). In Section 3, we study the periodic reachability and observability Gramians from [6] using a lifted representation. A balanced truncation model reduction method for periodic descriptor systems is presented in Section 4. Section 5 contains a numerical example that illustrates the properties of the suggested model reduction technique.

2 Periodic Descriptor Systems

Lifted representations of discrete-time periodic systems play an important role in extending many theoretical results and numerical algorithms for time-invariant systems to the periodic setting [4, 7, 33]. We consider here the cyclic lifted representation which was introduced first for standard periodic systems in [17]. The *cyclic lifted representation* of the periodic descriptor system (1) is given by

$$\mathscr{E}\,\mathscr{X}_{k+1} = \mathscr{A}\,\mathscr{X}_k + \mathscr{B}\,\mathscr{U}_k, \quad \mathscr{Y}_k = \mathscr{C}\,\mathscr{X}_k, \tag{4}$$

where

$$\mathscr{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \quad \mathscr{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathscr{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 0 & \cdots & 0 & C_0 \\ C_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & C_{K-1} & 0 \end{bmatrix}. \tag{5}$$

The descriptor vector, system input and output of (4) are related to those of (1) via

$$\mathscr{X}_k = [x_1^T, \ldots, x_{K-1}^T, x_0^T]^T, \quad \mathscr{U}_k = [u_0^T, u_1^T, \ldots, u_{K-1}^T]^T, \quad \mathscr{Y}_k = [y_0^T, y_1^T, \ldots, y_{K-1}^T]^T,$$

respectively.

A set of periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ is called *regular* if the pencil $z\mathscr{E} - \mathscr{A}$ is regular, i.e., $\det(z\mathscr{E} - \mathscr{A}) \not\equiv 0$. In this case, we can define a transfer function of the lifted system (4) as $\mathscr{H}(z) = \mathscr{C}(z\mathscr{E} - \mathscr{A})^{-1}\mathscr{B}$.

The regular set of periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ can be transformed into a periodic Kronecker canonical form [6, 32]. For $k = 0, 1, \ldots, K-1$, there exist nonsingular matrices $W_k \in \mathbb{R}^{\mu_{k+1} \times \mu_{k+1}}$ and $Z_k \in \mathbb{R}^{n_k \times n_k}$ such that

$$W_k E_k Z_{k+1} = \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \quad W_k A_k Z_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \tag{6}$$

where $Z_K = Z_0$, $A^f_{k+K-1}A^f_{k+K-2}\cdots A^f_k = J_k$ is an $n^f_k \times n^f_k$ matrix corresponding to the finite eigenvalues, $E^b_k E^b_{k+1}\cdots E^b_{k+K-1} = N_k$ is an $n^\infty_k \times n^\infty_k$ nilpotent matrix corresponding to an eigenvalue at infinity, $n_k = n^f_k + n^\infty_k$ and $\mu_{k+1} = n^f_{k+1} + n^\infty_k$. The index ν of the periodic descriptor system (1) is defined as $\nu = \max(\nu_0, \nu_1, \ldots, \nu_{K-1})$, where ν_k is the nilpotency index of N_k. Note that the finite eigenvalues of $\{E_k, A_k\}_{k=0}^{K-1}$ coincide with the finite eigenvalues of the lifted pencil $z\mathscr{E} - \mathscr{A}$.

For $k = 0, 1, \ldots, K-1$, the matrices

$$P_r(k) = Z_k \begin{bmatrix} I_{n^f_k} & 0 \\ 0 & 0 \end{bmatrix} Z_k^{-1}, \quad P_l(k) = W_k^{-1} \begin{bmatrix} I_{n^f_{k+1}} & 0 \\ 0 & 0 \end{bmatrix} W_k,$$

are the *spectral projectors* onto the k-th right and left deflating subspaces of the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues, and $Q_r(k) = I - P_r(k)$ and $Q_l(k) = I - P_l(k)$ are the complementary projectors. Let for every $k = 0, 1, \ldots, K-1$, the vector $Z_k^{-1}x_k = [\,(x^f_k)^T,\ (x^b_k)^T\,]^T$ and the matrices

$$W_k B_k = \begin{bmatrix} B^f_k \\ B^b_k \end{bmatrix}, \qquad C_k Z_k = \begin{bmatrix} C^f_k, & C^b_k \end{bmatrix},$$

be partitioned in blocks conformally to the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ in (6). Under this transformation, system (1) can be decoupled into forward and backward periodic subsystems

