A dynamic index for control sets

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Abstract. The controllability behavior of nonlinear control systems is described by associating semigroups to locally maximal subsets of complete controllability, i.e., local control sets. Periodic trajectories are called equivalent if there is a 'homotopy' between them involving only trajectories. The resulting object is a semigroup, which we call the *dynamic index* of the local control set. It measures the different ways the system can go through the local control set. A number of examples are considered.

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1. Introduction.

The aim of this paper is to contribute to the (formidable) task to classify the controllability behavior of nonlinear control systems. More precisely, we restrict our attention to certain subsets of complete controllability, local control sets as introduced in [3]. These are locally maximal subsets of complete controllability. They are composed of periodic trajectories.

The basic idea for the classification of local control sets is to call periodic trajectories equivalent if there is a 'homotopy' between them; however, these homotopies should involve only trajectories in order to capture the dynamic properties of the considered system. This leads to considerable technical difficulties. The resulting object is a semigroup, which we call the *dynamic index* of the local control set. It measures the "different" ways the system can go through the local control set. It turns out, that for linear systems with controllable (A, B) and admissible control range U the index is always trivial. If the control range is small enough, the same is true for local control sets around a hyperbolic equilibrium of the uncontrolled system. Furthermore, if the control range is small enough, we can also show that for a local control set around an attracting periodic solution of the uncontrolled system the index is isomorphic to the natural numbers $\mathbb{N}$. The index can distinguish such control sets from those occurring around a homoclinic orbit. Compare also San Martin and Santana [10], where the homotopy type of Lie semigroups and invariant control sets is studied. We remark that in our construction the direction of the trajectories plays a decisive role. This is a decisive difference of our semigroup from homotopy groups. Katok and Hasselblatt [6, p. 117] briefly discuss other constructions of topological invariants using trajectories of dynamical systems.

Perhaps closest in spirit to our paper are the papers [11, 12] by A. Sarychev. He studied homotopy properties of the space of trajectories. However, he was interested in the case, where the systems are completely controllable or, in our terminology, where the control set coincides with the whole state space.

In Section 2 we specify our assumptions on the considered control systems and recall some basic notions. In Section 3 we define the key notion for the construction of the index, the so-called 'strong inner pairs', and show some of their relevant properties. Section 4 is devoted to the construction of the index and some simple examples are provided, whereas in Section 5 we investigate the relation between the indices of nested local control sets. Section 6, finally, presents the explicit computation of the index in the case of the control set which arises, for a small control range, around an attracting periodic orbit of the uncontrolled system.

2. Preliminaries.

In this section we specify the considered class of control systems and recall some basic notions.

Throughout all the paper we will let U be a compact convex neighborhood of the origin in $\mathbb{R}^m$ and for $0 \le \rho \le 1$, we put $\rho U = \{\rho \cdot x : x \in U\}$. Moreover we denote by $\mathcal{U}^\rho$ the set of all $L_\infty(\mathbb{R}, \mathbb{R}^m)$ control functions taking values in ρU. For simplicity, when $\rho = 1$, we shall simply omit it. If not specified otherwise, the space $\mathcal{U}$ will be considered in the weak* topology inherited from the inclusion $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m) = \big(L_1(\mathbb{R}, \mathbb{R}^m)\big)^*$. Notice that $\mathcal{U}$ is in this topology a compact and separable metrizable space (see, e.g., Dunford/Schwartz [4]); an appropriate metric will be denoted by 'd'.

We will consider the following control-affine system in $\mathbb{R}^d$

$$\dot{x}(t) = f\big(x(t), u(t)\big) := f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad u \in \mathcal{U}^\rho. \tag{2.1}$$

with sufficiently smooth vector fields f_i, $i = 0, 1, ..., m$. We assume that for every control $u \in \mathcal{U}$ and every initial condition $x(0) = x_0 \in \mathbb{R}^d$ there exists a unique trajectory which we denote by $\varphi(t, x, u)$, $t \in \mathbb{R}$. Our results will also hold –with some technical modifications– for systems on manifolds. Note that for control affine systems, the trajectories $\varphi(t, x, u)$ depend continuously on (t, x, u), uniformly on bounded time intervals; here $\mathcal{U}$ is endowed with the weak* topology; see [1, Lemma 4.3.2].

The following definitions specify subsets of complete approximate controllability, which are our primary concern in this paper.

Definition 2.1. *A subset D with nonempty interior of the state space $\mathbb{R}^d$ is a precontrol set if for all $x, y \in D$ and every $\varepsilon > 0$ there exist $T > 0$ and $u \in \mathcal{U}$ such that*

$$\varphi(t, x, u) \in D \text{ for all } t \in [0, T] \quad \text{and} \quad |\varphi(T, x, u) - y| < \varepsilon.$$

Definition 2.2. *A precontrol set D of $\mathbb{R}^d$ is a local control set if there exists a neighborhood V of* cl D *such that for every precontrol set D' with $D \subset D' \subset V$ one has $D' = D$.*

Thus a local control set is a locally maximal precontrol set. Note also that control sets (with nonvoid interior) as discussed in [1] are globally maximal precontrol sets. The sets of reachable points from x and controllable to $x \in \mathbb{R}^d$ in time $T > 0$ are denoted by

$$\mathcal{O}^+_{\leq T}(x) = \left\{ y \in \mathbb{R}^d, \text{ there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u) \right\}$$

and

$$\mathcal{O}^-_{\leq T}(x) = \left\{ y \in \mathbb{R}^d, \text{ there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } x = \varphi(t, y, u) \right\},$$

respectively. Furthermore let

$$\mathcal{O}^+(x) = \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x) \text{ and } \mathcal{O}^-(x) = \bigcup_{T>0} \mathcal{O}^-_{\leq T}(x)$$

denote the reachable set from x and the set controllable to x, respectively. We also call $\mathcal{O}^{\pm}(x)$ the positive and negative orbits of x, respectively.

Throughout this paper we require local accessibility, that is, $\mathcal{O}^+_{\leq T}(x)$ and $\mathcal{O}^-_{\leq T}(x)$ have nonvoid interiors for all $x \in \mathbb{R}^d$ and all $T > 0$. Recall also that local accessibility is guaranteed by the following *accessibility rank condition*:

$$\dim \Delta_{\mathcal{L}}(x) = d \text{ for all } x \in \mathbb{R}^d, \tag{2.2}$$

where $\mathcal{L}$ denotes the Lie algebra generated by the vector fields $f_0, ..., f_m$, and $\Delta_{\mathcal{L}}(x)$ is the subspace of the tangent space (identified with $\mathbb{R}^d$) generated by the vector fields in $\mathcal{L}$.

3. Strong Inner Pairs.

In this section we specify the subclass of periodic trajectories which will be used for the construction of the dynamic index.

First note that for a control $u \in \mathcal{U} = \mathcal{U}^1$ there is $\delta_0 > 0$ such that $d(u(t), \partial U) > \delta_0$ for almost all $t > 0$ iff $u \in \mathcal{U}^\rho$ for some $\rho < 1$.

