CONTROL SYSTEMS WITH ALMOST PERIODIC EXCITATIONS

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Abstract. For control systems described by ordinary differential equations subject to almost periodic excitations the controllability properties depend on the specific excitation. Here these properties and, in particular, control sets and chain control sets are discussed for all excitations in the closure of all time shifts of a given almost periodic function. Then relations between heteroclinic orbits of an uncontrolled and unperturbed system and controllability for small control ranges and small perturbations are studied using Melnikov's method. Finally, a system with two-well potential is studied in detail.

Key words. Nonautonomous control systems, almost periodicity, control sets, Melnikov method

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1. Introduction. This paper analyzes controllability properties of control systems which are subject to almost periodic excitations. More precisely, we consider

$$\dot{x}(t) = f\big(x(t), z(t), u(t)\big),\ u \in \mathcal{U}, \tag{1.1}$$

in an open set $M \subset \mathbb{R}^d$ with admissible controls in $\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U$ for all $t \in \mathbb{R}\}$ and control range $U \subset \mathbb{R}^m$. We assume that z is an almost periodic function with values in a compact subset $Z \subset \mathbb{R}^k$. In particular, this includes periodic excitations and excitations with several incommensurable periods.

Instead of analyzing the behavior of system (1.1) for a single almost periodic excitation, we allow time shifts of z and, more generally, all excitations in the set $\mathcal{Z}$ of continuous functions which can uniformly be approximated by shifts of z (again, all elements of $\mathcal{Z}$ are almost periodic). Observe that the trajectories of (1.1) are determined by the initial states $x = x(0) \in M$, the excitation $z \in \mathcal{Z}$, and the control function $u : \mathbb{R} \to \mathbb{R}^m$.

There are various ways to look at this system:

(i) as a control system in M with states $x \in M$;

(ii) as a control system in $M \times \mathcal{Z}$ with extended states $(x, z) \in M \times \mathcal{Z}$;

(iii) as a dynamical system in $M \times \mathcal{Z} \times \mathcal{U}$ with states $(x, z, u) \in M \times \mathcal{Z} \times \mathcal{U}$.

Observe that the control system in (i) is nonautonomous; the evolution of the states x is only determined, if, in addition to the control function $u \in \mathcal{U}$, also the phase of the almost periodic function z is known. Hence here we have to distinguish between an analysis for fixed excitation $z \in \mathcal{Z}$ and the projections to M. In (ii), we can sometimes, if the almost periodic function is a solution of a differential equation on a compact manifold Z (e. g. if Z is a k-Torus) replace $\mathcal{Z}$ by Z. Here, however, exact controllability properties in the extended state space $M \times Z$ can only hold in the very special case of a periodic function z. Furthermore, the dimension of the state space of the control system is increased by k, which makes a global numerical analysis much more difficult. The formulation (iii) results in a continuous dynamical system (a control flow) provided that the system is control affine and the control range U is compact and convex. The analysis of this dynamical system (including time shifts on $\mathcal{Z}$ and on $\mathcal{U}$) may yield structural insights and, in particular, sheds light on subsets of complete controllability, i. e. control sets. In the present paper, we will analyze system (1.1) employing all three points of view above.

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Note that for T-periodically excited control systems, controllability properties in the extended state space (where also the phase in $\mathbb{R}/T\mathbb{Z}$ is part of the state) can to a large extent be characterized by a Poincaré section, i.e. the intersection with a fiber over a fixed phase (compare Gayer [8]). We will generalize some of these results. Using methods from ergodic theory, controllability properties of nonautonomous linear control systems have also been discussed by Johnson and Nerurkar [10]. Many further results in this direction have been obtained, in particular in connection with associated Riccati equations. For a different line of research, see San Martin and Patrao [16], who study control sets and chain control sets for semi-dynamical systems on fiber bundles (related to the third interpretation above of system (1.1)).

The main topic of this paper are the relations between hetero- or homoclinic orbits of an uncontrolled and unperturbed system and controllability for small control ranges. Here Melnikov's method plays an important role. In the case of a periodic excitation this was discussed from a numerical point of view in Colonius, Kreuzer, Marquardt and Sichermann [4]. In the present paper a characterization in the general almost periodic case will be given (the result is also new in the periodic case). Melnikov's method for such differential equations was, in particular, developed by Palmer [15], Scheurle [18] and Meyer and Sell [14]. Our paper is closer to the spirit of the latter reference, since we consider the hull of an almost periodic excitation. We would like to point out that we do not really need the strength of Melnikov's result here; existence of a chaotic set is not in our center of interest. Instead intersections of stable and unstable manifolds are relevant here. Note that basic references for almost periodic differential equations include Fink [7] and Levitan and Zhikov [12]; a nice discussion of almost periodic and quasi-periodic functions can also be found in § II.1 of [14], together with further references.

The paper is organized as follows: After preliminaries in § 2, we analyze chain control sets in § 3. Section 4 introduces control sets and presents relations to chain control sets and to almost periodic solutions of the uncontrolled system. Section 5 presents relevant results on almost periodic perturbations of hyperbolic equilibria and Melnikov's method. These results are essentially known in the literature (see Palmer [15], Scheurle [18], and also Meyer and Sell [14]). However, for the reader's convenience, we have included some arguments from the proofs. This is used in § 6 to study the relation between heteroclinic orbits of an unperturbed system and controllability for small control ranges. In the final section 7 we discuss a second order system with M-potential modelling ship roll motion. Note that here the controls u are interpreted as time-dependent perturbations.

2. Preliminaries. Consider the control system (1.1)

$$\dot{x}(t) = f\big(x(t), z(t), u(t)\big),\ u \in \mathcal{U},$$

in an open set $M \subset \mathbb{R}^d$ with admissible controls in $\mathcal{U}$ and assume that z is an almost periodic function. That is, we assume (compare e.g. Scheurle [18], Definition 2.6) that $z : \mathbb{R} \to \mathbb{R}^k$ is continuous and that for every $\varepsilon > 0$ there exists an $l = l(\varepsilon) > 0$ such that in any interval of length l there is a so-called *translation number* τ such that

$$\|z(t+\tau) - z(t)\| < \varepsilon \text{ for all } t \in \mathbb{R}.$$

Define θ as the time shift $(\theta_t z)(s) := z(t+s), s,t \in \mathbb{R}$. Let $\mathcal{Z}$ be the closure in the space $C_b(\mathbb{R}, \mathbb{R}^k)$ of bounded continuous functions of the shifts of an almost periodic function. Then $\mathcal{Z}$ is a minimal set, i.e. every trajectory is dense in $\mathcal{Z}$. Observe that for $z \in \mathcal{Z}$ it holds that $z(t) = (\theta_t z)(0)$. Assuming global existence and uniqueness, we denote by $\varphi(t, t_0, x, z, u)$ the solution of the initial value problem

$$\dot{x}(t) = f\big(x(t), z(t), u(t)\big), \ x(t_0) = x; \tag{2.1}$$

if $t_0 = 0$, we often omit this argument. The solution map of the coupled system is denoted by

$$\psi(t, x, z, u) = \big(\varphi(t, x, z, u), \theta_t z\big).$$

We assume that the set of admissible controls is given by

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), \ u(t) \in U \text{ for almost all } t\},$$

where $U \subset \mathbb{R}^m$. If we denote also the time shift on $\mathcal{U}$ by θ_t, we obtain the cocycle property

$$\varphi(t+s, x, z, u) = \varphi\big(s, \varphi(t, x, z, u), \theta_t z, \theta_t u\big), t, s \in \mathbb{R}.$$

Finally, the maps

$$\Phi_t : M \times \mathcal{Z} \times \mathcal{U} \to M \times \mathcal{Z} \times \mathcal{U}, \ \Phi_t(x, z, u) = \big(\psi(t, x, z, u), \theta_t u\big), t \in \mathbb{R},$$

define a continuous flow, the *control flow*, provided that $U \subset \mathbb{R}^m$ is convex and compact and

$$f(x, z, u) = f_0(x, z) + \sum_{i=1}^{m} u_i f_i(x, z)$$

with C^1-functions $f_i : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$; here $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ is endowed with the weak* topology. This follows by a minor extension of Proposition 4.1.1 in [3]. Throughout this paper, we assume that these conditions guaranteeing continuity of the control flow are satisfied. For convenience, we also assume that $0 \in U$, and we call the corresponding differential equation with $u \equiv 0$ the uncontrolled system.

For periodic and for quasi-periodic excitations we may be able to replace $\mathcal{Z}$ by a finite dimensional state space Z.

EXAMPLE 2.1. *For a smooth periodic excitation let $\zeta : \mathbb{S}^1 \to \mathbb{S}^1 =: Z$ be the solution map $\zeta_t z_0 = \omega(t + z_0), t \in \mathbb{R}$, of $\dot{z} = \omega, \ z(0) = z_0$; here $\omega > 0$ is the frequency and (2.1) may be written as*

$$\dot{x}(t) = f\big(x(t), \zeta_t(z_0), u(t)\big), \ x(0) = x_0.$$

For a quasi-periodic excitation, let $\zeta : \mathbb{S}^k \to \mathbb{S}^k =: Z$ be the solution map $\zeta_t z_0 = \big(\omega_1(t + z_{0,1}), \dots, \omega_k(t + z_{0,k})\big), t \in \mathbb{R}$, of

$$\dot{z}_1 = \omega_1, \dot{z}_2 = \omega_2, \dots, \dot{z}_k = \omega_k,$$

with initial condition $z(0) = (z_{0,1}, \dots, z_{0,k})$. Here $\omega_1, \dots, \omega_k > 0$ are the frequencies and we assume that they are rationally independent, i.e. if $q_i \in \mathbb{Q}$ with $q_1\omega_1 + \cdots + q_k\omega_k = 0$, then $q_i = 0$ for all i. Again (2.1) may be written as above.

3. Chain Control Sets. In this section we define and characterize chain control sets relative to a subset of the state space working in the general almost periodic case.

It will be convenient to write for a subset $A \subset M \times \mathcal{Z}$ the section with a fiber over $z \in \mathcal{Z}$ as

$$A_z := A \cap (M \times \{z\}).$$

Hence $A = \bigcup_{z\in\mathcal{Z}} A_z$. Where convenient, we identify A_z and $\{x \in M,\ (x,z) \in A_z\}$.