$$x^f_{k+1} = A^f_k x^f_k + B^f_k u_k, \quad y^f_k = C^f_k x^f_k, \tag{7}$$

$$E^b_k x^b_{k+1} = x^b_k + B^b_k u_k, \quad y^b_k = C^b_k x^b_k, \tag{8}$$

respectively, with $y_k = y^f_k + y^b_k$, $k = 0, 1, \ldots, K-1$. The state transition matrix for the forward subsystem (7) is given by $\Phi_f(i, j) = A^f_{i-1}A^f_{i-2}\cdots A^f_j$ for $i > j$ and $\Phi_f(i,i) = I_{n^f_i}$. For the backward subsystem (8), the state transition matrix is defined as $\Phi_b(i, j) = E^b_i E^b_{i+1}\cdots E^b_{j-1}$ for $i < j$ and $\Phi_b(i,i) = I_{n^\infty_i}$. Using these matrices we can now define the forward and backward fundamental matrices of the periodic descriptor system (1) as

$$\Psi_{i,j} = \begin{cases} Z_i \begin{bmatrix} \Phi_f(i, j+1) & 0 \\ 0 & 0 \end{bmatrix} W_j, & i > j, \\ Z_i \begin{bmatrix} 0 & 0 \\ 0 & -\Phi_b(i, j) \end{bmatrix} W_j, & i \le j. \end{cases}$$

These fundamental matrices play an important role in the definition of the reachability and observability Gramians of the periodic descriptor system (1) that we will consider in the next section.

3 Periodic Gramians and Matrix Equations

The dynamics of the periodic descriptor system (1) are often addressed by the eigenstructure of the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$.

Definition 1. The periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ are said to be *periodic stable* (*pd-stable*) if all the finite eigenvalues of $\{E_k, A_k\}_{k=0}^{K-1}$ lie inside the unit circle.

In balanced truncation model reduction, Gramians play a fundamental role [15, 25, 28, 31]. For periodic descriptor system (1), the reachability and observability Gramians have been first introduced in [6].

Definition 2. Suppose that the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ are pd-stable. For $k = 0, 1, \ldots, K-1$, we define the *causal* and *noncausal reachability Gramians* G_k^{cr} and G_k^{ncr} of system (1) as

$$G_k^{cr} = \sum_{j=-\infty}^{k-1} \Psi_{k,j} B_j B_j^T \Psi_{k,j}^T, \qquad G_k^{ncr} = \sum_{j=k}^{k+\nu K-1} \Psi_{k,j} B_j B_j^T \Psi_{k,j}^T.$$

The *complete reachability Gramian* G_k^r is the sum of the causal and noncausal Gramians, i.e., $G_k^r = G_k^{cr} + G_k^{ncr}$ for $k = 0, 1, \ldots, K-1$.

Definition 3. For the pd-stable matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ and $k = 0, 1, \ldots, K-1$, the *causal* and *noncausal observability Gramians* G_k^{co} and G_k^{nco} of system (1) are defined as

$$G_k^{co} = \sum_{j=k}^{\infty} \Psi_{j,k-1}^T C_j^T C_j \Psi_{j,k-1}, \qquad G_k^{nco} = \sum_{j=k-\nu K}^{k-1} \Psi_{j,k-1}^T C_j^T C_j \Psi_{j,k-1}.$$

The *complete observability Gramian* G_k^o is the sum of the causal and noncausal Gramians, i.e., $G_k^o = G_k^{co} + G_k^{nco}$ for $k = 0, 1, \ldots, K-1$.

Note that the causal and noncausal Gramians correspond to the forward and backward subsystems (7) and (8), respectively.

3.1 Periodic Projected Lyapunov Equations

It has been shown in [26] that the Gramians of discrete-time descriptor systems satisfy projected generalized discrete-time Lyapunov equations with special right-hand sides. A similar result also holds for periodic descriptor systems.

Theorem 1. [6] *Consider a periodic discrete-time descriptor system* (1), *where the periodic matrix pairs* $\{E_k, A_k\}_{k=0}^{K-1}$ *are pd-stable.*

1. *For* $k = 0, 1, \ldots, K-1$, *the causal reachability and observability Gramians* $\{G_k^{cr}\}_{k=0}^{K-1}$ *and* $\{G_k^{co}\}_{k=0}^{K-1}$ *are the unique symmetric, positive semidefinite solutions of the projected generalized discrete-time periodic Lyapunov equations (PGDPLEs)*

$$\begin{aligned} A_k G_k^{cr} A_k^T - E_k G_{k+1}^{cr} E_k^T &= -P_l(k) B_k B_k^T P_l(k)^T, \\ G_k^{cr} &= P_r(k) G_k^{cr} P_r(k)^T, \end{aligned} \tag{9}$$

and

$$\begin{aligned} A_k^T G_{k+1}^{co} A_k - E_{k-1}^T G_k^{co} E_{k-1} &= -P_r(k)^T C_k^T C_k P_r(k), \\ G_k^{co} &= P_l(k-1)^T G_k^{co} P_l(k-1), \end{aligned}$$

respectively, where $G_K^{cr} = G_0^{cr}$, $G_K^{co} = G_0^{co}$, $E_{-1} = E_{K-1}$, *and* $P_l(-1) = P_l(K-1)$.