Definition 3.1. *A pair $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ is called a strong inner pair, if:*

(i) there is $\rho < 1$ such that $u \in \mathcal{U}^\rho$;

(ii) the control u is piecewise constant and there is $\delta > 0$ such that for all $y \in \mathbb{R}^d$ with $|x - y| < \delta$ and all $\tau > 0$ small enough, the following property holds:

For all $0 < t \leq \tau$ there are neighborhoods $N_t^{\pm}(y)$ of $\varphi(\pm t, y, u)$ such that for any curve $\lambda \mapsto z_\lambda^+ \in N_t^+(y)$ and $\lambda \mapsto z_\lambda^- \in N_t^-(y)$, with $z_0^{\pm} = \varphi(\pm t, y, u)$, there are continuous maps

$$\lambda \mapsto (\pm t_\lambda^{\pm}, u_\lambda^{\pm}) : [0,1] \to (0,T) \times \mathcal{U},$$

with $u_\lambda^{\pm}$ piecewise constant for $\lambda \in [0,1]$, and

$$(\pm t_0^{\pm}, u_0^{\pm}) = (\pm t, u) \text{ and } \varphi(\pm t_\lambda^{\pm}, y, u_\lambda^{\pm}) = z_\lambda^{\pm}.$$

Moreover, we say that a strong inner pair is T-periodic if $(u, \varphi(\cdot, x, u))$ is T-periodic.

Remark 3.2. *Observe that for the point (ii) in Definition 3.1, one has*

$$\varphi(\pm t_0^{\pm}, y, u_0^{\pm}) = \varphi(\pm t, y, u);$$

and the neighborhoods $N_t^{\pm}(y)$ are contained in the reachable sets $\mathcal{O}^{\pm}(y)$ from y provided they are connected.

Remark 3.3. *In [1], inner pairs were defined as those pairs (u, x) satisfying*

$$\varphi(\pm t, x, u) \in \mathrm{int}\mathcal{O}^{\pm}(x)$$

for some $t > 0$. Here, in order to construct the dynamic index, we need the stronger properties required in Definition 3.1.

It is convenient to introduce the following notation (compare e.g. [1, 9]): When u is a constant control, we shall write $e^{tX}x$, with $X = f(\cdot, u)$, in place of $\varphi(t, x, u)$.

We now note that strong inner pairs are abundant provided that local accessibility holds.

Proposition 3.4. *Consider a pair $(u, x) \in \mathcal{U} \times \mathbb{R}^d$ with piecewise constant control $u \in \mathcal{U}^\rho$ with $\rho < 1$ which, on the intervals $[0, \sum_{i=1}^d s_i^+]$ and $[-\sum_{i=1}^d s_i^-, 0]$, with $s_i^{\pm} > 0$, takes the values*

$$u(t) = u_i^+ \in \mathrm{int}\, U \text{ for } t \in (s_1 + ... + s_i, s_1 + ... + s_{i+1}),$$
$$u(-t) = u_i^- \in \mathrm{int}\, U \text{ for } t \in (-s_1 - ... - s_i - s_{i+1}, -s_1 - ... - s_i).$$

Suppose that there is $\varepsilon > 0$ such that $s_1^{\pm}, ..., s_d^{\pm} \in (0, \varepsilon)$ and for $X_i^{\pm} := f(\cdot, u_i^{\pm})$ the two maps

$$(t_d, ..., t_1) \mapsto e^{\pm s_d X_d^{\pm}} \cdots e^{\pm s_1 X_1^{\pm}} x$$

have full rank on $(0, \varepsilon) \times ... \times (0, \varepsilon)$. Then (u, x) is a strong inner pair.

Proof. Obviously, property (i) of strong inner pairs holds. Property (ii) is satisfied, because the rank condition holds for y in a neighborhood of x and neighborhoods of $\varphi(\pm t, y, u)$ are of the form

$$\{e^{\pm t_d X_d} ... e^{\pm t_1 X_1} x, \text{ with } t_1, ..., t_d \in (0, \varepsilon)\}.$$

Hence the required continuous families are obtained by changing the times t_i. □

Remark 3.5. *Assume that accessibility rank condition (2.2) holds and fix $x \in \mathbb{R}^d$. Then, as in the proof of Krener's Theorem (cp. [7] or [1,Th. A.4.4]), one can show that there exist constants $u_1, \ldots, u_d \in \operatorname{int} U$ with the property that the two maps*

$$(t_d, ..., t_1) \mapsto e^{\pm t_d X_d} \cdots e^{\pm t_1 X_1} x,$$

$X_i = f(\cdot, u_i)$, have full rank on $(0, \varepsilon) \times \cdots \times (0, \varepsilon)$. Therefore, one can construct a piecewise constant function u as in Proposition 3.4, so that (u, x) is a strong inner pair.

A further class of strong inner pairs is obtained when the linearized control system is controllable. Recall that for two vector fields X, Y one defines $\mathrm{ad}_X^0 Y = Y$ and for $k = 1$, 2, ... one defines $\mathrm{ad}_X^k Y$ as the Lie bracket $\mathrm{ad}_X^k Y := [X, \mathrm{ad}^{k-1} Y]$.

Proposition 3.6. *Let $x \in \mathbb{R}^d$ and assume that*

$$\operatorname{span}\{\mathrm{ad}_{f_0}^k f_i(x),\ i = 1, ..., m,\ k = 0, 1, ...\} = \mathbb{R}^d. \tag{3.1}$$

Then for $\rho > 0$, small enough, each $(u, y) \in \mathcal{U}^\rho \times \mathbb{R}^d$ with u piecewise constant and $u \in \mathcal{U}^{\rho'}$, for some $\rho' < \rho$ and $|y - x| < \rho'$, is a strong inner pair.

Proof. The stated Lie algebraic assumption also holds for all $\varphi(T, y, u)$ with $\|u\|_\infty < \rho$ and all y in a neighborhood of x provided that $\rho > 0$ and $T > 0$ are small enough. It guarantees, for all $0 < \tau \leq T$, controllability for the linearized control system

$$\dot{z}(t) = D_1 f(\varphi(t, y, u), u(t)) z(t) + D_2 f(\varphi(t, y, u), u(t)) v(t),\ t \in [0, \tau],$$

with unbounded controls $v \in L_\infty([0, \tau], \mathbb{R}^m)$. Then a standard result in nonlinear control theory, see, e.g. [1, Theorem A.4.11 and Remark A.4.12] guarantees that the nonlinear control system with controls in $\mathcal{U}^\rho$ is locally controllable about the trajectory $\varphi(t, y, u)$, provided that $u \in \mathcal{U}^{\rho'}$ for some $\rho' < \rho$. This is based on an application of the inverse function theorem, which also provides the existence of neighborhoods $N_t^\pm$ as in Definition 3.1. □

Remark 3.7. *A slight modification of [1, Proposition 4.5.19] shows that in Proposition 3.6 one may consider, instead of condition (3.1), the following:*

$$\operatorname{span}\{f_0(x), \mathrm{ad}_{f_0}^k f_i(x),\ i = 1, ..., m,\ k = 0, 1, ...\} = \mathbb{R}^d.$$

This is based on a controllability condition due to Nam and Araposthatis [8].

We will need that the set of periodic strong inner pairs is open in the following sense.

Proposition 3.8. *Let (u_0, x_0) be a T_0-periodic strong inner pair. Then there exists $\delta > 0$ such that for every T_1-periodic strong inner pair $(u_1, x_1) \in \mathcal{U} \times \mathbb{R}^d$ with $|T_0 - T_1| < \delta$, $\mathrm{d}(u_0, u_1) < \delta$, and $|x_0 - x_1| < \delta$ there exists a continuous map $H : [0,1] \to \mathbb{R}^+ \times \mathcal{U} \times \mathbb{R}^d$, $H(\alpha) = (T_\alpha, x_\alpha, u_\alpha)$ with the following properties:*

1. *for all $\alpha \in [0,1]$, (u_α, x_α) is a T_α-periodic strong inner pair;*
2. *$H(0) = (T_0, u_0, x_0)$ and $H(1) = (T_1, u_1, x_1)$.*

Proof. As a first step we construct a 'homotopy' from (T_0, u_0, x_0) to an appropriate triple (T, v, x_0) where $\varphi(t, x_0, v)$, $t \in [0, T]$, is a T-periodic trajectory satisfying

$$\varphi(t, x_0, v_0) = \varphi(t, x_1, u_1)$$

for $t \in [\tau, T_0 - \tau]$ for a suitable time $\tau > 0$.