A controlled (ε, T)-chain along $z \in \mathcal{Z}$ is given by $T_0, \dots, T_{n-1} \geq T$, controls $u_0, \dots, u_{n-1} \in \mathcal{U}$ and points $x_0, \dots, x_n \in M$ with

$$\mathrm{d}\big(\varphi(T_j, x_j, \theta_{T_0+\cdots+T_{j-1}} z, u_j), x_{j+1}\big) < \varepsilon \text{ for all } j = 0, \dots, n-1.$$

Definition 3.1. *A chain control set relative to a closed set $Q \subset M \times \mathcal{Z}$ is a nonvoid maximal set $E \subset M \times \mathcal{Z}$ such that*

(i) *for all $(x,z), (y,w) \in E$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q along z from x to (y,w), i. e. $x_0 = x, x_n = y$ and $\mathrm{d}(\theta_{T_0+\cdots+T_{n-1}} z, w) < \varepsilon$, and*

$$\psi(t, x_j, \theta_{T_0+\cdots+T_{j-1}} z, u_j) \in Q \text{ for all } t \in [0, T_j] \text{ and for all } j; \tag{3.1}$$

(ii) *for all $(x,z) \in E$ there is $u \in \mathcal{U}$ with $\psi(t,x,z,u) \in E$ for all $t \in \mathbb{R}$.*

The condition in (3.1) can be written as

$$\varphi(t, x_j, \theta_{T_0+\cdots+T_{j-1}} z, u_j) \in Q_{\theta_t z_j}.$$

Note that the three components x, z and u are treated in different ways: jumps are allowed in x, approximate reachability is required for z and no condition on the controls is imposed. Observe that also Meyer and Sell [14] do not allow jumps in the almost periodic base flow. It is easy to show that chain control sets are closed.

Next we discuss the behavior for fixed 'phases' $z \in \mathcal{Z}$ by looking at the fibers of a chain control set.

Lemma 3.2. *Suppose that E is a chain control set relative to Q. Then the fibers $E_z := E \cap Q_z, z \in \mathcal{Z}$, satisfy the following properties:*

(i) *For every $z \in \mathcal{Z}$ and all $x, y \in E_z$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q from x along z to (y, z).*

(ii) *For every $z \in \mathcal{Z}$ and every $x \in E_z$ there exists a control $u \in \mathcal{U}$ such that*

$$\varphi(t, x, z, u) \in E_{\theta_t z} \text{ for all } t \in \mathbb{R}.$$

(iii) *If $x_n \in E_{z_n}$ with $(x_n, z_n) \to (x,z) \in M \times \mathcal{Z}$, then $x \in E_z$.*

Proof. Condition (iii) follows from closedness of E, (i) and (ii) are obvious. ∎

Remark 3.3. *In condition (ii) of Lemma 3.2 one does not have that a trajectory exists which after an appropriate time comes back to E_z (as for periodic excitations, where one comes back into the same fiber after the period). In the general almost periodic case the trajectory will never come back to the same fiber. Instead, the weaker property formulated in (ii) holds together with condition (iii), which locally connects different fibers and is an upper semi-continuity property of $z \mapsto E_z$.*

Next we discuss if the properties formulated in Lemma 3.2 characterize chain control sets.

Lemma 3.4. *Suppose Q is compact and that $E^z \subset Q_z, z \in \mathcal{Z}$, is a family of sets satisfying conditions (i), (ii), and (iii) in Lemma 3.2. Assume that*

$$E := \bigcup_{z\in\mathcal{Z}} E^z \subset \operatorname{int} Q.$$

Then E satisfies properties (i) and (ii) of chain control sets in Definition 3.1.

Proof. Let $(x,z),(y,w) \in E$ and $\varepsilon, T > 0$. Then $\omega(z) = \mathcal{Z}$ and there is a control $u \in \mathcal{U}$ such that $\psi(t,x,z,u) \in E$ for all $t \in \mathbb{R}$. In particular, this proves property (ii) of chain control sets. Furthermore, there are $S_k > T$ such that for $z_k := \theta_{S_k} z$ one has $\mathrm{d}(z_k, w) < 1/k$ and clearly $y_k := \varphi(S_k, x, z, u) \in E_{z_k}$. By compactness of Q we may assume that (y_k, z_k) converges to some $(y_0, w) \in Q$. By property (iii) it follows that $y_0 \in E_w$. By property (i) there is a controlled $(\varepsilon/2, T)$-chain in Q from y_0 along w to (y,w) satisfying $x_0 = y_0, x_n = y$ and $\mathrm{d}(\theta_{T_0+\cdots+T_{n-1}} w, w) < \varepsilon/2$, and

$$\psi(t, x_j, \theta_{T_0+\cdots+T_{j-1}} w, u_j) \in Q \text{ for all } t \in [0, T_j] \text{ and for all } j.$$

Introducing, if necessary, trivial jumps, we may assume that $T_j \in [T, 2T]$ for all j. By uniform continuity, there is $\delta > 0$ such that for all $x \in Q$ and all $u \in \mathcal{U}$

$$\mathrm{d}(z, z') < \delta \text{ implies } \mathrm{d}\big(\varphi(t,x,z,u), \varphi(t,x,z',u)\big) < \varepsilon/2, \ t \in [0, 2T]. \tag{3.2}$$

Choose k large enough such that

$$\mathrm{d}(z_k, w) = \mathrm{d}(\theta_{S_k} z, w) := \sup_{t \in \mathbb{R}} \|z(S_k + t) - w(t)\| < \delta \text{ and } \mathrm{d}(\varphi(S_k, x, z, u), y_0) < \varepsilon.$$

Hence for all j

$$\begin{aligned}
&\mathrm{d}\big(\varphi(T_j, x_j, \theta_{\hat{}S_k+T_0+\cdots+T_{j-1}} z, u_j), x_{j+1}\big) \\
&\le \mathrm{d}\big(\varphi(T_j, x_j, \theta_{S_k+T_0+\cdots+T_{j-1}} z, u_j), \varphi(T_j, x_j, \theta_{T_0+\cdots+T_{j-1}} w, u_j)\big) \\
&\quad + \mathrm{d}\big(\varphi(T_j, x_j, \theta_{T_0+\cdots+T_{j-1}} w, u_j), x_{j+1}\big) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

This shows that there is a controlled (ε, T)-chain from x along z to (y,w). Since by assumption $E \subset \operatorname{int} Q$ and by (3.2) this (ε, T)-chain is ε-close to an (ε, T)-chain in Q, we may choose $\varepsilon > 0$ small enough, such that this is a chain in Q. This proves property (i) of chain control sets. □

The following result clarifies the relations between chain control sets and their fibers.

PROPOSITION 3.5. *Consider system (1.1) in a closed subset $Q \subset M \times \mathcal{Z}$.*

(i) *Suppose that Q is compact and let $E^z \subset Q_z, z \in \mathcal{Z}$, be a maximal family of sets satisfying conditions (i)–(iii) in Lemma 3.2. If $E := \bigcup_{z\in\mathcal{Z}} E^z \subset \operatorname{int} Q$, then E is a chain control set.*

(ii) *Let E be a chain control set. Then the fibers $E_z, z \in \mathcal{Z}$, are contained in a maximal family $\tilde{E}^z \subset Q_z, z \in \mathcal{Z}$, of sets satisfying conditions (i)–(iii) in Lemma 3.2. If $\tilde{E} := \bigcup_{z\in\mathcal{Z}} \tilde{E}^z \subset \operatorname{int} Q$, then $E = \tilde{E}$.*

Proof. It only remains to discuss the maximality properties.

(i) The union E satisfies properties (i) and (ii) of chain control sets, since for $\varepsilon < \operatorname{dist}(E, \partial Q)$ the controlled (ε, T)-chains are in Q. Hence E is contained in the union $\tilde{E}$ of all sets containing E and satisfying these properties. Then $\tilde{E}$ is a chain control set and its fibers $\tilde{E}_z$ contain the sets E^z and satisfy properties (i)–(iii) in Lemma 3.2. By maximality, it follows that $E = \tilde{E}$.

(ii) Let E be a chain control set. Then the fibers E_z satisfy properties (i)–(iii) in Lemma 3.2. Clearly, the family $E_z, z \in \mathcal{Z}$, is contained in a maximal family $\tilde{E}^z, z \in \mathcal{Z}$, with these properties. If $\tilde{E} \subset \operatorname{int} Q$, the first assertion shows that $\tilde{E}$ is a chain control set and hence $E = \tilde{E}$. □

It is of great interest to see if the behavior in a single fiber determines chain control sets. In the periodic case, one can reconstruct chain control sets from their intersection with a fiber. More precisely, the following is a minor modification of Gayer [8], Taubert [21, Satz 2.2.5].

PROPOSITION 3.6. *Assume that in system (1.1) the set $\mathcal{Z}$ consists of the shifts of a T-periodic function and write $Z := \mathbb{R}/T\mathbb{Z}$. Let $Q \subset M \times Z$ be closed and pick $z_0 \in Z$. Suppose that $E^{z_0} \subset Q_{z_0}$ is a maximal set such that*

(i) *for all $x, y \in E^{z_0}$ and all $\varepsilon > 0$ there are $(x_j, z_j) \in Q \times Z$ and controls $u_j \in \mathcal{U}$ with $(x_0, z_0) = (x, z_0), (x_n, z_n) = (y, z_0)$ such that for all $j = 0, \ldots, n-1$*

$$\mathrm{d}\big(\psi(T, (x_j, z_j, u_j)), (x_{j+1}, z_{j+1})\big) < \varepsilon \text{ and } \psi(t, x_j, z_j, u_j) \in Q \text{ for } t \in [0, T],$$

(ii) *for all $x \in E^{z_0}$ there is $u \in \mathcal{U}$ with $\varphi(T, x, z_0, u), \varphi(-T, x, z_0, u) \in E^{z_0}$.*

Then the set

$$E := \left\{ (x, z) \in M \times Z, \quad \begin{array}{c} \text{there are } x_0 \in E^{z_0}, u \in \mathcal{U}, t \in [0, T) \text{ with} \\ (x, z) = \psi(t, x_0, z_0, u) \text{ and } \varphi(T, x_0, z_0, u) \in E^{z_0} \end{array} \right\}$$

is a chain control set relative to Q.

Conversely, for a chain control set $E \subset Q \times Z$, every fiber $E_{z_0}, z_0 \in Z$, is maximal with properties (i) and (ii).

In order to derive an analogous result in the almost periodic case, we have to modify property (ii) in Proposition 3.6, since it cannot be satisfied.

THEOREM 3.7. *Consider system (1.1) and assume that $Q \subset M \times \mathcal{Z}$ is compact. For some $z_0 \in \mathcal{Z}$ let $E^{z_0} \subset Q \times \{z_0\}$ be a nonvoid maximal set such that for all $x_0, y_0 \in E^{z_0}$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q from x_0 along z_0 to (y_0, z_0).*

Then the set

$$E := \mathrm{cl} \left\{ (x, z) \in M \times Z, \quad \begin{array}{c} \text{for all } \varepsilon, T > 0 \text{ there are } x_0, y_0 \in E^{z_0} \text{ and controlled} \\ (\varepsilon, T)\text{-chains in } Q \text{ from } x_0 \text{ along } z_0 \text{ to } (y_0, z_0) \text{ such} \\ \text{that } (x, z) = \psi(t, x_j, \theta_{T_0 + \cdots + T_{j-1}} z_0, u_j) \text{ for some } j \\ \text{and } t \in [0, T_j] \end{array} \right\}.$$

is a chain control set relative to Q.