2. *For* $k = 0, 1, \ldots, K-1$, *the noncausal reachability and observability Gramians* $\{G_k^{ncr}\}_{k=0}^{K-1}$ *and* $\{G_k^{nco}\}_{k=0}^{K-1}$ *are the unique symmetric, positive semidefinite solutions of the PGDPLEs*

$$\begin{aligned} A_k G_k^{ncr} A_k^T - E_k G_{k+1}^{ncr} E_k^T &= Q_l(k) B_k B_k^T Q_l(k)^T, \\ G_k^{ncr} &= Q_r(k) G_k^{ncr} Q_r(k)^T, \end{aligned}$$

and

$$\begin{aligned} A_k^T G_{k+1}^{nco} A_k - E_{k-1}^T G_k^{nco} E_{k-1} &= Q_r(k)^T C_k^T C_k Q_r(k) \\ G_k^{nco} &= Q_l(k-1)^T G_k^{nco} Q_l(k-1), \end{aligned}$$

respectively, where $G_K^{ncr} = G_0^{ncr}$, $G_K^{nco} = G_0^{nco}$ *and* $Q_l(-1) = Q_l(K-1)$.

Numerical solution of the PGDPLEs has been considered in [6]. The method proposed there extends the periodic Schur method [5, 29, 30] and the generalized Schur-Hammarling method [23] developed for periodic standard and projected generalized Lyapunov equations, respectively. This method is based on an initial reduction of the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ to the generalized periodic Schur form [11, 32] and solving the resulting generalized periodic Sylvester and Lyapunov equations of (quasi)-triangular structure using the recursive blocked algorithms [9]. Due to the computational complexity, the periodic generalized Schur-Hammarling method is restricted to problems of small and medium size. In [12, 29, 27], iterative methods based on Smith iterations [22] have been developed for periodic standard Lyapunov equations and also for projected generalized Lyapunov equations. These methods can also be extended to periodic projected Lyapunov equations by using the lifted repsentation of these equations. Therefore, we will discuss these lifted representations in the following subsection.

3.2 Lifted Representation of Periodic Lyapunov Equations

It is known that the Gramians of standard periodic systems satisfy the lifted form of the periodic Lyapunov equations and the solutions of these equations are diagonal matrices [12, 29].

The following theorem describes the block structures of the solutions of periodic Lyapunov equations in lifted form and their relations to the corresponding solutions of PGDPLEs in Theorem 1.

Theorem 2. *Consider the periodic discrete-time descriptor system* (1) *and its lifted representation* (4)*, where the periodic matrix pairs* $\{E_k, A_k\}_{k=0}^{K-1}$ *are pd-stable. The causal and noncausal reachability Gramians* $\mathscr{G}^{cr}$ *and* $\mathscr{G}^{ncr}$ *satisfy the lifted projected Lyapunov equations*

$$\begin{aligned} \mathscr{A}\mathscr{G}^{cr}\mathscr{A}^T - \mathscr{E}\mathscr{G}^{cr}\mathscr{E}^T &= -\mathscr{P}_l\mathscr{B}\mathscr{B}^T\mathscr{P}_l^T, \quad & \mathscr{G}^{cr} &= \mathscr{P}_r\mathscr{G}^{cr}\mathscr{P}_r^T, \\ \mathscr{A}\mathscr{G}^{ncr}\mathscr{A}^T - \mathscr{E}\mathscr{G}^{ncr}\mathscr{E}^T &= \mathscr{Q}_l\mathscr{B}\mathscr{B}^T\mathscr{Q}_l^T, \quad & \mathscr{G}^{ncr} &= \mathscr{Q}_r\mathscr{G}^{ncr}\mathscr{Q}_r^T, \end{aligned} \tag{10}$$

respectively, where $\mathscr{E}$, $\mathscr{A}$ *and* $\mathscr{B}$ *are as in* (5) *and*

$$\mathscr{G}^{cr} = diag(G_1^{cr}, \ldots, G_{K-1}^{cr}, G_0^{cr}), \quad \mathscr{G}^{ncr} = diag(G_1^{ncr}, \ldots, G_{K-1}^{ncr}, G_0^{ncr}),$$

$$\begin{aligned} \mathscr{P}_l &= diag(P_l(0), P_l(1), \ldots, P_l(K-1)), \qquad & \mathscr{Q}_l &= I - \mathscr{P}_l, \\ \mathscr{P}_r &= diag(P_r(1), \ldots, P_r(K-1), P_r(0)), \qquad & \mathscr{Q}_r &= I - \mathscr{P}_r. \end{aligned} \tag{11}$$