Let $\tau > 0$ and $N_t^{\pm}(x_0)$, for $0 < t < \tau$, be as in Definition 3.1. Take for short

$$N^+ = N_\tau^+(x_0) \quad \text{and} \quad N^- = N_\tau^-(x_0).$$

Since N^+ and N^- are neighborhoods of $\varphi(\tau, x_0, u_0)$ and $\varphi(T_0 - \tau, x_0, u_0)$ respectively, by continuous dependence on the control function (cp. [1, Lemma 4.3.2]), choosing $\delta > 0$ small enough, we can assume

$$\sup_{t \in [0, \max\{T_0, T_1\}]} |\varphi(t, x_0, u_0) - \varphi(t, x_1, u_1)|$$

as small as we please. Therefore we can take

$$x_\lambda^- := \varphi(T_0 - \tau, x_1, u_\lambda) \in N^- \text{ and } x_\lambda^+ := \varphi(\tau, x_1, u_\lambda) \in N^+,$$

where $u_\lambda := \lambda u_1 + (1 - \lambda) u_0$. (Recall that also $|T_0 - T_1| < \delta$.) As in Definition 3.1 (ii), there are continuous maps $\lambda \longmapsto (\pm t_\lambda^{\pm}, v_\lambda^{\pm})$, with $v_\lambda^{\pm}$ piecewise constant for $\lambda \in [0,1]$, and

$$(\pm t_0^{\pm}, v_0^{\pm}) = (\pm\tau, u_0) \text{ and } \varphi(\pm t_\lambda^{\pm}, x_0, v_\lambda^{\pm}) = x_\lambda^{\pm} \text{ for all } \lambda \in [0,1].$$

The concatenations

$$\lambda \mapsto v_\lambda^- \circ u_\lambda|[t_\lambda^+, T_0 - t_\lambda^-] \circ v_\lambda^+, \text{ and } \lambda \mapsto T_0 - \tau + t_\lambda^+ + t_\lambda^-,$$

yield the desired continuous family of periodic trajectories.

As a second step, reducing δ if necessary, we essentially repeat the construction above and connect (T_1, u_1, x_1) with the triple (T, v, x_0) that we have just constructed. More precisely, if δ is small enough, we can find a point $z = \varphi(\tau, x_1, u_1)$ near x_0 lying on $\varphi([0,T], x_0, v)$ and such that $(\varphi(\tau, x_1, u_1), u_1(\tau + \cdot))$ is a strong inner pair. Moreover, reducing δ if necessary, one can find $\tau_- > 0$ such that for $\lambda \in [0,1]$ one has

$$x_\lambda := \varphi(T - \tau - \tau_-, z, v_\lambda) \in N_\tau^-(z) \text{ and } v_\lambda := \lambda u_1 + (1-\lambda)v.$$

Then using property (ii) in the definition of strong inner pairs, one finds a continuous family of controls connecting these points to z. Concatenating the elements of this family with v_λ as in the first step, one gets a homotopy between (T_1, u_1, x_1) and (T, v, x_0).

The two steps together yield the desired homotopy. □

The following lemma establishes a local controllability property around the trajectory of a periodic strong inner pair.

Lemma 3.9. *Let (u, x) be a T-periodic strong inner pair. Then every neighborhood V of $\{\varphi(t, x, u),\ t \in [0,T]\}$ contains a neighborhood D which is a precontrol set.*

Proof. First observe that, trivially, the periodic trajectory is a precontrol set. By assumption, there are $T \geq \tau > 0$, arbitrarily small, and open neighborhoods $N^\pm$ of $\varphi(\pm\tau, x, u)$ contained in $\mathcal{O}^\pm(x)$, respectively. By continuous dependence on initial values, we may assume that for every $x_1 \in N^+$ one has $\varphi(T - 2\tau, x_1, u(\tau + \cdot)) \in N^-$. Hence, one can steer x into every point of N^+ and one can steer every point of N^+ into N^- (using the control u) and from there into x. By continuous dependence on the initial value, the piece of the periodic trajectory $\{\varphi(t, x, u),\ t \in [\tau, T - \tau]\}$ is contained in a precontrol set contained in V. Now consider $\{\varphi(t, x, u),\ t \in [-\tau, \tau]\}$. Again, by continuous dependence on the initial value, the set N^- is mapped via the shifted control $u(T - \tau + \cdot)$ onto a neighborhood of any point $\varphi(t, x, u)$ in time $\tau + t$, and similarly, a neighborhood of this point is mapped into N^+ via $u(t + \cdot)$ in time $\tau - t$. We conclude that V contains a precontrol set D which is a neighborhood of $\{\varphi(t, x, u),\ t \in [0,T]\}$. □

Lemma 3.10. *Let D be a local control set for (2.1) and assume that the accessibility rank condition holds in D. Then, for any $x, y \in \operatorname{int} D$, there are $T > 0$ and a T-periodic control function $u \in \mathcal{U}$ such that (u, x) is a strong inner pair and $y \in \varphi([0,T], x, u)$.*

Proof. By the accessibility rank condition, as in the proof of Krener's Theorem, it follows that there exist $u_1, \ldots, u_d \in \operatorname{int} U$ and $\delta > 0$ such that,

$$\begin{aligned} N^+ &= \operatorname{int}\left\{e^{t_d X_d} \cdots e^{t_1 X_1} x : 0 \leq t_i \leq \delta,\ i = 1, ..., d\right\} \neq \emptyset, \\ N^- &= \operatorname{int}\left\{e^{t_d X_d} \cdots e^{t_1 X_1} x : -\delta \leq t_i \leq 0,\ i = 1, ..., d\right\} \neq \emptyset, \end{aligned}$$

where $X_i = f(\cdot, u_i)$ for $i = 1, ..., d$.

Take $x^+ \in N^+$. Since in the interior of D approximate controllability holds, one can find a control function v_0 and a time S_0 such that $x^- := \varphi(S_0, x^+, v) \in N^-$. By continuous dependence we can assume that v is a piecewise constant function belonging to $\mathcal{U}^\rho$ for some $\rho < 1$. Let $v^+, v_- \in \mathcal{U}$ and $S^+, S_- > 0$ be such that

$$x^+ = \varphi(S^+, x, v^+) \quad \text{and} \quad x = \varphi(S_-, x^-, v_-).$$

Concatenating v_-, v^+ and v_0, and taking $T = S_+ + S_0 + S_-$ one gets a T-periodic trajectory driven by some T-periodic piecewise constant control function u. One can also construct u as a control function which connects x^+ to y and y to x_-, in a way that essentially follows the line of the first part of the proof. □

1. The Dynamic Index.

In this section we construct a dynamic index for local control sets.

We consider a local control set D for (2.1) and assume throughout that the accessibility rank condition holds. Define the set

$$\mathcal{P}(D) = \left\{ (T, u, x) \in (0, \infty) \times \mathcal{U} \times \mathbb{R}^d : \begin{array}{l} (u, x) \text{ is a } T\text{-periodic} \\ \text{strong inner pair, } T > 0, \text{ and} \\ \varphi(t, x, u) \in D, \ \forall t \in [0, T] \end{array} \right\},$$

endowed with the metric topology given by

$$\operatorname{dist}\big((T, u, x), (S, v, y)\big) = |T - S| + \|x - y\|_{\mathbb{R}^d} + \mathrm{d}(u, v).$$

Remark 4.1. *Although the above definition is valid for any subset of the state space $\mathbb{R}^d$, the theory that we are developing is relevant only for (local and global) control sets in which the accessibility rank condition holds. In fact, by Lemma 3.10, if D is such a control set, then $\mathcal{P}(D) \neq \emptyset$; and, by Lemma 3.9, for every T-periodic strong inner pair (u, x) the point x is in some control set.*

Below, when no confusion can possibly arise, we shall omit the explicit dependence on the base set D.