Proof. Consider the fibers $E_z, z \in \mathcal{Z}$, of E. By closedness of E it is clear that $x_n \in E_{z_n}$ with $(x_n, z_n) \to (x, z) \in M \times \mathcal{Z}$ implies $x \in E_z$. Since E^{z_0} is nonvoid and E is contained in the compact set Q, hence also compact, every fiber E_z of E is nonvoid.

Let $(x, z), (y, w) \in E$ and $\varepsilon, T > 0$. Then there exists a controlled (ε, T)-chain in Q from x along z to (y, w). This follows for elements on controlled chains from E^{z_0} to E^{z_0} by concatenating appropriate chains and using continuity (in order to guarantee $T_j \geq T$). Again by continuity, this also follows for elements in the closure of the set of these points. It remains to show that for every $z \in \mathcal{Z}$ and every $x \in E_z$ there exists a control $u \in \mathcal{U}$ such that

$$\varphi(t, x, z, u) \in E_{\theta_t z} \text{ for all } t \in \mathbb{R}.$$

For $(x, z) \in E$ and $k \in \mathbb{N}$ choose controlled $(1/k, T)$-chains ζ^k from x along z to (x, z) with controls $u_j^k \in \mathcal{U}$. Then a subsequence of u_0^k converges to some control $v_0 \in \mathcal{U}$ and, by continuity,

$$\varphi(T, x, z, u_0^k) \to \varphi(T, x, z, v_0) \text{ for } k \to \infty.$$

Then one finds that $\varphi(T,x,z,v_0) \in E_{\theta_T z}$, since E is closed. Iterating this procedure, one constructs a control $u^+ \in \mathcal{U}$ with $\varphi(t,x,u^+) \in E$ for all $t \geq 0$. For negative times, consider the last members of the chains ζ^k. We may assume that the corresponding controls $u^k_{n_k}$ converge to a control $v \in \mathcal{U}$ and, by definition,

$$\psi(T_{n_k}, x_{n_k}, \theta_{T_0^k+\dots+T_{n_k}^k} z, u^k_{n_k}) \to (x,z) \text{ for } k \to \infty.$$

By continuity, we may assume that $T^k_{n_k} \in [T, 2T]$, and hence that $T^k_{n_k} \to S \geq T$. Then $\theta_{T^k_{n_k}} u^k_{n_k} \to \theta_S v$ and continuity implies

$$\begin{aligned}
&\psi(T_{n_k} - T, x_{n_k}, \theta_{T_0^k+\dots+T_{n_k}^k} z, u^k_{n_k}) \\
&= \psi(-T, \psi(T_{n_k}, x_{n_k}, \theta_{T_0^k+\dots+T_{n_k}^k} z, u^k_{n_k}), \theta_{T^k_{n_k}} u^k_{n_k}) \\
&\to \psi(-T, x, z, \theta_S v) \text{ for } k \to \infty.
\end{aligned}$$

With $u^- := \theta_S v$ one finds that $\varphi(-T,x,z,v_1) \in E_{\theta_T z}$, since E is closed. Iterating this procedure, one constructs a control $u^- \in \mathcal{U}$ with $\varphi(t,x,z,u^-) \in E$ for all $t \leq 0$. Combining u^+ and u^- the desired control u is found. $\square$

REMARK 3.8. *Theorem 3.7 shows that, up to closure, one can find chain control sets by looking at a single fiber, i. e. a single almost periodic excitation. This significantly simplifies numerical computations, since only one almost periodic excitation $z(t), t \geq 0$, has to be considered. Then the resulting sets must be considered for those times T where z and $\theta_T z$ are close. In the quasi-periodic case (cp. Example 2.1), one has to look for (large) times t where all $\omega_i t$ are close to zero modulo 2π.*

In addition to chain control sets E, also their projection to M defined as

$$\pi_M E := \{x \in M,\ (x,z) \in E \text{ for some } z \in \mathcal{Z}\}$$

is of interest. Obviously, for all $(x_1, x_2) \in \pi_M E$ there are $z_1, z_2 \in \mathcal{Z}$ such that $(x_1, z_1), (x_2, z_2) \in E$, hence there are controlled (ε, T)-chains from x_1 along z_1 to (x_2, z_2).

4. Controllability and Chain Controllability.

The main aim in this section is to analyze, when an almost periodic solution of the uncontrolled system is contained in the interior of a subset of complete controllability. For this purpose, we ask when a reachable point is contained in the interior of the reachable set and discuss chain controllability. This leads us to control sets and their relation to chain control sets.

Again, consider control system (1.1). For a closed subset $Q \subset M \times \mathcal{Z}$, a point $x \in Q$ and $z \in \mathcal{Z}$ we define the positive and negative orbits along z relative to Q as

$$\begin{aligned}
\mathcal{O}^+(x;z,Q) &:= \{\varphi(t,x,z,u), \text{ with } \psi(s,x,z,u) \in Q, s \in [0,t] \text{ for some } t \geq 0, u \in \mathcal{U}\}, \\
\mathcal{O}^-(x;z,Q) &:= \{\varphi(t,x,z,u), \text{ with } \psi(s,x,z,u) \in Q, s \in [t,0] \text{ for some } t \leq 0, u \in \mathcal{U}\}.
\end{aligned}$$

Observe that $\varphi(t,x,z,u) \in Q_{\theta_t z}$. Analogously $\mathcal{O}^+_t(x;z,Q), \mathcal{O}^-_t(x;z,Q)$ etc. are defined, if we restrict the times accordingly. If $Q = M$, we omit the argument Q.

In addition to chain control sets it is also of interest to discuss control sets, i. e. maximal subsets of approximate controllability.

DEFINITION 4.1. *For a closed subset $Q \subset M \times \mathcal{Z}$ a subset $D \subset Q$ is a control set relative to Q if it is maximal with the following properties:*

(i) *For all $(x,z), (y,w) \in D$ there are $T_n \geq 0, u_n \in \mathcal{U}$ with $\psi(T_n, x, z, u_n) \to (y,w)$ and $\psi(t,x,z,u_n) \in Q$ for $t \in [0, T_n]$.*

(ii) *For every* $z \in \mathcal{Z}$ *and every* $x \in D_z$ *there exists a control* $u \in \mathcal{U}$ *such that*

$$\psi(t, x, z, u) \in D \text{ for all } t \geq 0.$$

In condition (i), it is clear that $T_n \to \infty$, unless the excitation is periodic. Condition (ii) immediately implies that the projection of the control set is dense in $\mathcal{Z}$; the inclusion may be rewritten as $\varphi(t, x, z, u) \in D_{z(t+\cdot)}$ for all $t \geq 0$.

For periodic excitations, one can characterize control sets by looking at the discrete time system defined by the Poincaré map (Gayer [8]). We will show that also in the almost periodic case, it is possible to characterize control sets fiberwise.

LEMMA 4.2. *Suppose that* $D \subset Q$ *is a control set. Then the fibers* $D_z := D \cap Q_z$, $z \in \mathcal{Z}$, *satisfy the following properties:*

(i) *For every* $z \in \mathcal{Z}$ *and all* $x, y \in D_z$ *there are* $T_n \to \infty$ *and* $u_n \in \mathcal{U}$ *with* $\psi(T_n, x, z, u_n) \to (y, z)$ *and* $\psi(t, x, z, u_n) \in Q$ *for all* $t \in [0, T_n]$.

(ii) *For every* $z \in \mathcal{Z}$ *and every* $x \in D_z$ *there exists a control* $u \in \mathcal{U}$ *such that*

$$\varphi(t, x, z, u) \in D_{\theta_t z} \text{ for all } t \geq 0.$$

Proof. This obviously follows from properties (i) and (ii) of control sets. □

The following lemma shows, that the properties in Lemma 4.2 characterize control sets.

LEMMA 4.3. *Suppose* $Q \subset M \times \mathcal{Z}$ *is closed and that* $D^z \subset Q_z$, $z \in \mathcal{Z}$, *is a family of sets satisfying conditions (i) and (ii) in Lemma 4.2 and, additionally,*

(iii) *For every* $(x, z) \in D^z$ *and all* $T_n > 0$ *with* $\theta_{T_n} z \to w \in \mathcal{Z}$ *there are* $y \in M$ *and* $u_n \in \mathcal{U}$ *such that* $\psi(T_n, x, z, u_n) \to (y, w) \in D^w$ *and* $\psi(t, x, z, u_n) \in Q$ *for all* $t \in [0, T_n]$.

Then $D := \bigcup_{z \in \mathcal{Z}} D^z$ *satisfies properties (i) and (ii) of control sets in Definition 4.1.*

Proof. Property (ii) of control sets is clearly satisfied due to property (ii) of the fibers. In order to prove property (i), let $(x, z), (y, w) \in D$. Since $\omega(z) = \mathcal{Z}$ there are $S_k \to \infty$ with $\theta_{S_k} z \to w$. By Property (iii) we may assume that, for some controls $u_k \in \mathcal{U}$ and some $(y_0, w) \in D$

$$\psi(S_k, x, z, u_k) \to (y_0, w) \text{ in } Q. \tag{4.1}$$

By property (i) of the fibers there are $T_n \to \infty$ and $v_n \in \mathcal{U}$ with

$$\psi(T_n, y_0, w, v_n) \to (y, w) \text{ in } Q. \tag{4.2}$$

Let $\varepsilon > 0$ and denote here and in the following the open ε-ball around x by $\mathbf{B}_\varepsilon(x)$. For every $n \in \mathbb{N}$ there is an $\eta_n > 0$ such that

$$\psi\big(T_n, \mathbf{B}_{\eta_n}(y_0, w), v_n\big) \subset \mathbf{B}_{\varepsilon/2}\big(\psi(T_n, y_0, w, v_n)\big) \tag{4.3}$$

due to continuous dependence on initial conditions. Convergence in (4.2) implies that $\psi(T_n, y_0, w, v_n) \in \mathbf{B}_{\varepsilon/2}(y, w)$ for sufficiently large n. Together, this yields

$$\psi\big(T_n, \mathbf{B}_{\eta_n}(y_0, w), v_n\big) \subset \mathbf{B}_\varepsilon(y, w)$$

for n large enough.

By convergence in (4.1), there is a sequence $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\psi(S_{k_n}, x, z, u_{k_n}) \in \mathbf{B}_{\eta_n}(y_0, w).$$

Let $\tilde{T}_n := S_{k_n} + T_n$ and

$$\tilde{u}_n(t) := \begin{cases} u_n(t) & \text{if } t < S_{k_n}, \\ v_n(t - S_{k_n}) & \text{otherwise.} \end{cases}$$

Then inclusion (4.3) implies $\psi(\tilde{T}_n, x, z, \tilde{u}_n) \in \mathbf{B}_\varepsilon(y, w)$ for all $n \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, this implies $\psi(\tilde{T}_n, x, z, \tilde{u}_n) \to (y, w)$. Furthermore $\psi(t, x, z, \tilde{u}_n) \in Q$ for all $t \in [0, \tilde{T}_n]$, $n \in \mathbb{N}$, by construction. □

The following result clarifies the relations between control sets and their fibers.