Proof. We will only sketch the proof due to space limitation of the paper. Using the block structure of matrix coefficients, a straightforward computation shows that the projected Lyapunov equation (10) is equivalent to the periodic projected Lyapunov equation (9). Since the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ are pd-stable, the pencil $z\mathscr{E} - \mathscr{A}$ is regular and all its eigenvalues lie inside the unit circle. Then (10) has a unique solution [24]. The proof for $\mathscr{G}^{ncr}$ can be treated similarly. □

For the observability Gramians, the situation becomes a bit more complex. The reason is that we do not want to destroy the block diagonal structure of the lifted solutions and we would like to use the lifted solution to find a balanced realization of the original system.

Theorem 3. *Consider the periodic discrete-time descriptor system* (1) *and its lifted representation* (4)*. The causal and noncausal observability Gramians* $\mathscr{G}^{co}$ *and* $\mathscr{G}^{nco}$ *satisfy the lifted projected Lyapunov equations*

$$\begin{aligned} \mathscr{A}^T\mathscr{G}^{co}\mathscr{A} - \mathscr{E}^T\mathscr{G}^{co}\mathscr{E} &= -\mathscr{P}_r^T\hat{\mathscr{C}}^T\hat{\mathscr{C}}\mathscr{P}_r, \quad & \mathscr{G}^{co} &= \mathscr{P}_l^T\mathscr{G}^{co}\mathscr{P}_l, \\ \mathscr{A}^T\mathscr{G}^{nco}\mathscr{A} - \mathscr{E}^T\mathscr{G}^{nco}\mathscr{E} &= \mathscr{Q}_r^T\hat{\mathscr{C}}^T\hat{\mathscr{C}}\mathscr{Q}_r, \quad & \mathscr{G}^{nco} &= \mathscr{Q}_l^T\mathscr{G}^{nco}\mathscr{Q}_l, \end{aligned}$$

respectively, where $\mathscr{E}$ *and* $\mathscr{A}$ *are as in* (5)*, the projectors* $\mathscr{P}_l$, $\mathscr{P}_r$, $\mathscr{Q}_l$ *and* $\mathscr{Q}_r$ *are as in* (11)*,* $\hat{\mathscr{C}} = diag(C_1, \ldots, C_{K-1}, C_0)$ *and*

$$\mathscr{G}^{co} = diag(G_1^{co}, \ldots, G_{K-1}^{co}, G_0^{co}), \quad \mathscr{G}^{nco} = diag(G_1^{nco}, \ldots, G_{K-1}^{nco}, G_0^{nco}).$$

Proof. The proof is analogous to the previous proof of Theorem 2. □

3.3 Hankel Singular Values

Let the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ be pd-stable. Then the causal and noncausal matrices $M_k^c = G_k^{cr} E_{k-1}^T G_k^{co} E_{k-1}$ and $M_k^{nc} = G_k^{ncr} A_k^T G_{k+1}^{nco} A_k$, $k=0,1,\dots,K-1$, have real and nonnegative eigenvalues. These eigenvalues are used to define the causal and noncausal Hankel singular values of system (1).

Definition 4. Let the set of periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ be pd-stable. For $k = 0,1,\dots,K-1$, the square roots of the largest n_k^f eigenvalues of the matrix M_k^c, denoted by $\sigma_{k,j}$, are called the *causal Hankel singular values* and the square roots of the largest n_k^∞ eigenvalues of M_k^{nc}, denoted by $\theta_{k,j}$, are called the *noncausal Hankel singular values* of the periodic descriptor system (1).

Since the causal and noncausal reachability and observability Gramians are symmetric and positive semidefinite, there exist the Cholesky factorizations

$$G_k^{cr} = R_k R_k^T, \quad G_k^{co} = L_k^T L_k, \quad G_k^{ncr} = \check{R}_k \check{R}_k^T, \quad G_k^{nco} = \check{L}_k^T \check{L}_k, \tag{12}$$

Simple calculations show that $\sigma_{k,j} = \zeta_j(L_k E_{k-1} R_k)$ and $\theta_{k,j} = \zeta_j(\check{L}_{k+1} A_k \check{R}_k)$, where $\zeta_j(.)$ denotes the singular values of the corresponding matrices.