Let us now introduce a relation on $\mathcal{P}$.

Definition 4.2. *$(T, u, x) \sim (S, v, y)$ in $\mathcal{P}$ if there are $k + 1$ elements $(T_0, u_0, x_0), \ldots, (T_k, u_k, x_k)$ in $\mathcal{P}$ with the following properties:*

(i) $(T_0, u_0, x_0) = (T, u, x)$ *and* $(T_k, u_k, x_k) = (S, v, y)$;

(ii) *for* $i = 0, ..., k$ *there are*

$$0 = \tau_i^0 < ... < \tau_i^{k_i} = T_i \text{ and } 0 = \sigma_{i+1}^0 < ... < \sigma_{i+1}^{k_i} = T_{i+1},$$

such that $\varphi(\tau_i^j, x_i, u_i) = x_i$ *and* $\varphi(\sigma_{i+1}^j, x_{i+1}, u_{i+1}) = x_{i+1}$ *for all* i *and all* j;

(iii) there are continuous maps $H_i^j : [0,1] \to \mathcal{P}$ such that for $i = 0, ..., k$ and $j = 0, ..., k_i - 1$

$$H_i^j(0) = \left(\tau_i^{j+1} - \tau_i^j, u_i(\tau_i^j + \cdot), x_i\right), \quad \textit{and}$$
$$H_i^j(1) = \left(\sigma_{i+1}^{j+1} - \sigma_{i+1}^j, u_{i+1}(\sigma_{i+1}^j + \cdot), x_{i+1}\right).$$

In other words, (T_i, u_i, x_i) and $(T_{i+1}, u_{i+1}, x_{i+1})$ are chopped into k_i periodic pieces of period $\tau_i^{j+1} - \tau_i^j$ and $\sigma_{i+1}^{j+1} - \sigma_{i+1}^j$ respectively, and the corresponding pieces are homotopic via trajectories.

Notice that the relation introduced above is an equivalence relation. Then, consider on $\mathcal{P}/\sim$, the set $\mathcal{Q}$ of all the formal (juxtaposition) products, i.e. the free semigroup on $\mathcal{P}/\sim$. (See, e.g., Howie [5] for some general facts about the algebraic theory of semigroups.) As usual, we write $[T, u, x]^n$ instead of

$$\underbrace{[T, u, x] \cdots [T, u, x]}_{n \text{ times}},$$

for any $n \geq 0$. Here the square parentheses denote the equivalence classes.

Clearly $\mathcal{Q}$ is a semigroup which, besides its non-commutativity, is far too large for being of any use. Below we factorize it over the congruence induced by two families of equations among the elements of $\mathcal{Q}$. Recall that a congruence on a semigroup $(S, \cdot)$ is an equivalence relation '$\equiv$' such that

$$a \equiv a' \text{ and } b \equiv b' \text{ imply } a \cdot b \equiv a' \cdot b',$$

for any $a, a', b, b' \in S$.

Consider the following families of relations:

$$\mathcal{F} = \Big\{[T, u, x][S, v, x] = [T + S, u \circ v, x] : (T, u, x), (S, v, x) \in \mathcal{P}\Big\},$$
$$\mathcal{G} = \Big\{[T, u, x][S, v, y] = [S, v, y][T, u, x] : (T, u, x), (S, v, y) \in \mathcal{P}\Big\}.$$

Notice that the elements of $\mathcal{F}$ are well defined. In fact, by the definition of '$\sim$' one has that

$$(T, u, x) \sim (\bar{T}, \bar{u}, \bar{x}) \text{ and } (S, v, x) \sim (\bar{S}, \bar{v}, \bar{x})$$

imply

$$(T + S, u \circ v, x) \sim (\bar{T} + \bar{S}, \bar{u} \circ \bar{v}, \bar{x}).$$

The union of the families $\mathcal{F}$ and $\mathcal{G}$ clearly can be seen as a relation on $\mathcal{Q}$, i.e., as a subset of $\mathcal{Q} \times \mathcal{Q}$. Now, since the intersection of congruences is again a congruence, it makes sense to consider the congruence $(\mathcal{F} \cup \mathcal{G})^{\#}$ generated by the set $\mathcal{F} \cup \mathcal{G}$, namely the intersection of all the congruences containing $\mathcal{F} \cup \mathcal{G}$ (see e.g. [5]).

Remark 4.3. *An alternative definition for the congruence $(\mathcal{F}\cup\mathcal{G})^{\#}$ is the following (see [5, Proposition 5.9]):*

Let $\mathcal{Q}^1$ be the semigroup obtained from $\mathcal{Q}$ by formally adjoining (if necessary) a unity element, and define the relation in $\mathcal{Q}$ given by

$$\mathcal{R} = \big\{(\xi\alpha\eta, \xi\beta\eta) : \xi, \eta \in \mathcal{Q}^1, (\alpha,\beta) \in \mathcal{F}\cup\mathcal{G}\big\}.$$

Then $(\mathcal{F}\cup\mathcal{G})^{\#}$ is the equivalence relation generated by $\mathcal{R}$ (i.e. the intersection of all the equivalence relations containing $\mathcal{R}$).

Finally, we define the *dynamic index* $\mathcal{I}(D)$ of D as the quotient

$$\mathcal{I}(D) := \mathcal{Q}(D)/(\mathcal{F}\cup\mathcal{G})^{\#} \ .$$

Notice that, $\mathcal{I}(D)$ is a commutative semigroup.

Remark 4.4. *Instead of the family $\mathcal{G}$ above, we could take*

$$\mathcal{G}' = \Big\{[T,u,x][S,v,y] = [S,v,y][T,u,x] : (T,u,x), (S,v,y) \in \mathcal{P},\ x \neq y\Big\}.$$

In fact, for $(T,u,x), (S,v,x) \in \mathcal{P}$, it follows that

$$(T+S, u\circ v, x) \sim (S+T, v\circ u, x),$$

as one can see with the 'homotopy'

$$H(\lambda) = \big(T+S, (u\circ v)(\lambda T + \cdot), \varphi(\lambda T, p, u)\big), \quad \lambda \in [0,1].$$

Example 4.5. (Linear Systems) *Consider the following linear control system with restricted control range*

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{ in } \mathbb{R}^d,\ u \in \mathcal{U},$$

where $U \subset \mathbb{R}^m$ is convex and compact with $0 \in$ int U and A and B are constant matrices of dimensions $d\times d$ and $d\times m$, respectively. We assume that the pair (A,B) is controllable, i.e., that rank $[B, AB, ...A^{d-1}B] = d$. *Then the index $\mathcal{I}(D)$ of the unique control set D reduces to the unity.*

This follows from the uniqueness proof of D (cp. [1]): Consider a T-periodic strong inner pair (u,x) in the interior of D. Define a homotopy to the origin via

$$H(\alpha) := (T, \alpha u, \alpha x),\ \alpha \in [0,1].$$

Linearity implies that $\varphi(T, \alpha x, \alpha u) = \alpha x$ for all $\alpha \in [0,1]$. Hence this is a periodic solution, and for $\alpha = 0$ one obtains the equilibrium.