THEOREM 4.4. *Consider system* (1.1) *in a closed subset* $Q \subset M \times \mathcal{Z}$.

(i) *Let* $D^z \subset Q_z$, $z \in \mathcal{Z}$, *be a maximal family of sets satisfying conditions (i) and (ii) in Lemma 4.2 and condition (iii) in Lemma 4.3. Then* $D := \bigcup_{z \in \mathcal{Z}} D^z$ *is a control set.*

(ii) *Let* D *be a control set. Then the fibers* D_z *form a maximal family of sets satisfying conditions (i) and (ii) in Lemma 4.2*

Proof. By Lemmas 4.2 and 4.3 only maximality has to be shown.

(i) By Lemma 4.3 the set $D := \bigcup_{z \in \mathcal{Z}} D^z$ satisfies the two defining properties of control sets and is thus contained in a control set $\tilde{D}$. The fibers $\tilde{D}_z$, $z \in \mathcal{Z}$, satisfy conditions (i) and (ii) in Lemma 4.2. So by maximality $\tilde{D}_z = D^z$ for every $z \in \mathcal{Z}$, which implies $D = \tilde{D}$.

(ii) By Lemma 4.2 the fibers D_z satisfy conditions (i) and (ii) and are thus contained in a maximal family D^z, $z \in \mathcal{Z}$, of sets satisfying these properties. By Lemma 4.3 the set $\tilde{D} := \bigcup_{z \in \mathcal{Z}} D^z$ is a control set. Clearly $D \subset \tilde{D}$. Maximality implies $D = \tilde{D}$ and so $D_z = D^z$ for all z. □

We note the following simple property of control sets.

PROPOSITION 4.5. *Let* D_1 *and* D_2 *be control sets relative to* Q *and assume that there are* $z \in \mathcal{Z}$, *times* $T_2 > T_1 > 0$, *a point* $x \in D_1^z$, *and a control* $u \in \mathcal{U}$ *such that*

$$\varphi(T_1, x, z, u) \in D_{2, z(T_1 + \cdot)} \text{ and } \varphi(T_1 + T_2, x, z, u) \in D_{1, z(T_1 + T_2 + \cdot)},$$
$$\text{and } \psi(t, x_1, z, u) \in Q \text{ for all } t \in [0, T_1 + T_2].$$

Then $D_1 = D_2$.

Proof. This follows by maximality of D_1, since $D_1 \cup \{\psi(t, x, z, u), t \in [0, T_1 + T_2]\}$ satisfies properties (i) and (ii) of control sets. □

Our next aim is to prove that under an inner-pair condition every almost periodic solution of the uncontrolled equation is contained in the interior of a control set. For a periodic excitation as considered in Example 2.1, the state space $Z = \mathbb{S}^1$ is (trivially) completely controllable. However, already for a quasi-periodic excitation with two noncommensurable (i. e. rationally independent) frequencies ω_1, ω_2, this is no longer true. Hence it does not make sense to consider exact controllability properties in the z-component. This is different in the x-component as shown by the following proposition.

PROPOSITION 4.6. *Let* $\psi(t, x^0, z^0, 0) \in Q, t \in \mathbb{R}$, *be an almost periodic solution of the uncontrolled system and define* $A := \mathrm{cl}\{\psi(t, x^0, z^0, 0), t \in \mathbb{R}\}$. *Assume that there are* $\varepsilon, T > 0$ *such that for every* $(x, z) \in A$

$$\mathbf{B}_\varepsilon\big(\varphi(T, x, z, 0)\big) \subset \mathcal{O}_T^+(x; z, Q).$$

Then for all $(x, z), (y, w) \in A$ *there is* $\tau > 0$ *such that* $\mathbf{B}_{\varepsilon/2}(y) \subset \mathcal{O}_\tau^+(x; z, Q)$ *and for every* $y_0 \in \mathbf{B}_{\varepsilon/2}(y)$ *there are* $\tau_n \geq 0$ *and* $u_n \in \mathcal{U}$ *with* $\varphi(\tau_n, x, z, u_n) = y_0$ *in* Q *and* $\theta_{\tau_n} z \to w$.

Proof. Let $(x,z),(y,w) \in A$. Note that by uniform continuity, there is $\delta > 0$ such that

$$\mathrm{d}\big((x_1,z_1),(x_2,z_2)\big) < \delta \text{ implies } \mathrm{d}\big(\psi(T,x_1,z_1,0),\psi(T,x_2,z_2,0)\big) < \varepsilon/2.$$

By almost periodicity one has $\omega(x,z) = A$, hence there are $S_n \to \infty$ such that $\psi(S_n,x,z,0) \to \psi(-T,y,w,0)$ in $A \subset Q$. Choose n large enough such that for $S_0 := S_n$

$$\mathrm{d}(\psi(-T,y,w,0),\psi(S_0,x,z,0)) < \delta. \tag{4.4}$$

This implies

$$\mathrm{d}((y,w),\psi(S_0+T,x,z,0)) = \mathrm{d}(\psi(T,\psi(-T,y,w,0),0),\psi(T,\psi(S_0,x,z,0),0)) < \varepsilon/2$$

and we conclude for $\varepsilon > 0$, small enough,

$$\begin{aligned}\mathbf{B}_{\varepsilon/2}(y) \subset \mathbf{B}_{\varepsilon}\big(\varphi(S_0+T,x,z,0)\big) &= \mathbf{B}_{\varepsilon}\Big(\varphi\big(T,\varphi(S_0,x,z,0),\theta_T z\big)\Big)\\ &\subset \operatorname{int}\mathcal{O}^+_T\big(\varphi(S_0,x,z,0);\theta_T z,Q\big) \subset \operatorname{int}\mathcal{O}^+_{S_0+T}(x;z,Q).\end{aligned}$$

This yields the first assertion with $\tau = S_0 + T$ and the second assertion follows with $\tau_n := S_n + T$ if we consider $\delta_n \to 0$ in (4.4). ☐

This proposition allows us to show that almost periodic solutions of the uncontrolled system are contained in the interior of control sets. In other words, around an almost periodic solution we have complete controllability along the almost periodic excitations.

THEOREM 4.7. *Let $\psi(t,x^0,z^0,0) \in Q, t \in \mathbb{R}$, be an almost periodic solution of the uncontrolled system and let $A := \mathrm{cl}\{\psi(t,x^0,z^0,0), t \in \mathbb{R}\}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x,z) \in A$*

$$\mathbf{B}_{\varepsilon}(\varphi(T,x,z,0)) \subset \mathcal{O}^+_T(x;z,Q) \text{ and } \mathbf{B}_{\varepsilon}(\varphi(-T,x,z,0)) \subset \mathcal{O}^-_T(x;z,Q). \tag{4.5}$$

Then there exists a control set D such that for every $(x,z) \in A$ one has $x \in \operatorname{int} D^z$.

Proof. It is clear that the set A satisfies properties (i) and (ii) of Definition 4.1. Hence it is contained in a maximal set with these properties, i. e. a control set D. The assertion follows, if we can show that for all $(x,z) \in A$ the neighborhoods $\mathbf{B}_{\varepsilon/2}(x)$ also satisfy these properties. Let $(x,z),(y,w) \in A$. For property (i) it suffices to show that for $x_0 \in \mathbf{B}_{\varepsilon/2}(x), y_0 \in \mathbf{B}_{\varepsilon/2}(y)$ there are $T_n \geq 0$ and $u_n \in \mathcal{U}$ with $\psi(T_n,y_0,w,u_n) \to (x_0,z)$ in Q. Since $\psi(T,x,z,0) \in A$, condition (4.5) implies

$$\mathbf{B}_{\varepsilon/2}(x) \subset \mathcal{O}^-_T\big(\psi(T,x,z,0)\big).$$

Hence for every $(x_0,z) \in \mathbf{B}_{\varepsilon/2}(x) \times \{z\}$ there is a control $u_0 \in \mathcal{U}$ with $\psi(T,x,z,0) = \psi(T,x_0,z,u_0)$. Similarly, $\psi(-T,y,w,0) \in A$ implies

$$\mathbf{B}_{\varepsilon/2}(y) \subset \mathcal{O}^+_T\big(\psi(-T,y,w,0)\big),$$

and hence there is a control $v_0 \in \mathcal{U}$ with $(y_0,w) = \psi(T,\psi(-T,y,w,0),v_0)$.

Since $\psi(T,x,z,0),\psi(-T,y,w,0) \in A$ there are $S_n \geq 0$ and $v_n \in \mathcal{U}$ with

$$\psi\big(S_n,\psi(T,x,z,0),v_n\big) \to \psi(-T,y,w,0) \text{ in } Q.$$

By continuity, this implies

$$\psi\Big(T, \psi\big(S_n, \psi(T, x, z, 0), v_n\big), v_0\Big) \to \psi\big(T, \psi(-T, y, w, 0), v_0\big) = (y_0, w).$$

Define the concatenated controls

$$u_n(t) := \begin{cases} u_0(t) & \text{for} \quad t \in [0, T] \\ v_n(t - T) & \text{for} \quad t \in (T, T + S_n] \\ v_0(t - T - S_n) & \text{for} \quad t \in (T + S_n, 2T + S_n]. \end{cases}$$

Then, with $T_n := 2T + S_n$,

$$\begin{aligned} \psi(T_n, x_0, z, u_n) &= \psi(2T + S_n, x_0, z, u_n) \\ &= \psi\Big(T, \psi\big(S_n, \psi(T, x_0, z, u_0), v_n\big), v_0\Big) \\ &= \psi\Big(T, \psi\big(S_n, \psi(T, x, z, 0), v_n\big), v_0\Big) \\ &\to (y_0, w). \end{aligned}$$

This proves property (i). Then property (ii) is obvious. □

REMARK 4.8. *Condition (4.5) is analogous to the inner-pair condition (but slightly stronger) for autonomous control systems, see Definition 4.1.5 in [3]*

Assumption (4.5) in Theorem 4.7 can be guaranteed for a large class of systems, as shown by Gayer [8]: Consider the following nth order systems on $\mathbb{R}^m$

$$\begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_m^{(n)} \end{pmatrix} + \begin{pmatrix} f_1(t, x, \ldots, x^{(n-1)}) \\ \vdots \\ f_m(t, x, \ldots, x^{(n-1)}) \end{pmatrix} = \begin{pmatrix} b_1(t, x, \ldots, x^{(n-1)}) \;\; u_1(t) \\ \vdots \\ b_m(t, x, \ldots, x^{(n-1)}) \;\; u_m(t) \end{pmatrix}. \tag{4.6}$$

Here $x = (x_i) \in C^{n-1}(\mathbb{R}, \mathbb{R}^m)$, its nth derivative exists but is not necessarily continuous, and $x^{(k)}$ denotes its kth derivative. Assume $f_i : \mathbb{R} \times \mathbb{R}^{nm} \to \mathbb{R}$ and $b_i : \mathbb{R} \times \mathbb{R}^{nm} \to \mathbb{R}$ are C^1 and consider controls

$$u = (u_i) \in \mathcal{U}^\rho := \{u : \mathbb{R} \to \mathbb{R}^m, \; u(t) \in U^\rho \text{ for all } t\}.$$

We assume that the control ranges U^ρ are compact and convex and that mapping $\rho \mapsto U^\rho$ is strictly increasing, i. e. $U^{\rho_1} \subset \operatorname{int} U^{\rho_2}$ for $0 \leq \rho_1 \leq \rho_2$. As before, assume that for all initial values and all controls the solutions are unique and exist for all times.