4 Balanced Truncation Model Reduction

In this section, we present a generalization of a balanced truncation model reduction method to periodic descriptor systems. For a balanced system, the reachability and observability Gramians are both equal to a diagonal matrix [15, 25]. Balanced realizations for periodic descriptor system have been considered in [6].

Definition 5. A realization (E_k, A_k, B_k, C_k) of a periodic descriptor system (1) is called *balanced* if

$$G_k^{cr} = G_k^{co} = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix}, \quad G_k^{ncr} = G_{k+1}^{nco} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta_k \end{bmatrix},$$

where $\Sigma_k = \operatorname{diag}(\sigma_{k,1},\dots,\sigma_{k,n_k^f})$ and $\Theta_k = \operatorname{diag}(\theta_{k,1},\dots,\theta_{k,n_k^\infty})$, $k = 0,1,\dots,K-1$.

Consider the Cholesky factorizations (12) of the reachability and observability Gramians and let

$$L_k E_{k-1} R_k = U_k \Sigma_k V_k^T, \quad \check{L}_{k+1} A_k \check{R}_k = \check{U}_k \Theta_k \check{V}_k^T \tag{13}$$

be the singular value decompositions of the matrices $L_k E_{k-1} R_k$ and $\check{L}_{k+1} A_k \check{R}_k$ for $k = 0,1,\dots,K-1$. Here $U_k, V_k, \check{U}_k, \check{V}_k$ are orthogonal, and Σ_k and Θ_k are diagonal. If a realization (E_k, A_k, B_k, C_k) with pd-stable matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ is minimal, i.e., Σ_k and Θ_k are nonsingular, then there exist nonsingular periodic matrices

$$S_k = [L_{k+1}^T U_{k+1} \Sigma_{k+1}^{-1/2},\ \check{L}_{k+1}^T \check{U}_k \Theta_k^{-1/2}], \quad T_k = [R_k V_k \Sigma_k^{-1/2},\ \check{R}_k \check{V}_k \Theta_k^{-1/2}],$$

such that the transformed realization $(S_k^T E_k T_{k+1}, S_k^T A_k T_k, S_k^T B_k, C_k T_k)$ is balanced [6]. Note that as in the case of standard state space systems, the balancing transformation matrices for periodic discrete-time descriptor system (1) are not unique.

Model reduction via balanced truncation is discussed very widely for standard discrete-time periodic systems [12, 31] and also for continuous-time descriptor systems [14, 25]. For a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially. Unfortunately, we can not do the same for the noncausal Hankel singular values. If we truncate the states that correspond to the small non-zero noncausal Hankel singular values, then the pencil for the reduced-order system may get finite eigenvalues outside the unit circle that will lead to additional errors in the system approximation.

Assume that the periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ are pd-stable. Consider the Cholesky factorizations in (12). Let

$$L_k E_{k-1} R_k = [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \quad \check{L}_{k+1} A_k \check{R}_k = \check{U}_k \Theta_k \check{V}_k^T,$$

be singular value decompositions of $L_k E_{k-1} R_k$ and $\check{L}_{k+1} A_k \check{R}_k$, where

$$\Sigma_{k,1} = \operatorname{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r_k^f}), \quad \Sigma_{k,2} = \operatorname{diag}(\sigma_{k,r_{k+1}^f}, \ldots, \sigma_{n_k^f}),$$

with $\sigma_{k,1} \geq \cdots \geq \sigma_{k,r_k^f} > \sigma_{k,r_{k+1}^f} \geq \ldots \geq \sigma_{k,n_k^f} > 0$, and $\Theta_k = \operatorname{diag}(\theta_{k,1}, \ldots, \theta_{k,r_k^\infty})$ is nonsingular for $k = 0, 1, \ldots, K-1$. Then the reduced-order system can be computed as

$$\tilde{E}_k = S_{k,r}^T E_k T_{k+1,r}, \quad \tilde{A}_k = S_{k,r}^T A_k T_{k,r}, \quad \tilde{B}_k = S_{k,r}^T B_k, \quad \tilde{C}_k = C_k T_{k,r}, \tag{14}$$

where

$$S_{k,r} = [L_{k+1}^T U_{k+1,1} \Sigma_{k+1,1}^{-1/2},\ \check{L}_{k+1}^T \check{U}_k \Theta_k^{-1/2}] \in \mathbb{R}^{\mu_{k+1}, r_{k+1}},$$
$$T_{k,r} = [R_k V_{k,1} \Sigma_{k,1}^{-1/2}, \quad \check{R}_k \check{V}_k \Theta_k^{-1/2}] \in \mathbb{R}^{n_k, r_k},$$