It is also of interest to consider the following pointed notion of the index. For x in a local control set D define

$$\mathcal{I}_x(D) = \{[T,u,x] : [T,u,x] \in \mathcal{I}(D)\}\,.$$

Clearly $\mathcal{I}_x$ is a subsemigroup of $\mathcal{I}$. An important property enjoyed by this new object is the following.

Theorem 4.6. *If $x_0 \in D$ is an equilibrium for f, i.e., there exists a constant $\bar{u} \in \operatorname{int} U$ such that $f(x_0, \bar{u}) = 0$, then $\mathcal{I}_{x_0}$ is a monoid (i.e. admits unity).*

Proof. The unity can be written as $[1, \bar{u}, x_0]$. In fact, if $[T, u, x_0]$ is any element of $\mathcal{I}_{x_0}$, then the homotopy $H(\lambda) = (T + \lambda, u_\lambda, x_0)$, where u_λ is the $(T + \lambda)$-periodic extension to $\mathbb{R}$ of the following function:

$$\begin{cases} \bar{u} & \text{for } t \in [0, \lambda] \\ u(t - \lambda) & \text{for } t \in [\lambda, T + \lambda], \end{cases}$$

shows that

$$[1, \bar{u}, x_0][T, u, x_0] = [T + 1, \bar{u} \circ u, x_0] = [T, u, x_0].$$

Analogously, one can see that $[T, u, x_0][1, \bar{u}, x_0] = [T, u, x_0]$. □

5. Changing the base set.

If D and D' are local control sets with $D \subset D'$ then the inclusion $i : D \hookrightarrow D'$ determines a natural homomorphism $i_* : \mathcal{I}(D) \to \mathcal{I}(D')$. Analogously, if $x_0 \in D$ one has a natural homomorphism $i_{*,x_0} : \mathcal{I}_{x_0}(D) \to \mathcal{I}_{x_0}(D')$. Indeed, it is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{I}(D') & \xleftarrow{\supset} & \mathcal{I}_{x_0}(D') \\ {\scriptstyle i_*}\big\uparrow & & \big\uparrow{\scriptstyle i_{*,x_0}} \\ \mathcal{I}(D) & \xleftarrow{\supset} & \mathcal{I}_{x_0}(D) \end{array}$$

By commutativity of this diagram, injectivity of i_* implies that $i_{*.x_0}$ is injective as well. Furthermore, the following fact holds.

Theorem 5.1. *Let D and D' be local control sets for (2.1) such that $D \subset D'$. Then i_* is injective and, if $D \neq D'$, then i_* is not surjective.*

The proof is based on the following lemma

Lemma 5.2 *Take $(T, u, x) \in \mathcal{P}(D') \setminus \mathcal{P}(D)$. Then, if $(T', u', x') \sim (T, u, x)$, one necessarily has $(T', u', x') \in \mathcal{P}(D') \setminus \mathcal{P}(D)$.*

Proof. By the definition of local control sets, there exists an open neighborhood V of $\operatorname{cl} D$ such that D is the maximal subset of complete controllability of V. Without loss of generality, we can assume that $\operatorname{cl} D \subset V$ (and, clearly, $D' \not\subset V$). Assume by contradiction that $(T', u', x') \in \mathcal{P}(D)$. Since the relation

$$(T', u', x') \sim (T, u, x)$$

holds, by our definition of '$\sim$', there exists a continuous $H : [0,1] \to \mathcal{P}(D')$, $\lambda \mapsto (T_\lambda, u_\lambda, x_\lambda)$, such that $H(0) \in \mathcal{P}(D)$ and $H(1) \in \mathcal{P}(D') \setminus \mathcal{P}(D)$.

Put $O_\lambda = \varphi([0,T_\lambda], x_\lambda, u_\lambda)$. By continuity, there exists some $\bar\lambda \in [0,1]$ such that $O_\lambda \subset V$ for all $0 \le \lambda \le \bar\lambda$ and $O_{\bar\lambda} \not\subset D$. Let $\delta > 0$ be the distance between $\bigcup_{\lambda\le\bar\lambda} O_\lambda$ and the boundary of V. Lemma 3.9 implies that there exists a neighborhood W of this union that is a precontrol set contained in $\bigcup_{\lambda\le\bar\lambda} O_\lambda + B(0,\delta/2)$. Obviously, $W \cup D \subset V$ is a precontrol set containing D properly. This contradicts the choice of V. $\square$

Proof of Theorem 5.1. If $D = D'$ there is nothing to prove, thus we assume $D \neq D'$. To prove that i_* is injective we have to show that given any (T,u,x) and (T',u',x') in $\mathcal{P}(D)$, with $[T,u,x] \neq [T',u',x']$ in $\mathcal{I}(D)$, they cannot be joined by a continuous curve in $\mathcal{P}(D')$.

In fact, if they were connected by some $H : [0,1] \to \mathcal{P}(D')$, $\lambda \mapsto (T_\lambda, u_\lambda, x_\lambda)$, there would exist $\lambda_0 \in [0,1]$ such that $(T_{\lambda_0}, u_{\lambda_0}, x_{\lambda_0}) \in \mathcal{P}(D') \setminus \mathcal{P}(D)$, but this is impossible by Lemma 5.2.

As for the non-surjectivity of i_*, it is enough to notice that, by Lemma 5.2, no element of $\mathcal{P}(D') \setminus \mathcal{P}(D)$ can be joined to any one of $\mathcal{P}(D)$ by a continuous curve in $\mathcal{P}(D')$. This means that, given any $(T,u,x) \in \mathcal{P}(D') \setminus \mathcal{P}(D)$ one has $[T,u,x] \notin i_*\big(\mathcal{I}(D)\big)$. $\square$

The above theorem allows us to drop the 'i_*' and consider $\mathcal{I}(D)$ as a subsemigroup of $\mathcal{I}(D')$. When $D \subsetneqq D'$, Theorem 5.1 just says that $\mathcal{I}(D)$ is a *proper subsemigroup* of $\mathcal{I}(D)$.

6. The index of a control set near a periodic orbit.

This section is devoted to the computation of the index of the control set for (2.1) which arise for a small control range around an isolated attracting periodic orbit $\gamma = \varphi([0,T], x_0, 0)$, with (minimal) period $T > 0$, of the uncontrolled system, assuming that the linearized system along γ is controllable. Recall that a periodic orbit (of an autonomous differential equation) is called attracting, if the eigenvalues of the linearized Poincaré map are strictly smaller than one in modulus; compare [9].

Proposition 6.1. *Let γ be a attracting orbit of the uncontrolled system, and let A be a neighborhood of γ. Assume that the controllability rank condition (3.1) holds. Then there exist ρ_0 such that for any $0 < \rho \le \rho_0$ there exists a unique control set D^ρ with $\gamma \subset D^\rho \subset A$.*

Proof. The controllability rank condition implies by Proposition 3.6 that all pairs $(x,0) \in \gamma \times \mathcal{U}^\rho$ are strong inner pairs, hence inner pairs. Then Corollary 4.7.6 in [1] implies the assertion. $\square$

We shall prove that, when ρ is small enough, the index of the control set D^ρ, relative to system (2.1) containing γ, is isomorphic to $\mathbb{N}$. To prove this result we need to show that when $(T_1,u_1,x_1) \in \mathcal{P}(D^\rho)$ is such that $\varphi([0,T_1],x_1,u_1)$ goes n times around γ, then $(T_1,u_1,x_1) \sim (nT,0,x_0)$ and therefore $[T_1,u_1,x_1] = [T,0,x_0]^n$. To make this precise we shall introduce Definition 6.3 below.