We consider the associated first order systems. So for initial values $y_0, \ldots, y_{n-1} \in \mathbb{R}^m$ at time $t_0 = 0$ and a control $u \in \mathcal{U}^\rho$ denote by $\lambda(t, y_0, \ldots, y_{n-1}, u)$ the corresponding solution of (4.6). We set $y^0 := (y_0, \ldots, y_{n-1}) \in \mathbb{R}^{nm}$ and define the set reachable from y^0 at time $T > 0$ by

$$\mathcal{O}_T^{+,\rho}(y^0) := \left\{ \begin{array}{l} (z_0, \ldots, z_{n-1}) \in \mathbb{R}^{nm}, \text{ there is } u \in \mathcal{U} \\ \text{with } z_i = \lambda^{(i)}(t, y^0) \text{ for } 0 \leq i \leq n-1 \end{array} \right\}.$$

PROPOSITION 4.9. *Consider system (4.6) and assume that there is some $\alpha > 0$ such that $|b_i(t, y)| \geq \alpha$ for all $i \in \{1, \ldots, m\}$ and all $(t, y) \in \mathbb{R} \times \mathbb{R}^{nm}$. Let $0 \leq \rho_1 \leq \rho_2$*

and consider a compact subset $B \subset \mathbb{R}^{nm}$. Then for every $T > 0$ there is $\varepsilon > 0$ such that for all $(y^0, u) \in B \times \mathcal{U}^{\rho_1}$

$$\mathbf{B}\Big((\lambda(T, y^0, u), \dots, \lambda^{(n-1)}(T, y^0, u); \varepsilon\Big) \subset \mathcal{O}_T^{+,\rho_2}(y^0).$$

Proof. This follows from [8, Theorem 3] and its proof. Here arbitrary time dependence of the right hand side is allowed and the theorem is formulated a bit differently (in terms of inner pairs for varying control range), but the proof shows the stronger result formulated above. □

In particular, under the assumptions of Proposition 4.9, one obtains for $\rho_1 = 0$ that condition (4.5) is satisfied (applying the theorem also to the time reversed system).

Next we generalize Theorem 4.7 in order to show a relation between chain controllability and controllability. We begin with the following lemma.

LEMMA 4.10. *Let $0 \le \rho_1 \le \rho_2$ and consider a compact subset $Q \subset M \times \mathcal{Z}$. Let E^{ρ_1} be a chain control set relative to Q for system (1.1) with controls in $\mathcal{U}^{\rho_1}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x, z) \in E^{\rho_1}$ and $u \in \mathcal{U}^{\rho_1}$*

$$\mathbf{B}_\varepsilon(\varphi(T, x, z, u)) \subset \mathcal{O}_T^{+,\rho_2}(x; z, Q). \tag{4.7}$$

Then for all $(x, z), (y, w) \in E^{\rho_1}$ there is $\tau > 0$ such that $\mathbf{B}_{\varepsilon/2}(y) \subset \mathcal{O}_\tau^{+,\rho_2}(x; z, Q)$ and for every $y_0 \in \mathbf{B}_{\varepsilon/2}(y)$ there are $\tau_n \ge 0$ and $u_n \in \mathcal{U}^{\rho_2}$ with $\varphi(\tau_n, x, z, u_n) = y_0$ in Q and $\theta_{\tau_n} u_n \to w$.

Proof. Let $(x, z), (y, w) \in E^{\rho_1}$. By uniform continuity, there is δ with $0 < \delta < \varepsilon/2$ such that for all u

$$\mathrm{d}\big((x_1, z_1), (x_2, z_2)\big) < \delta \text{ implies } \mathrm{d}\big(\psi(T, x_1, z_1, u), \psi(T, x_2, z_2, u)\big) < \varepsilon/2.$$

There is $u_0 \in \mathcal{U}^{\rho_1}$ such that $\psi(-T, y, w, u_0) \in E^{\rho_1}$. By chain controllability, there exists a controlled (δ, T)-chain in Q along z from x to $\psi(-T, y, w, u_0)$, i.e. $x_0 = x, x_n = \varphi(-T, y, w, u_0)$, and

$$\begin{gathered}\mathrm{d}(\theta_{T_0+\dots+T_{n-1}} z, \theta_{-T} w) < \delta,\ \mathrm{d}\big(\varphi(T_j, x_j, \theta_{T_0+\dots+T_{j-1}} z, v_j), x_{j+1}\big) < \delta \text{ for all } j,\\ \psi(t, x_j, \theta_{T_0+\dots+T_{j-1}} z, v_j) \in Q \text{ for all } t \in [0, T_j] \text{ and for all } j;\end{gathered}$$

For every j one finds by induction

$$\begin{aligned} x_{j+1} &\in \mathbf{B}_\delta\big(\varphi(T_j, x_j, \theta_{T_0+\dots+T_{j-1}} z, v_j)\big)\\ &= \mathbf{B}_\delta\big(\varphi(T, \varphi(T_j - T, x_j, \theta_{T_0+\dots+T_{j-1}} z, v_j), \theta_{T_0+\dots+T_{j-1}+T_j-T} z, \theta_{T_j-T} v_j\big)\\ &\subset \mathcal{O}_T^{+,\rho_2}\big(\varphi(T_j - T, x_j, \theta_{T_0+\dots+T_{j-1}} z, v_j); \theta_{T_0+\dots+T_{j-1}+T_j-T} z, Q\big)\\ &\subset \mathcal{O}_{T_0+\dots+T_j}^{+,\rho_2}(x_0; z, Q). \end{aligned}$$

Hence there is a control $v \in \mathcal{U}^{\rho_2}$ with

$$x_n = \varphi(T_0 + \dots + T_{n-1}, x, z, v) \text{ and } \mathrm{d}(\theta_{T_0+\dots+T_{n-1}} z, \theta_{-T} w) < \delta. \tag{4.8}$$

By choice of δ we find

$$\begin{aligned} &\mathrm{d}\big(\psi(T, x_n, \theta_{T_0+\dots+T_{n-1}} z, \theta_{-T} u_0), (y, w)\big)\\ &= \mathrm{d}\Big(\psi(T, x_n, \theta_{T_0+\dots+T_{n-1}} z, \theta_{-T} u_0), \psi\big(T, \psi(-T, y, w, u_0), \theta_{-T} u_0\big)\Big) < \varepsilon/2. \end{aligned}$$

We conclude for $\varepsilon > 0$, small enough,

$$\begin{aligned}\mathbf{B}_{\varepsilon/2}(y) &\subset \mathbf{B}_\varepsilon\big(\varphi(T, x_n, \theta_{T_0+\cdots+T_{n-1}}z, \theta_{-T}u_0)\big)\\ &= \mathbf{B}_\varepsilon\Big(\varphi\big(T, \varphi(T_0+\cdots+T_{n-1}, x, z, v), \theta_{T_0+\cdots+T_{n-1}}z, \theta_{-T}u_0\big)\Big)\\ &\subset \mathcal{O}^{+,\rho_2}_{T_0+\cdots+T_{n-1}+T}(x; z, Q).\end{aligned}$$

This yields the first assertion with $\tau = T_0 + \cdots + T_{n-1} + T$. The second assertion follows with $\tau_n = T_0 + \cdots + T_{n-1} + T$ if we consider $\delta_n \to 0$ in (4.8). ∎

This lemma allows us to show that chain control sets are contained in the interior of control sets for larger control ranges.

THEOREM 4.11. *Let $0 \le \rho_1 \le \rho_2$ and consider a compact subset $Q \subset M \times \mathcal{Z}$. Let E^{ρ_1} be a chain control set relative to Q for system (1.1) with controls in $\mathcal{U}^{\rho_1}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x, z) \in E^{\rho_1}$ and $u \in \mathcal{U}^{\rho_1}$*

$$\mathbf{B}_\varepsilon\big(\varphi(T, x, z, u)\big) \subset \mathcal{O}^{+,\rho_2}_T(x; z, Q) \text{ and } \mathbf{B}_\varepsilon\big(\varphi(-T, x, z, u)\big) \subset \mathcal{O}^{-,\rho_2}_T(x; z, Q). \quad (4.9)$$

Then there exists a control set D^{ρ_2} such that for every $(x, z) \in E^{\rho_1}$ one has $x \in \operatorname{int} D^{\rho_2}_z$.

Proof. The assertion follows, if we can show that for all $(x, z) \in E^{\rho_1}$ the neighborhoods $\mathbf{B}_{\varepsilon/2}(x)$ satisfy conditions (i) and (ii) in Definition 4.1 for controls in $\mathcal{U}^{\rho_2}$. Then E^{ρ_1} is contained in a maximal set with these properties, i. e. a control set D^{ρ_2}. Let $(x, z), (y, w) \in E^{\rho_1}$. For property (i) it suffices to show that for $x_0 \in \mathbf{B}_{\varepsilon/2}(x), y_0 \in \mathbf{B}_{\varepsilon/2}(y)$ there are $T_n \ge 0$ and $u_n \in \mathcal{U}^{\rho_2}$ with $\psi(T_n, y_0, w, u_n) \to (x_0, z)$ in Q. There is a control $v_0 \in \mathcal{U}^{\rho_1}$ with $\psi(T, x, z, v_0) \in E^{\rho_1}$, hence condition (4.5) implies

$$\mathbf{B}_{\varepsilon/2}(x) \subset \mathcal{O}^{-,\rho_2}_T\big(\psi(T, x, z, v_0)\big).$$

Hence for every $x_0 \in \mathbf{B}(x, \varepsilon/2)$ there is a control $u_0 \in \mathcal{U}^{\rho_2}$ with $\psi(T, x, z, v_0) = \psi(T, x_0, z, u_0)$. Similarly, there is a control $v_1 \in \mathcal{U}^{\rho_1}$ with $\psi(-T, y, w, v_1) \in E^{\rho_1}$ and

$$\mathbf{B}_{\varepsilon/2}(y) \subset \mathcal{O}^{+,\rho_2}_T\big(\psi(-T, y, w, v_1)\big),$$

and hence there is a control $u_1 \in \mathcal{U}^{\rho_2}$ with $(y_0, w) = \psi\big(T, \psi(-T, y, w, v_1), u_1\big)$.