with $r_k = r_k^f + r_k^\infty$. Let $\tilde{\mathscr{H}}(z)$ be the transfer function of the reduced-order lifted system formed from the reduced-order subsystems in (14). Then we have the following $\mathbb{H}_\infty$-norm error bound

$$\|\mathscr{H} - \tilde{\mathscr{H}}\|_{\mathbb{H}_\infty} = \sup_{\omega \in [0, 2\pi]} \|\mathscr{H}(e^{i\omega}) - \tilde{\mathscr{H}}(e^{i\omega})\|_2 \leq 2 \sum_{k=0}^{K-1} \operatorname{trace}(\Sigma_{k,2}), \tag{15}$$

where $\|.\|_2$ denotes the matrix spectral norm and $\Sigma_{k,2}$ contains the truncated causal Hankel singular values. This error bound can be obtained similarly to the standard state space case [13, 31].

5 Example

We consider a periodic discrete-time descriptor system with $\mu_k = n_k = 10$, $m_k = 2$, $p_k = 3$, and period $K = 3$ as presented in [6, Example 1]. The periodic matrix pairs $\{E_k, A_k\}_{k=0}^{K-1}$ are pd-stable with $n_k^f = 8$ and $n_k^\infty = 2$ for $k = 0, 1, 2$. The norms of the computed solutions of the periodic Lyapunov equations and the corresponding residuals, e.g.,

$$\rho_k^{cr} = \|A_k G_k^{cr} A_k^T - E_k G_{k+1}^{cr} E_k^T + P_l(k) B_k B_k^T P_l(k)^T\|_2,$$

are shown in Table 1 and Table 2.

Table 1: Norms and relative residuals for the reachability Gramians

k	$\Vert G_k^{cr} \Vert_2$	ρ_k^{cr}	$\Vert G_k^{ncr} \Vert_2$	ρ_k^{ncr}
0	5.8182×10^2	6.1727×10^{-12}	1.3946×10^1	1.5444×10^{-14}
1	8.2981×10^4	8.2172×10^{-12}	1.3660×10^1	1.7508×10^{-14}
2	7.1107×10^3	3.0961×10^{-12}	1.4308×10^1	3.3847×10^{-14}

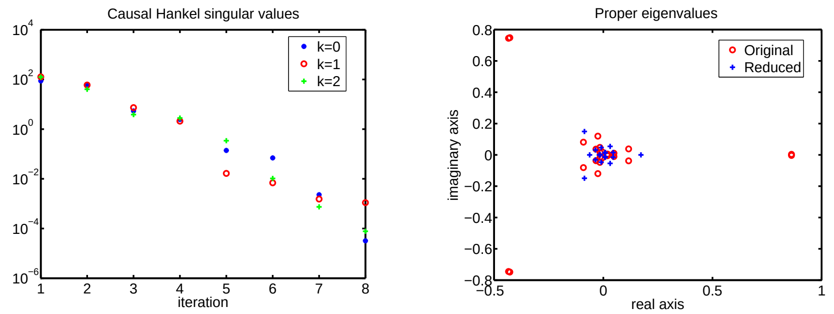
Table 2: Norms and relative residuals for the observability Gramians

k	$\Vert G_k^{co} \Vert_2$	ρ_k^{co}	$\Vert G_k^{nco} \Vert_2$	ρ_k^{nco}
0	9.7353×10^1	2.7678×10^{-13}	1.6866×10^0	1.3372×10^{-15}
1	1.1373×10^3	7.7003×10^{-14}	1.7406×10^0	2.1113×10^{-15}
2	9.6984×10^0	1.7859×10^{-14}	1.6866×10^0	1.1626×10^{-15}

The original lifted system has order $n = 30$. Figure 1(a) shows the causal Hankel singular values of the different subsystems for $k = 0, 1, 2$. We see that they decay fast, and, hence system (1) can be well approximated by a reduced-order model. We have 24 causal Hankel singular values for the original lifted system and the remaining 6 are noncausal Hankel singular values which are positive. We approximate system (1) to the tolerance 10^{-2} by truncating the states corresponding to the smallest 7 causal Hankel singular values.