However, it is first necessary to establish some preliminaries on the Poincaré map for control systems. We begin the following notion from Colonius/Sieveking [2].

Definition 6.2. *Let $x_0 \in \mathbb{R}^n$, $L : \mathbb{R}^n \to \mathbb{R}$ linear and $\alpha > 0$. If $Lf(x, u) > \alpha$ for all x in a neighborhood W of x_0 and all $u \in \rho U$ then the connected component of $W \cap L^{-1}(x_0)$ containing x_0 is called a* local transversal section *through x_0.*

The definition above allows us to formulate precisely what is meant by saying that an orbit goes n times around another.

Definition 6.3. *Let Ω be a neighborhood of γ. We say that a closed orbit $\gamma_1 = \varphi([0, T_1], x_1, u_1) \subset \Omega$ goes n times around γ (relatively to Ω) if there exists a linear map L as in Definition 6.2 such that*

1. *$S := \Omega \cap L^{-1}(x_0)$ is a local transversal section to γ,*
2. *$\gamma \cap S = \{x_0\}$,*
3. *$x_1 \in S$, and*
4. *there exist exactly n times $t_i \in (0, T_1]$, $i = 1, ..., n$, such that*

$$\varphi(t_i, x_1, u_1) \in S.$$

An important fact about local transversal sections is the following (see [2, Proposition 2.14]).

Lemma 6.4. *If $0 \notin f(x_0, \rho U)$ then x_0 admits a local transversal section.*

Therefore, if $0 \neq f(x_0, 0)$ then x_0 admits a local transversal section for ρ small enough. Another useful notion from [2] is that of a *flow box* for control systems.

Definition 6.5. *Let S be a local transversal section through x_0, and let $V_1 \subset V_0$ be neighborhoods of x_0. The triple (V_0, V_1, S) is a* flow box *around x_0 if it has the following property:*

If $\varphi(\cdot, x_0, u)$ satisfies

$$\varphi(t_0, x_0, u) \notin V_0, \quad \varphi(t_1, x_0, u) \in V_1, \quad \varphi(t_2, x_0, u) \notin V_0$$

for some $0 \leq t_0 < t_1 < t_2$, then there exists $t \in (t_0, t_2)$ such that $\varphi(t, x_0, u) \in S$ and $\varphi(s, x_0, u) \in V_0$ for all s between t and t_1.

From the proof of Theorem 2.16 in [2], one immediately gets the following result.

Lemma 6.6. *Let S be a local transversal section through x_0. Then, for any neighborhood W of S there are neighborhoods V_0 and V_1 of x_0 contained in W such that (V_0, V_1, S) is a flow box around x_0.*

We now turn to the Poincaré map.

Proposition 6.7. *Let S be a local transversal section through $x_0 \in \gamma$. If ρ is small enough, there exists a neighborhood V of x_0 in S such that the Poincaré first return map $P : V \times \mathcal{U}^\rho \to S$ is well-defined and continuous. Moreover, the map that takes (x, u) into the 'first return time' $\tau(x, u)$ is continuous.*

Proof. Let us first show that P is well-defined. Notice that the orbits can cross S only from one side; therefore it is sufficient to show that there exists a neighborhood $V \subset S$ of x_0 such that the orbits return to S after a finite time.

Let W be a neighborhood of x_0 in $\mathbb{R}^d$ and (V_0, V_1, S) be a flow box around x_0 with $\operatorname{cl} V_0 \subset W$. Taking if necessary a smaller W, we can assume that there are times t_0 and t_1, with $0 < t_0 < T < t_1$, for which $\varphi(t_0, x_0, 0)$ and $\varphi(t_1, x_0, 0)$ are in $W \setminus \operatorname{cl} V_0$.

By continuous dependence on initial data there exist a neighborhood $V \subset V_1$ of x_0 in S and $\rho_0 > 0$ such that, if $0 < \rho < \rho_0$

$$\left.\begin{array}{l} \varphi(t_0, x, u) \in W \setminus \operatorname{cl} V_0 \\ \varphi(t_1, x, u) \in W \setminus \operatorname{cl} V_0 \\ \varphi(T, x, u) \in V_1 \end{array}\right\} \text{ for every } (x, u) \in V \times \mathcal{U}^\rho.$$

Since (V_0, V_1, S) is a flow box, for each $(x, u) \in V \times \mathcal{U}^\rho$ there exists a time $\tau(x, u)$, with $t_0 < \tau(x, u) < t_1$ such that $\varphi\big(\tau(x, u), x, u\big) \in S$. For W small enough this time is unique proving that $P(x, u) := \varphi\big(\tau(x, u), x, u\big)$ is well-defined.

We shall now prove continuity of the map $(x, u) \mapsto P(x, u)$. Consider a sequence $\{(\xi_n, u_n)\}$ in $S \times \mathcal{U}^\rho$ converging to (ξ_0, u_0). Fix a neighborhood W of $P(\xi_0, u_0)$ in S and let $\hat{W}$ be a neighborhood of $P(\xi_0, u_0)$ in $\mathbb{R}^d$ such that $W = \hat{W} \cap S$. Let (V_0, V_1, S) be a flow box around $P(\xi_0, u_0)$ with $\operatorname{cl} V_0 \subset \hat{W}$.

Let $\tau = \tau(\xi_0, u_0)$. As in the first part of the proof, taking W smaller if necessary, one can find times $0 < \tau_0 < \tau < \tau_1$ such that

$$\varphi(\tau_0, x_0, u_0), \varphi(\tau_1, x_0, u_0) \in W \setminus \operatorname{cl} V_1.$$

From [1, Lemma 4.3.2] one has

$$\begin{aligned} \lim_{n\to\infty} \varphi(\tau, \xi_n, u_n) &= \varphi(\tau, \xi_0, u_0) = P(\xi_0, u_0), \\ \lim_{n\to\infty} \varphi(\tau_0, \xi_n, u_n) &= \varphi(\tau_0, \xi_0, u_0), \\ \lim_{n\to\infty} \varphi(\tau_1, \xi_n, u_n) &= \varphi(\tau_1, \xi_0, u_0). \end{aligned}$$

Therefore, for n large enough,

$$\varphi(\tau_0, x_n, u_n), \varphi(\tau_1, x_n, u_n) \notin V_0 \quad \text{and} \quad \varphi(\tau, x_n, u_n) \in V_1.$$

Since (V_0, V_1, S) is a flow box there exists $\tau_n \in (\tau_0, \tau_1)$ such that $P(x_n, u_n) = \varphi(\tau_n, x_n, u_n) \in S \cap W$. This proves that, for n large, $P(x_n, u_n) \in W$ and continuity follows. Notice also that, in the construction above, $\tau_n = \tau(\xi_n, u_n)$ satisfies

$$\tau_1 - \tau_0 > |\tau - \tau_n|,$$

and that, by shrinking W, we can make the differences $\tau_1 - \tau_0$ as small as we please, therefore proving the continuity of the map $(x,u) \mapsto \tau(x,u)$. $\square$

Given a control function u and a time $T > 0$ it is convenient to denote by $[u]_T$ the function $u \mid [0,T]$ extended periodically to $\mathbb{R}$.

Proposition 6.8. *Let $\lambda \mapsto T_\lambda : [0,1] \to \mathbb{R}$ be continuous. For a (fixed) control function $u \in \mathcal{U}^\rho$, the map $\lambda \mapsto u_\lambda := \lambda[u]_{T_\lambda}$ is continuous.*

Proof. Suppose $\lambda_n \to \lambda$ as $n \to \infty$. For notational simplicity, assume that $T_{\lambda_n} < T_\lambda$ and $\rho = 1$. In order to show that $u_{\lambda_n} \to u_\lambda$ let W be a neighborhood of u_λ; we shall show that u_{λ_n} belongs to W for n large enough.