Since $\psi(T, x, z, v_0), \psi(-T, y, w, v_1) \in E^{\rho_1}$, Proposition 4.10 implies that there are $\tau_n \ge 0$ and $v_n \in \mathcal{U}^{\rho_2}$ with $\psi\big(\tau_n, \psi(T, x, z, v_0), v_n\big) \to \psi(-T, y, w, v_1)$ in Q.

Together, one obtains

$$\psi\Big(T, \psi\big(\tau_n, \psi(T, x, z, v_0), v_n\big), u_1\Big) \to \psi\big(T, \psi(-T, y, w, v_1), u_1\big) = (y_0, w).$$

Define the concatenated control $u_n \in \mathcal{U}^{\rho_2}$ by

$$u_n(t) := \begin{cases} u_0(t) & \text{for } t \in [0, T] \\ v_n(t - T) & \text{for } t \in (T, T + \tau_n] \\ u_1(t - T - \tau_n) & \text{for } t \in [T + \tau_n, 2T + \tau_n]. \end{cases}$$

Then, with $T_n := 2T + \tau_n$

$$\begin{aligned}\psi(T_n, x_0, z, u_n) &= \psi(2T + \tau_n, x_0, z, u_n)\\ &= \psi\Big(T, \psi\big(\tau_n, \psi(T, x_0, z, u_n), \theta_T u_n\big), \theta_{T+\tau_n} u_n\Big)\\ &= \psi\Big(T, \psi\big(\tau_n, \psi(T, x_0, z, u_0), v_n\big), u_1\Big)\\ &\to (y_0, w).\end{aligned}$$

This proves property (i) of control sets. Now property (ii) is obvious. ∎

REMARK 4.12. *Using this theorem we can, as in [3, Theorem 4.7.5], show that for all up to at most countably many ρ-values the closures of control sets and the chain control sets coincide. The proof is based on Scherbina's Lemma [17] for continuity of monotonically increasing set valued functions. Hence, by Theorem 3.7 one may also determine the fibers of control sets via the fibers of the chain control sets. For this purpose, one has to consider 'long' times, since these fibers are determined only on long time intervals, cp. Remark 3.8. At first sight, this is different, if the excitation is periodic; here only the Poincaré map, and hence the period length, is needed, Proposition 3.6. Nevertheless, also in this case approximate controllability is relevant (the entrance boundary of a control set is reached from the interior only for time tending to infinity), and hence also these objects are only determined on long time intervals.*

5. Almost Periodic Solutions and Heteroclinic Orbits. In this section we recall results on almost periodic perturbations of hyperbolic equilibria and Melnikov's method. Since in the literature they are not precisely stated in the form needed here, we recall the relevant concepts and some arguments for the proofs.

It is well-known that, under small periodic perturbations, a hyperbolic fixed point of an autonomous differential equation becomes a periodic solution, see e. g. [1, Theorem 25.2] for details on this result, which is known as *Poincaré continuation.* This result can be generalized to almost periodic perturbations, in which case the existence of an almost periodic solution can be shown. Consider the differential equation

$$\dot{x} = g(x) + \mu h(t, x, \mu) \tag{5.1}$$

for $g : \mathbb{R}^d \to \mathbb{R}^d$ and $h : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. The parameter $\mu \in \mathbb{R}$ is interpreted as a small perturbation. Setting $\mu = 0$ in system (5.1) leads to the equation $\dot{x} = g(x)$ which will be referred to as the *unperturbed* system. Throughout we assume that (5.1) satisfies the following conditions:

The function g is C^1 and h is continuous and h_x exists and there are a bounded and open subset $V \subset \mathbb{R}^d$ containing x_0 and a constant $\bar{\mu} > 0$, such that h and h_x are almost periodic in t, uniformly with respect to $(x, \mu) \in \operatorname{cl} V \times [-\bar{\mu}, \bar{\mu}]$, and solutions of (5.1) exist for all starting points in V, all $\mu \in [-\bar{\mu}, \bar{\mu}]$ and all times.

As noted in Scheurle [18], Remark 2.7, almost periodicity of h_x uniformly with respect to (x, μ) is equivalent with h_x being uniformly continuous on $\mathbb{R} \times \operatorname{cl} V \times [-\bar{\mu}, \bar{\mu}]$.

Next recall the notion of exponential dichotomies, which generalize the idea of hyperbolicity to nonautonomous systems, cf. Coppel [5].

DEFINITION 5.1. *Consider the system*

$$\dot{x} = A(t)x \tag{5.2}$$

for a piecewise continuous matrix function $A : J \to \mathbb{R}^{d \times d}$ defined on an interval $J \subset \mathbb{R}$ and let $X(t)$ be a fundamental matrix function for (5.2). System (5.2) has an exponential dichotomy *on J if there is a projection $P : \mathbb{R}^d \to \mathbb{R}^d$ and constants $K \geq 1$, $\alpha > 0$ such that*

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq Ke^{-\alpha(t-s)} && \textit{for } s \leq t, \\ \|X(t)(I-P)X^{-1}(s)\| &\leq Ke^{-\alpha(s-t)} && \textit{for } s \geq t. \end{aligned}$$

Then the following perturbation result (Lemma 2.4 in [18]) holds.

LEMMA 5.2. *Let $g(t, x)$ and $h(t, x)$ be functions which are defined and continuous on $\mathbb{R} \times V$ with values in $\mathbb{R}^d$, where V is an open subset of $\mathbb{R}^d$. Furthermore, assume* *that the partial derivatives g_x and h_x exist and that g_x is uniformly continuous and h_x continuous in $\mathbb{R} \times V$. Finally assume that the equation $\dot{x} = g(t,x)$ has a solution $x = x_0(t)$ defined and contained in V for all $t \in \mathbb{R}$, and strictly bounded away from the boundary of V, such that the variational equation $\dot{x} = g_x\big(t, x_0(t)\big)x$ has an exponential dichotomy on $\mathbb{R}$ with constants K and α. Then there exist a positive constant η_0 and a function $\eta_1(\eta)$ depending only on g, K, and α such that, if $0 < \eta \leq \eta_0$,*

$$\sup_{(t,x)\in\mathbb{R}\times V} \|h(t,x)\| < \eta_1(\eta) \text{ and } \sup_{(t,x)\in\mathbb{R}\times V} \|h_x(t,x)\| < K\ \alpha/2,$$

then the equation $\dot{x} = g(t,x) + h(t,x)$ has a unique solution $x(t)$ satisfying $\|x(t) - x_0(t)\| \leq \eta$, $t \in \mathbb{R}$.

A slight modification of Bohr's proof for the boundedness of almost periodic functions in [2] shows uniform boundedness of uniformly almost periodic functions.

LEMMA 5.3. *Let Λ be a compact topological space, M a normed vector space with norm $\|\cdot\|$ and $f : \mathbb{R} \times \Lambda \to M$ continuous and almost periodic in t uniformly with respect to $x \in \Lambda$. Then*

$$\sup{}_{(t,x)\in\mathbb{R}\times\Lambda}\|f(t,x)\| < \infty.$$

Proof. Since f is uniformly almost periodic, there is an interval length L such that for every interval $J \subset \mathbb{R}$ of length L there exists a translation number $\tau(J) \in J$ satisfying $\big\|f\big(t+\tau(J),x\big) - f(t,x)\big\| < 1$ for all $(t,x) \in \mathbb{R} \times \Lambda$. Here L and τ are independent of x due to uniformity.

Since f is continuous and Λ compact, $c := \sup_{(t,x)\in[0,L]\times\Lambda} \|f(t,x)\| < \infty$. For every $t \in \mathbb{R}$ any translation number τ_t in the interval $J = [-t, -t+L]$ satisfies $t + \tau_t \in [0, L]$. Therefore for every $t \in \mathbb{R}$ and $x \in \Lambda$

$$\|f(t,x)\| \leq \|f(t+\tau_t)\| + \|f(t) - f(t+\tau_t)\| \leq c+1. \qquad \square$$

The previous lemmas imply the following result (this is essentially Lemma 2.8 in [18]).

PROPOSITION 5.4. *Suppose that the unperturbed system corresponding to (5.1) has a hyperbolic fixed point x_0, i. e. $g(x_0) = 0$ and the real parts of the eigenvalues of $g_x(x_0)$ are different from 0. For all (small) $\eta > 0$ there is $\mu_0 = \mu_0(\eta) > 0$ such that for $|\mu| \leq \mu_0$ there exists a unique solution $\zeta^\mu(t)$ of system (5.1) satisfying $\|\zeta^\mu(t) - x_0\| \leq \eta$ for all $t \in \mathbb{R}$. This solution is almost periodic.*

Proof. First we show that system (5.1) satisfies the assumptions of Lemma 5.2. The functions g and h are continuous and the derivatives g_x and h_x exist and g_x is uniformly continuous on the compact set $\operatorname{cl} V$. As x_0 is a hyperbolic equilibrium of the unperturbed equation, the corresponding linearized equation $\dot{x} = g_x(x_0)x$ trivially has an exponential dichotomy on $\mathbb{R}$. Finally, $\sup_{(t,x)\in\mathbb{R}\times V} \|\mu h(t,x,\mu)\|$ and $\sup_{(t,x)\in\mathbb{R}\times V} \|\mu h_x(t,x,\mu)\|$ can be made arbitrarily small by choosing μ small enough, since h and h_x are uniformly almost periodic and thus uniformly bounded, due to Lemma 5.3.

This means that for sufficiently small perturbations μ there is a unique solution ζ^μ which stays near the original fixed point x_0 for all times. For sufficiently small μ the equation

$$\dot{x} = \big[g_x\big(\zeta^\mu(t)\big) + \mu h_x\big(t, \zeta^\mu(t), \mu\big)\big]x$$

has an exponential dichotomy on $\mathbb{R}$. This follows from roughness of exponential dichotomies with respect to small perturbations; see Proposition 2.2 in [18] or [5,

p. 34]. Finally, it remains to show almost periodicity of the perturbed solution ζ^μ. For this purpose consider the shifted system

$$\dot{x} = g(x) + \mu h(t+\tau, x, \mu) \tag{5.3}$$

for $\tau \in \mathbb{R}$. Lemma 5.2 applied to (5.3) shows that for small η and $|\mu| \le \mu_0(\eta)$ there is a unique solution $\zeta_\tau^\mu(t)$ which satisfies $\|\zeta_\tau^\mu(t) - x_0\| \le \eta$ for all $t \in \mathbb{R}$. Obviously $\zeta_\tau^\mu(t) = \zeta^\mu(t+\tau)$ for all $t, \tau \in \mathbb{R}$.