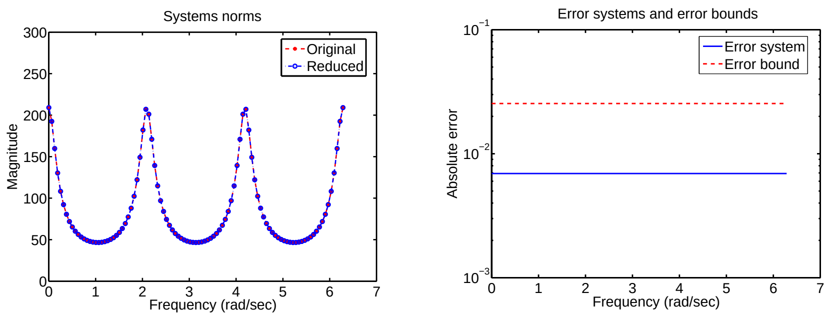
Figure 1(b) shows the finite eigenvalues of the original and reduced-order lifted systems. We observe that stability is preserved for the reduced-order system. In Figure 2(a), we present the norms of the frequency responses $\mathscr{H}(e^{i\omega})$ and $\tilde{\mathscr{H}}(e^{i\omega})$ of the original and reduced-order lifted systems for a frequency range $[0, 2\pi]$. We observe nice match of the system norms.

In Figure 2(b), we display the absolute error $\|\mathscr{H}(e^{i\omega}) - \tilde{\mathscr{H}}(e^{i\omega})\|_2$ and the error bound (15). One can see that the absolute error is smaller than the error bound.



(a) (b)

Fig. 1: (a) Causal Hankel singular values of different subsystems, (b) finite eigenvalues of the original and the reduced-order lifted systems.



(a) (b)

Fig. 2: (a) The frequency responses of the original and the reduced-order lifted systems; (b) absolute error and error bound.

6 Conclusion

In this paper, we have considered the reachability and observability Gramians as well as Hankel singular values for periodic discrete-time descriptor systems. For such systems, a balanced truncation model reduction method has been presented. The proposed method delivers a reduced-order model that preserves the regularity and stability properties of the original system. A computable global error bound for the approximate system is also available.

Acknowledgements This work was supported by the Research Network SyreNe - *System Reduction for Nanoscale IC Design* funded by the German Federal Ministry of Education and Science (BMBF), grants 03BEPAE1 and 03STPAE3. Responsibility for the contents of this publication rests with the authors.

References

1. Arcara, P., Bittanti, S.: Periodic control of helicopter rotors for attenuation of vibrations in forward flight. IEEE Trans. on Control Systems Tech. **8**(6), 883–894 (2000)
2. Benner, P., Mehrmann, V., Xu, H.: Perturbation analysis for the eigenvalue problem of a formal product of matrices. BIT **42**(1), 1–43 (2002)
3. Benner, P., Quintana-Ortí, E., Quintana-Ortí, G.: Computing passive reduced-order models for circuit simulation. In: Proc. Intl. Conf. Parallel Comp. in Elec. Engrg. PARELEC 2004, pp. 146–151. IEEE Computer Society, Los Alamitos, CA (2004)
4. Bittanti, S., Colaneri, P.: Invariant representations of discrete-time periodic systems. Automatica **36**, 1777–1793 (2000)
5. Byers, R., Rhee, N.: Cyclic Schur and Hessenberg-Schur numerical methods for solving periodic Lyapunov and Sylvester equations. Technical report, Department of Mathematics, University of Missouri at Kansas City (1995)
6. Chu, E.W., Fan, H.Y., Lin, W.W.: Projected generalized discrete-time periodic Lyapunov equations and balanced realization of periodic descriptor systems. SIAM J. Matrix Anal. Appl. **29**(3), 982–1006 (2007)
7. Farhood, M., Beck, C., Dullerud, G.: Model reduction of periodic systems: a lifting approach. Automatica, **41**, 1085–1090 (2005)
8. Freund, R.W.: Padé-type model reduction of second-order and higher-order linear dynamical systems. In: P. Benner, V. Mehrmann, D. Sorensen (eds.) Dimension Reduction of Large-Scale Systems, *Lecture Notes in Computational Science and Engineering*, vol. 45, pp. 191–223. Springer-Verlag, Berlin/Heidelberg, Germany (2005)
9. Granat, R., Jonsson, I., Kågström, B.: Recursive blocked algorithms for solving periodic triangular Sylvester-type matrix equations. In: B. Kågström, E. Elmroth, J. Dongarra, J. Waśniewski (eds.) Applied Parallel Computing. State of the Art in Scientific Computing, *Lecture Notes in Computer Science*, vol. 4699, pp. 531–539. Springer-Verlag (2007)
10. Günther, M., Feldmann, U.: CAD-based electric-circuit modeling in industry. II. Impact of circuit configurations and parameters. Surveys Math. Indust. **8**(2), 131–157 (1999)
11. Kressner, D.: An efficient and reliable implementation of the periodic QZ algorithm. In: S. Bittanti, P. Colaneri (eds.) Periodic Control Systems 2001. A Proceedings volume of the IFAC Workshop, Cernobbio-Como, Italy, 27–28 August 2001. Elsevier Science, Oxford, UK (2001)
12. Kressner, D.: Large periodic Lyapunov equations: Algorithms and applications. In: Proc. of ECC03, Cambridge, UK (2003)
13. Lall, S., Beck, C., Dullerud, G.: Guaranteed error bounds for model reduction of linear time-varying systems. In: Proceedings of the Americal Control Conference, Philadelphia, PA, pp. 634–638 (1998)
14. Mehrmann, V., Stykel, T.: Balanced truncation model reduction for large-scale systems in descriptor form. In: P. Benner, V. Mehrmann, D. Sorensen (eds.) Dimension Reduction of Large-Scale Systems, *Lecture Notes in Computational Science and Engineering*, vol. 45, pp. 83–115. Springer-Verlag, Berlin/Heidelberg, Germany (2005)
15. Moore, B.: Principal component analysis in linear systems: controllability, observability, and model reduction. IEEE Trans. Automat. Control **AC-26**(1), 17–32 (1981)
16. Nakhla, M., Gad, E.: Efficient model reduction of linear periodically time-varying systems via compressed transient system function. IEEE Trans. Automat. Control **52**(6) (2005)
17. Park, B., Verriest, E.: Canonical forms of discrete linear periodically time-varying systems and a control application. In: Proc. of the 28th Conference on Decision and Control, Tampa, FL, pp. 1220–1225 (1989)
18. Phillips, J., Daniel, L., Miguel Silveira, L.: Guaranteed passive balancing transformations for model order reduction. IEEE Trans. Computer-Aided Desing Integr. Circuits Syst. **22**, 1027–1041 (2003)
19. Roychowdhury, J.: Reduced-order modelling of linear time-varying systems. Design Automation Conference, 1999.Proceedings of the ASP-DAC '99. Asia and South Pacific **1**, 53–56 (18-21 Jan.1999)