There are $\varepsilon > 0$ and $x_1, ..., x_N \in L_1(\mathbb{R}, \mathbb{R}^m)$ with

$$\left\{ v \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ \begin{array}{l} \left|\int_{\mathbb{R}} \langle u_\lambda(t) - v(t), x_j(t)\rangle dt\right| < \varepsilon \text{ for} \\ j = 1, ..., N \text{ and } v(t) \in U \text{ a.e.} \end{array} \right\} \subset W,$$

because the sets of this form constitute a subbase of the neighborhoods of u_λ in the weak* topology (see, e.g., Dunford/Schwartz [4]).

Since $x_j \in L_1(\mathbb{R}, \mathbb{R}^m)$ there is $k \in \mathbb{N}$ such that for $j = 1, ..., N$

$$\int_{\mathbb{R}\setminus[-kT_\lambda, kT_\lambda]} |x_j(t)| dt < \frac{\varepsilon}{2 \operatorname{diam} U} \ ,$$

where $\operatorname{diam} U = \sup\{|u_1 - u_2|, u_1, u_2 \in U\}$. Then, for $j = 1, ..., N$,

$$\begin{aligned} &\left| \int_{\mathbb{R}} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t)\rangle dt \right| \\ &\leq \left| \int_{-kT_\lambda}^{kT_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t)\rangle dt \right| + \left| \int_{\mathbb{R}\setminus[-kT_\lambda, kT_\lambda]} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t)\rangle dt \right| \end{aligned}$$

The second summand is bounded from above by

$$\operatorname{diam} U \int_{\mathbb{R}\setminus[-kT_\lambda, kT_\lambda]} |x_j(t)| dt < \varepsilon/2.$$

The first summand is

$$\left| \int_{-kT_\lambda}^{kT_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t)\rangle dt \right| \leq \sum_{i=-k}^{k-1} S_i,$$

where

$$S_i := \left| \int_{iT_\lambda}^{(i+1)T_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right|.$$

We have, for n large enough, that

$$\begin{aligned} S_0 &= \left| \int_0^{T_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| \\ &\leq \left| \int_0^{T_{\lambda_n}} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| + \left| \int_{T_{\lambda_n}}^{T_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| \\ &\leq [T_\lambda \left|\lambda_n - \lambda\right| + \left|T_\lambda - T_{\lambda_n}\right| (\lambda_n + |\lambda - \lambda_n|) \operatorname{diam} U] \max_{j=1,\dots,N} \|x_j\|_{L_1} < \frac{\varepsilon}{4k} \ . \end{aligned}$$

Now consider S_1. By definition

$$u_\lambda(t) = \lambda u(t - T_\lambda) \text{ for } t \in [T_\lambda, 2T_\lambda],$$

and

$$u_{\lambda_n}(t) = \begin{cases} \lambda_n u(t - T_{\lambda_n}) & \text{for } t \in [T_{\lambda_n}, 2T_{\lambda_n}], \\ \lambda_n u(t - 2T_{\lambda_n}) & \text{for } t \in [2T_{\lambda_n}, 3T_{\lambda_n}]. \end{cases}$$

For n large enough, one has $T_\lambda < 2T_{\lambda_n}$. Thus

$$\begin{aligned} S_1 &= \left| \int_{T_\lambda}^{2T_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| \\ &\leq \left| \int_{T_\lambda}^{2T_{\lambda_n}} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| + \left| \int_{2T_{\lambda_n}}^{2T_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| \\ &\leq \left| \int_{T_\lambda}^{2T_{\lambda_n}} \langle \lambda_n u(t - T_{\lambda_n}) - \lambda u(t - T_\lambda), x_j(t) \rangle dt \right| \\ &\qquad + 2|T_\lambda - T_{\lambda_n}| \, (\lambda_n + |\lambda - \lambda_n|) \operatorname{diam} U \max_{j=1,\dots k} \|x_j\|_{L_1}. \end{aligned}$$

For n large enough, the first summand can be made less than $\varepsilon/(8k)$ since the shift in $\mathcal{U}$ is continuous (see [1, Lemma 4.2.4]); and, as in the case of S_0, we can assume

$$2|T_\lambda - T_{\lambda_n}| \, (\lambda_n + |\lambda - \lambda_n|) \operatorname{diam} U \max_{j=1,\dots,N} \|x_j\|_{L_1} < \frac{\varepsilon}{8k}.$$

Hence, one has $S_1 < \varepsilon/(4k)$.

Proceeding analogously for all summands S_i, $i = -k, \dots, k-1$, we see that for n large enough $S_i < \varepsilon/(4k)$. Thus

$$\left| \int_{-kT_\lambda}^{kT_\lambda} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| \leq \sum_{i=-k}^{k-1} S_i \leq \frac{2k\varepsilon}{4k} = \varepsilon/2.$$

We have proved that, for $j = 1, ..., N$ and n large enough,

$$\left| \int_{\mathbb{R}} \langle u_{\lambda_n}(t) - u_\lambda(t), x_j(t) \rangle dt \right| < \varepsilon.$$

This implies that, for n large enough, u_{λ_n} belongs to the neighborhood W. □

We also need the following fact which can be proved by standard arguments.

Lemma 6.9. *The set of continuous functions is dense in* $\mathcal{U}^\rho$.

The next fact is crucial for the construction of the homotopy between the orbits that wind n times around γ and $[T, 0, x]^n$. We shall make use of the following parametrized version of the Implicit Function Theorem:

Theorem 6.10. *Let $\mathcal{T}$, X and Y be Banach spaces and let $\mathcal{F}$ be a topological space. For any $u \in \mathcal{F}$ let $\Psi_u : \mathcal{T} \times X \to Y$ be C^1 and let $(t, x, u) \mapsto \Psi_u(t, x)$ and $(t, x, u) \mapsto \Psi'_u(t, x)$ be continuous. Assume in addition that:*

1. *there exist $x_0 \in X$ such that $\Psi_u(0, x_0) = 0$ for all $u \in \mathcal{F}$;*
2. *there exists $\delta > 0$ such that for every $u \in \mathcal{F}$, $D_1\Psi_u(0, x_0)$ is invertible, and*
$$\left\| \big(D_1\Psi_u(0, x_0) \big)^{-1} \right\| \le \delta \quad \text{for any } u \in \mathcal{F},$$
3. *it holds*
$$\lim_{(t,x)\to(0,x_0)} \left\| \Psi'_u(t, x) - \Psi'_u(0, x_0) \right\| = 0$$
uniformly in u.

Then, there exist an open neighborhood W of x_0 in X and a (unique) C^1 function $\tau_u : W \to Y$ such that $\tau_u(x_0) = 0$ and $\Psi_u(\tau_u(x), x) = 0$, for any $u \in \mathcal{F}$.