Now we apply Lemma 5.2 to (5.3) again, setting $g(t,x) = g(x) + \mu h(t,x,\mu)$, $h(t,x) = \mu[h(t+\tau,x,\mu) - h(t,x,\mu)]$ and $x_0(t) = \zeta^\mu(t)$. For sufficiently small μ and $\eta > 0$ there is an $\varepsilon = \varepsilon(\mu,\eta) > 0$ such that $\|\zeta^\mu(t) - \zeta_\tau^\mu(t)\| \le \eta$, provided that

$$|\mu| \sup_{(t,x)\in\mathbb{R}\times V} \|h(t+\tau,x,\mu) - h(t,x,\mu)\| < \varepsilon$$

and

$$|\mu| \sup_{(t,x)\in\mathbb{R}\times V} \|h_x(t,x,\mu) - h_x(t+\tau,x,\mu)\| < \varepsilon.$$

Hence uniform almost periodicity of h and h_x implies almost periodicity of $\zeta^\mu(t)$. $\square$

If we suppose that in our setting there exist two hyperbolic fixed points $x_\pm \in \mathbb{R}^d$ of the unperturbed system, Proposition 5.4 implies the existence of almost periodic solutions $\zeta_\pm^\mu$ near $x_\pm$ for sufficiently small μ. If there is a heteroclinic orbit ζ from x_- to x_+, the question arises how the system behaves near ζ for small perturbations μ.

For time-periodic perturbations Melnikov's method gives a handy criterion for the existence of transversal heteroclinic points. K. J. Palmer has developed a generalization of Melnikov's method in [15] which, in our setting, yields the following.

THEOREM 5.5. *Consider the system $\dot{x} = g(x) + \mu h(t,x,\mu)$ and let the following assumptions be satisfied:*

(i) *There is a bounded and open subset $V \subset \mathbb{R}^d$ and a constant $\bar{\mu} > 0$ such that $g : V \to \mathbb{R}^d$ is C^2 and $h : \mathbb{R} \times V \times [-\bar{\mu}, \bar{\mu}] \to \mathbb{R}^d$ is continuous. The partial derivatives h_t, h_x, h_μ, h_{xx}, $h_{x\mu}$, $h_{\mu x}$, $h_{\mu\mu}$ exist, are bounded, continuous in t for each fixed x, μ and continuous in x, μ uniformly with respect to t, x, μ.*

(ii) *The functions h and h_x are almost periodic in t, uniformly with respect to $(x,\mu) \in \operatorname{cl} V \times [-\bar{\mu}, \bar{\mu}]$.*

(iii) *The unperturbed equation $\dot{x} = g(x)$ has hyperbolic fixed points $x_\pm \in V$ with stable and unstable manifolds of the same dimensions.*

(iv) *There is a heteroclinic orbit ζ from x_- to x_+ contained in V.*

(v) *The function*

$$\Delta(t_0) := \int_{-\infty}^{\infty} \varphi(t) \cdot h\big(t + t_0, \zeta(t), 0\big)\ \mathrm{d}t$$

has a simple zero at some $t_0 \in \mathbb{R}$, where $\varphi(t)$ is the unique (up to a scalar multiple) bounded solution of the adjoint system $\dot{x} = g_x\big(\zeta(t)\big)^T x$ and "$\cdot$" denotes the inner product in $\mathbb{R}^d$.

Then there exists $\delta_0 > 0$ such that for sufficiently small μ the perturbed system (5.1) has a unique solution $x(t,\mu)$ satisfying $\|x(t,\mu) - \zeta(t-t_0)\| \le \delta_0$ for all $t \in \mathbb{R}$. Furthermore

$$\sup_{t\in\mathbb{R}} \|x(t,\mu) - \zeta(t-t_0)\| = O(\mu) \text{ for } \mu \to 0$$

and

$$\dot{x} = \big[g_x\big(x(t,\mu)\big) + \mu h_x\big(t, x(t,\mu), \mu\big)\big]x$$

has an exponential dichotomy on $\mathbb{R}$.

Finally, it holds that

$$\lim_{t\to\pm\infty} \|x(t,\mu) - \zeta_\pm^\mu(t)\| = 0 \tag{5.4}$$

for sufficiently small μ, *where* $\zeta_\pm^\mu$ *are the almost periodic solutions near* $x_\pm$.

Proof. This follows from [15, Corollary 4.3] and the remark on pp. 251–252 in [15] combined with the ideas of the proof of [15, Corollary 4.4] using the fact, that $\dot{x} = g_x\big(\zeta(t)\big)x$ has an exponential dichotomy on both half-lines and that the dimensions of the stable and unstable subspaces sum up to d.

More precisely, Corollary 4.4 in [15] shows (5.4) for the *periodic* case. But in fact, periodicity is only needed there to prove periodicity of $\zeta_\pm^\mu$. So (5.4) holds for the almost periodic case, too, cf. Remark 2.9 in [18]. In detail, there is a $\delta > 0$ independent of μ such that if

$$\|x(t,\mu) - \zeta_\pm^\mu(t)\| \le \delta \tag{5.5}$$

for sufficiently large $|t|$ (positive for "+", negative for "−"), then (5.4) holds, cf. [9, Theorem 3.1]. For sufficiently small μ and large $|t|$

$$\|x(t,\mu) - \zeta_\pm^\mu(t)\| \le \|x(t,\mu) - \zeta(t-t_0)\| + \|\zeta(t-t_0) - x_\pm\| + \|x_\pm - \zeta_\pm^\mu(t)\| \le \delta,$$

hence (5.5) holds.

The fact, that the variational system $\dot{x} = g_x\big(\zeta(t)\big)x$ has an exponential dichotomy and that the dimensions sum up to d, follows from standard perturbation theory, and from the assumption that the stable and unstable manifolds of x_- and x_+ have the same dimensions. □

REMARK 5.6. *This theorem is also applicable to* homoclinic *orbits by letting* $x_- = x_+$.

REMARK 5.7. *If in the two-dimensional case* g *is Hamiltonian,* $\Delta(t_0)$ *coincides with the Melnikov function up to a scalar multiple, Marsden [13].*

6. Heteroclinic Orbits and Controllability. In this section, we show that existence of a heteroclinic solution of the unperturbed uncontrolled equation implies a controllability condition for perturbed systems with small control influence. Conversely, if the controllability condition holds for small control influence, existence of a heteroclinic solution of the unperturbed equation follows. These results are used to relate heteroclinic cycles to the existence of control sets.

Consider the following family of control systems depending on a parameter μ

$$\dot{x} = g(x) + \mu h(x, z(t), \mu, u(t)), u \in \mathcal{U}, \tag{6.1}$$

with continuous functions g and h and control range $U \subset \mathbb{R}^m$ containing the origin; the functions z are in the hull $\mathcal{Z}$ of a single almost periodic function. We refer to $\dot{x} = g(x)$ and $\dot{x} = g(x) + \mu h(t, x, \mu, 0)$ as the unperturbed uncontrolled system and the perturbed uncontrolled system, respectively. For fixed μ this is a special case of the control system (1.1); we use the notation introduced in § 2, § 3 and § 4 with a superfix

μ to indicate dependence on this parameter. In particular, solutions (whose existence we always assume) are denoted by $\varphi^\mu(t, x_0, z, u), t \in \mathbb{R}, x_0 \in \mathbb{R}^d, z \in \mathcal{Z}, u \in \mathcal{U}$.

PROPOSITION 6.1. *Assume that system (6.1) with control $u = 0$ satisfies the assumptions (i) to (v) of Theorem 5.5. Let $\zeta^\mu_\pm$ be the almost periodic solutions near the hyperbolic equilibria $x_\pm$ of the unperturbed uncontrolled system and let $x(t,\mu) := \varphi^\mu(t, x^\mu, z_0, 0)$ be the solution near the heteroclinic orbit ζ from x_- to x_+ for some $x^\mu \in \mathbb{R}^d, z_0 \in \mathcal{Z}$. Let μ be a parameter value such that the conclusions of Theorem 5.5 hold, and assume that there are $\varepsilon = \varepsilon(\mu), T = T(\mu) > 0$ such that for every $(x, z) \in Q := \mathrm{cl}\, V \times \mathcal{Z}$*

$$\mathbf{B}_\varepsilon(\varphi^\mu(T, x, z, 0)) \subset \mathcal{O}^{\mu,+}_T(x; z, Q) \text{ and } \mathbf{B}_\varepsilon(\varphi^\mu(-T, x, z, 0)) \subset \mathcal{O}^{\mu,-}_T(x; z, Q). \quad (6.2)$$

Then there are a control function $u^\mu \in \mathcal{U}$ and times $t^\mu_- < t^\mu_+$ such that the corresponding solution $\varphi^\mu(t, x^\mu, z_0, u^\mu)$ of (6.1) satisfies

$$\varphi^\mu(t, x^\mu, z_0, u^\mu) = \begin{cases} \zeta^\mu_-(t) & \text{if } t \le t^\mu_-, \\ \zeta^\mu_+(t) & \text{if } t \ge t^\mu_+. \end{cases}$$

Proof. Pick μ as stated and denote the constants from condition (6.2) by $\varepsilon, T > 0$. The solution $x(t, \mu)$ for the uncontrolled system satisfies (5.4). In particular, there are times $t^\mu_- < 0 < t^\mu_+$, arbitrarily large, such that

$$\|x(t^\mu_-, \mu) - \zeta^\mu_-(t^\mu_-)\| < \varepsilon \text{ and } \|x(t^\mu_+, \mu) - \zeta^\mu_+(t^\mu_+)\| < \varepsilon.$$

Together with (6.2) and the cocycle property this means

$$\begin{aligned} \zeta^\mu_-(t^\mu_-) &\in \mathbf{B}_\varepsilon(\varphi^\mu(t^\mu_-, x^\mu, z_0, 0)) \\ &= \mathbf{B}_\varepsilon(\varphi^\mu(-T, \varphi^\mu(t^\mu_- + T, x^\mu, z_0, 0), z_0(t^\mu_- + T + \cdot), 0)) \\ &\subset \mathcal{O}^{\mu,-}_T(\varphi^\mu(t^\mu_- + T, x^\mu, z_0, 0); z_0(t^\mu_- + T + \cdot), Q) \end{aligned}$$

and, analogously,

$$\begin{aligned} \zeta^\mu_+(t^\mu_+) &\in \mathbf{B}_\varepsilon(\varphi^\mu(t^\mu_+, x^\mu, z_0, 0)) \\ &= \mathbf{B}_\varepsilon(\varphi^\mu(T, \varphi^\mu(t^\mu_+ - T, x^\mu, z_0, 0), z_0(t^\mu_+ - T + \cdot), 0)) \\ &\subset \mathcal{O}^{\mu,+}_T(\varphi^\mu(t^\mu_+ - T, x^\mu, z_0, 0); z_0(t^\mu_+ - T + \cdot), Q) \end{aligned}$$