20. Roychowdhury, J.: Reduced-order modeling of time-varying systems. IEEE Control Systems Magazine **46**, 1273 – 1288 (1999)
21. Shokoohi, S., Silverman, L., Van Dooren, P.: Linear time-variable systems: Balancing and model reduction. IEEE Trans. Automat. Control **AC-28**(8), 810–822 (1983)
22. Smith, R.: Matrix equation $XA+BX=C$. SIAM J. Appl. Math. **16**, 198–201 (1968)
23. Stykel, T.: Numerical solution and perturbation theory for generalized Lyapunov equations. Linear Algebra Appl. **349**, 155–185 (2002)
24. Stykel, T.: Stability and inertia theorems for generalized Lyapunov equations. Linear Algebra Appl. **355**, 297–314 (2002)
25. Stykel, T.: Gramian-based model reduction for descriptor systems. Math. Control Signals Systems **16**, 297–319 (2004)
26. Stykel, T.: On some norms for descriptor systems. IEEE Trans. Automat. Control **51**(5), 842–847 (2006)
27. Stykel, T.: Low-rank iterative methods for projected generalized Lyapunov equations. Electron. Trans. Numer. Anal. **30**, 187–202 (2008)
28. Van Dooren, P.: Gramian based model reduction of large-scale dynamical systems. In: Numerical Analysis 1999, Chapman & Hall/CRC Research Notes in Mathematics, 420, pp. 231–247. Chapman & Hall/CRC, Boca Raton, FL (2000)
29. Varga, A.: Periodic Lyapunov equations: some applications and new algorithms. Internat. J. Control **67**(1), 69–87 (1997)
30. Varga, A.: Balancing related methods for minimal realization of periodic systems. Systems Control Lett. **36**, 339–349 (1999)
31. Varga, A.: Balanced truncation model reduction of periodic systems. In: Proc. CDC'2000, Sydney, Australia (2000)
32. Varga, A.: Computation of Kronecker-like forms of periodic matrix pairs. In: Proc. of Mathematical Theory of Networks and Systems (MTNS 2004, Leuven, Belgium, July 5-9, 2004) (2004)
33. Varga, A.: A PERIODIC SYSTEMS Toolbox for MATLAB. In: Proc. of IFAC'05 World Congress, Prague, Czech Republik, July 3-8 (2005)
34. Wisniewski, R., Blanke, M.: Fully magnetic attitude control for spacecraft subject to gravity gradient. Automatica **35**(7), 1201–1214 (1999)
35. Yang, X., Kawamata, M., Higuchi, T.: Balanced realisations and model reduction of periodically time-varying state-space digital filters. In: IEE Proc. Vision, Image & Signal Proc., vol. 143, pp. 370–376 (1996)