Lemma 6.11. *Assume that the T-periodic orbit $\gamma = \varphi([0, T], x_0, 0)$ is attracting, and let S be a local transversal section for γ through x_0. Then there exists $\rho > 0$ and a neighborhood V of x_0 such that $P(\cdot, u)$ is a contraction, uniformly for $u \in \mathcal{U}^\rho$.*

Proof. Without loss of generality we can assume that S lies on the hyperplane $X := \{x^d = 0\} \subset \mathbb{R}^d$. Here and along all this proof the exponent d denotes the d-th component in $\mathbb{R}^d$. Take $\mathcal{T} = \mathbb{R}$, $Y := \mathbb{R}$ and

$$\mathcal{F} := \mathcal{U}^\rho \cap C^1(\mathbb{R}, \mathbb{R}^m)$$

with the L_∞ topology, and define the C^1 function $\Psi_u : \mathcal{T} \times X \to Y$ as

$$\Psi_u(t, y) := \varphi^d(t + t_u, y, u) - y^d,$$

where $t_u = \tau(x_0, u)$. One has $\Psi_u(0, x_0) = 0$. Moreover, reducing ρ if necessary, one can find $\delta > 0$ such that

$$D_1\Psi_u(0, x_0) = f^d\big(\tau(x_0, u), u\big) \geq 1/\delta.$$

Continuous dependence, ensures that also condition 3 in Theorem 6.10 is satisfied. Therefore we get the existence of a neighborhood V of x_0 in S and of a C^1 function $\tau_u : V \to \mathbb{R}$ such that $\Psi_u\big(\tau_u(x), x\big) = 0$ for every $x \in V$.

Clearly, if ρ is small enough and $x \in V$, then the time $\tau(x, u)$ for the Poincaré map coincides with $\tau_u(x) + t_u$. Thus $D_1\tau(x, u)$ is well defined and the map

$$(x, u) \mapsto D_1P(x, u) = D_1\varphi\big(\tau(x, u), x, u\big)D_1\tau(x, u) + D_2\varphi\big(\tau(x, u), x, u\big)$$

is continuous. Since γ is attracting, the eigenvalues of $D_1P(x_0, 0)$ are strictly smaller than one in modulus. Thus there exists a norm on S such that the operator $D_1P(x_0, 0)$ has norm smaller than one. By continuity and restricting V and ρ if necessary, we can assume that the same is true for $D_1P(x, u)$ for every $x \in V$ and $u \in \mathcal{U}^\rho \cap C^1(\mathbb{R}, \mathbb{R}^m)$. Whence it follows that $P(\cdot, u)$ is a contraction with constant

$$k = \sup_{(\xi,\mu)\in V\times\mathcal{U}^\rho} \|D_1P(\xi, \mu)\| < 1.$$

Let us show that $P(\cdot, u)$ remains a k-contraction when u is a general (not necessarily continuous) element of $\mathcal{U}^\rho$. Since the C^1 functions are dense in $\mathcal{U}^\rho$ in the weak* topology, there is a sequence $\{u_n\}$ of C^1 functions in $\mathcal{U}^\rho$ converging to u_0 in the weak* topology. Take x and y in V, by Proposition 6.7 we know that P is continuous when $\mathcal{U}^\rho$ is endowed with the weak* topology. Therefore, for $\varepsilon > 0$, one has

$$|P(x, u) - P(x, u_n)| + |P(y, u) - P(y, u_n)| < \varepsilon,$$

for n sufficiently large. Therefore

$$\begin{aligned} |P(x, u) - P(y, u)| &\leq |P(x, u) - P(x, u_n)| + |P(x, u_n) - P(y, u_n)| \\ &\qquad + |P(y, u_n) - P(y, u)| \\ &\leq k|x - y| + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the assertion. $\square$

Proposition 6.12. *Assume that the T-periodic orbit $\gamma = \varphi([0, T], x_0, 0)$ is attracting, and let S be a local transversal section for γ through x_0. Then there exists $\rho > 0$ and a neighborhood V of x_0 such that for every $n \in \mathbb{N}$ and every $(x, u) \in V \times \mathcal{U}^\rho$ the map $(x, u) \to P^n(x, u)$ is well defined.*

Moreover, for every $u \in \mathcal{U}^\rho$, there exists a $T_u > 0$ and a unique T_u-periodic solution $\varphi(\cdot, x_u, [u]_{T_u})$ winding n times around γ, and the functions $u \mapsto T_u$ and $u \mapsto x_u$ are continuous.

Proof. In Lemma 6.11 we proved that for $u \in \mathcal{U}^\rho$, with ρ sufficiently small, $P(\cdot, u)$ is a contraction on cl V. Consequently, we can assume that $P\big(\text{cl } V, u\big) \subset \text{cl } V$ for any $u \in \mathcal{U}^\rho$. In particular $P^n(\cdot, u)$ is well-defined for any $n \in \mathbb{N}$ and $u \in \mathcal{U}^\rho$. Notice also that for every $n \in \mathbb{N}$ and $u \in \mathcal{U}^\rho$ also $P^n(\cdot, u)$ is a contraction. Therefore, given n and u as above, there exist a unique fixed point x_u for $P^n(\cdot, u)$ in N which depends continuously on u. Define T_u as the time needed for $\varphi(\cdot, x_u, u)$ to reach x_u after winding n times around γ. Continuous dependence of T_u on u follows from continuity of the first return time as shown in Proposition 6.7. □

Notice that in $P^n(x, u)$ the control u restricted to $[0, T_u]$ is applied n times. We are finally in a position to prove the claim we made at the beginning of this section.

Theorem 6.13. *Assume that the uncontrolled system has an attracting T-periodic solution $\varphi(\cdot, x_0, 0)$ with $T > 0$, and that the controllability condition (3.1) is satisfied. Then, when ρ is small enough, the dynamic index $\mathcal{I}(D^\rho)$ of the control set D^ρ containing $\gamma := \varphi([0, T], x_0, 0)$ is isomorphic to $\mathbb{N}$.*

Proof. Let $N = \text{cl } V$ be the compact neighborhood of x_0 found in the proof of Proposition 6.12 above. Consider a T_1-periodic orbit $\varphi(\cdot, x, u)$ with $x \in N$, $u \in \mathcal{U}^{\rho'}$ for some $0 < \rho' < \rho$ and u piecewise constant. There exists n such that $\varphi(T_1, x, u) = P^n(x, u)$. By Proposition 6.12, there exist $T_\lambda > 0$ and a unique T_λ-periodic solution $\varphi(\cdot, x_\lambda, [\lambda u]_{T_\lambda})$ winding n times around γ. By Proposition 6.8 the map $\lambda \mapsto u_\lambda := [\lambda u]_{T_\lambda}$ is continuous. Hence, again by Proposition 6.12, it follows that T_λ and x_λ depend continuously on λ. In particular, $T_0 = nT$. Since, by Proposition 3.6, $([v]_{T_\lambda}, x_\lambda)$ is a strong inner pair for each λ, this yields the desired homotopy between (T_1, u_1, x_1) and $(T_0, 0, x_0)$. □

We conclude the paper with a remark showing that the dynamic index allows us to distinguish control sets around an attracting periodic orbit as above from control sets around a homoclinic orbit.

Remark 6.14. *Suppose that the uncontrolled system has a homoclinic orbit given by*

$$\varphi(t, x_1, u_1),\ t \in \mathbb{R},\ \text{with} \lim_{t \to \pm\infty} \varphi(t, x_1, u_1) = x_0,$$

where x_0 is an equilibrium of the uncontrolled system. If the controllability condition (3.1) holds for all points in $\gamma := \{x_0\} \cup \{\varphi(t, x_1, u_1),\ t \in \mathbb{R}\}$ and this is a chain recurrent component of the uncontrolled system, then for every $\rho > 0$ there is a control set D^ρ containing this set in its interior and

$$\bigcap_{\rho > 0} D^\rho = \gamma;$$

see Corollary 4.7.6 in [1]. For any small ρ, the index $\mathcal{I}(D^\rho)$ contains an element $[T, x_0, 0]$ which is idempotent, i.e., $[T, x_0, 0]^2 = [T, x_0, 0]$. Hence $\mathcal{I}(D^\rho)$ is not isomorphic to $\mathbb{N}$.

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