This ensures the existence of control functions $u^\mu_\pm \in \mathcal{U}$ satisfying

$$\begin{aligned} \zeta^\mu_-(t^\mu_-) &= \varphi\big(-T, \varphi^\mu(t^\mu_- + T, x^\mu, z_0, 0), z_0(t^\mu_- + T + \cdot), u^\mu_-\big), \\ \zeta^\mu_+(t^\mu_+) &= \varphi\big(T, \varphi^\mu(t^\mu_+ - T, x^\mu, z_0, 0), z_0(t^\mu_+ - T + \cdot), u^\mu_+\big). \end{aligned}$$

Setting

$$u_\mu(t) := \begin{cases} u_-(t - t^\mu_- - T) & \text{if } t \in [t^\mu_-, t^\mu_- + T], \\ u_+(t - t^\mu_+ + T) & \text{if } t \in [t^\mu_+ - T, t^\mu_+], \\ 0 & \text{otherwise} \end{cases}$$

completes the proof. $\square$

The previous proposition shows that existence of a heteroclinic orbit for the unperturbed uncontrolled equation implies the existence of a control steering the system with almost periodic excitation from the almost periodic solution near one equilibrium to the almost periodic solution near the other equilibrium. The following result considers a converse situation where the unperturbed equation has equilibria x_+ and x_- and we want to conclude from existence of controlled trajectories of the perturbed system from points near x_- to x_+ that a heteroclinic orbit of the unperturbed equation exists.

PROPOSITION 6.2. *Suppose that g and $h(x, z(t), \mu, 0)$ satisfy assumptions (i) and (ii) of Theorem 5.5 for all $z \in \mathcal{Z}$, i. e. these assumptions hold for system (6.1) with $u = 0$. Moreover, assume that the chain recurrent set of the unperturbed uncontrolled system $\dot{x} = g(x)$ relative to $\operatorname{cl} V$ is equal to $\{x_+, x_-\}$.*

Suppose furthermore that the control range U is bounded and there are $\mu_n \to 0$, almost periodic excitations $z_n \in \mathcal{Z}$, control functions $u_n \in \mathcal{U}$, times $t_-^n < t_+^n$, and points $x_n \in \operatorname{cl} V$ such that the solution $\varphi_n(t) := \varphi^{\mu_n}(t, x_n, z_n, u_n), t \in \mathbb{R}$, of (6.1) is contained in $\operatorname{cl} V$ and satisfies $\varphi_n(t_-^n) \to x_-$ and there is $\delta > 0$ with $\|\varphi_n(t) - x_-\| \geq \delta$ for all $t \geq t_+^n$ and all n.

Then the unperturbed, uncontrolled system has a heteroclinic orbit from x_- to x_+.

Proof. For every $n \in \mathbb{N}$ let $T_n \geq t_-^n$ be the largest time satisfying $\varphi_n(T_n) \in \operatorname{cl} \mathbf{B}_r(x_-)$, where $r > 0$ is chosen such that $\mathbf{B}_r(x_-) \subset \operatorname{cl} V$. We may assume the limit $\xi_0 := \lim_{n\to\infty} \varphi_n(T_n) \in \operatorname{cl} \mathbf{B}_r(x_-)$ exists. It suffices to prove that ξ_0 lies on a heteroclinic orbit in $\operatorname{cl} V$ from x_- to x_+.

By compactness of $\mathcal{Z}$, we may assume that $z_n(T_n + \cdot)$ converges to some $z^0 \in \mathcal{Z}$. In order to show that the orbit through ξ_0 lies in $\operatorname{cl} V$, fix $t \in \mathbb{R}$ and $\varepsilon > 0$. By assumption

$$\varphi_n(T_n) = \varphi^{\mu_n}(T_n, x_n, z_n, u_n) \to \xi_0,$$

and $\mu_n h(x, z, \mu_n, u)$ converges to zero, uniformly in (x, z, u) by continuity of h and boundedness of U. Then continuous dependence on the right hand side and the initial value implies

$$\begin{aligned}&\varphi^{\mu_n}(T_n + t, x_n, z_n, u_n)\\ &\qquad = \varphi^{\mu_n}(t, \varphi^{\mu_n}(T_n, x_n, z_n, u_n), z_n(T_n + \cdot), u_n(T_n + \cdot)) \to \varphi^0(t, \xi_0, z^0, 0).\end{aligned}$$

Hence the orbit through ξ_0 is contained in $\operatorname{cl} V$. Since the α- and ω-limit sets of x_0 are connected and in the chain recurrent set, they consist either of x_- or x_+. Since $\varphi^{\mu_n}(T_n + t, x_n, z_n, u_n) \in \operatorname{cl} \mathbf{B}_r(x_-)$ for $t \leq 0$, it follows that the α-limit set of ξ_0 is given by x_-. Similarly, $\varphi^{\mu_n}(T_n + t, x_n, z_n, u_n) \notin \operatorname{cl} \mathbf{B}_r(x_-)$ for $t > 0$, by maximality of T_n. Thus the ω-limit set is given by x_+. $\square$

Next we discuss consequences of these results for control sets of systems with almost periodic excitations. Roughly, the results above imply that the existence of a heteroclinic cycle of the unperturbed, uncontrolled system is equivalent to the existence of a control set containing all almost periodic solutions near the equilibria for the systems with almost periodic excitation and small control ranges.

Recall that a heteroclinic cycle of the unperturbed equation is given by a finite set $x_0, x_1, \ldots, x_n = x_0$ of equilibria together with heteroclinic solutions ζ_i from x_i to x_{i+1} for $i = 0, \ldots, n-1$. Existence of heteroclinic cycles can be expected in the presence of symmetries.

THEOREM 6.3. *Let $x_0, x_1, \ldots, x_n = x_0$ be pairwise different hyperbolic equilibria of the unperturbed uncontrolled system $\dot{x} = g(x)$ and consider control system (6.1)* *with a bounded control range U containing the origin. For $|\mu| \neq 0$, small, and $z \in \mathcal{Z}$ denote the almost periodic solutions near x_i for excitation z by $\zeta_i^\mu(z)$. Assume that system (6.1) with $u = 0$ satisfies assumptions (i) and (ii) of Theorem 5.5 for all $z \in \mathcal{Z}$ on an open set V containing all equilibria x_i.*

(i) *Assume that for all i there are open subsets $V_i \subset \mathbb{R}^d$ containing the equilibria $x_- = x_i$ and $x_+ = x_{i+1}$ such that assumptions (iii) to (v) of Theorem 5.5 are satisfied for (6.1) with $u = 0$, and let $x_i(t, \mu, z) = \varphi^\mu(t, x_i^\mu, z, 0)$ be the solution near the heteroclinic orbit $\zeta_i(z)$ from x_i to x_{i+1} . Assume that for all sufficiently small $|\mu| \neq 0$ there are $\varepsilon_i, T_i > 0$ such that for every $(x, z) \in Q_i := \operatorname{cl} V_i \times \mathcal{Z}$*

$$\mathbf{B}_{\varepsilon_i}(\varphi^\mu(T_i, x, z, 0)) \subset \mathcal{O}_{T_i}^{\mu,+}(x; z, Q_i) \text{ and } \mathbf{B}_{\varepsilon_i}(\varphi^\mu(-T_i, x, z, 0)) \subset \mathcal{O}_{T_i}^{\mu,-}(x; z, Q_i). \tag{6.3}$$

Then for all $|\mu| \neq 0$, small, there exists a control set D^μ such that for all $z \in \mathcal{Z}$ and all i the almost periodic solution satisfy $\zeta_i^\mu(t) \in D^\mu_{z(t+\cdot)}$ and the heteroclinic solutions satisfy $x_i(t, \mu, z) \in D^{\mu, z(t+\cdot)}$.

(ii) *Conversely, suppose for all i there are open subsets V_i containing x_i and x_{i+1} such that the chain recurrent set of the unperturbed uncontrolled system $\dot{x} = g(x)$ relative to $\operatorname{cl} V_i$ is equal to $\{x_i, x_{i+1}\}$. Furthermore, suppose that for a sequence $0 \neq \mu_n \to 0$ there are control sets D^{μ_n} containing the almost periodic solutions $\zeta_i^{\mu_n}$ near x_i for almost periodic excitations $z_n \in \mathcal{Z}$. Then the unperturbed system has a heteroclinic cycle through the x_i.*

Proof.

(i) For all i, Theorem 4.7 implies that there are control sets D_i^μ such that the almost periodic solutions $\zeta_i^\mu(z)$ are contained in the interior of $D^\mu_{i,z}$. It remains to show that all D_i^μ coincide. Fix $z \in \mathcal{Z}$ and consider the almost periodic solutions $\zeta_i(z)$ near x_i (we suppress dependence on μ in our notation). By Proposition 6.1 there are $y_1 \in \mathbb{R}^d$, a control function $u_1 \in \mathcal{U}$, and times $t_1 < t_2$ such that the corresponding solution $\varphi(t, y_1, z, u_1)$ of (6.1) satisfies

$$\varphi(t, y_1, z, u_1) = \begin{cases} \zeta_1(t) & \text{if } t \leq t_1, \\ \zeta_2(t) & \text{if } t \geq t_2. \end{cases}$$

There are $y_2 \in \mathbb{R}^d$, a control function $u_2 \in \mathcal{U}$, and times $\tau_2 > t_2$ and $t_3 > \tau_2$ such that the corresponding solution $\varphi(\cdot, y_2, z, u_1)$ of (6.1) satisfies

$$\varphi(t, y_2, z, u_2) = \begin{cases} \zeta_2(t) & \text{if } t \leq \tau_2, \\ \zeta_3(t) & \text{if } t \geq t_3. \end{cases}$$

Proceeding in this way and using $x_n = x_0$, one finds times $T_2 > T_1 > 0$, a point $x \in D_1^z$, and a control $u \in \mathcal{U}$ such that

$$\begin{aligned} &\varphi(T_1, x, z, u) \in D_{2, z(T_1+\cdot)} \text{ and } \varphi(T_1 + T_2, x, z, u) \in D_{1, z(T_1+T_2+\cdot)}, \\ \text{and } &\psi(t, x_1, z, u) \in Q \text{ for all } t \in [0, T_1 + T_2]. \end{aligned}$$

Then Proposition 4.5 shows $D_1 = D_2$ and, repeating this argument, one concludes that all control sets D_i coincide.

(ii) The assumptions allows us to apply Proposition 6.2. Hence, for all i, the unperturbed uncontrolled system has a heteroclinic orbit from x_i to x_{i+1}. ◻

7. An Oscillator with M-Potential. In this section we will apply our results to a second order system with M-potential, which models ship roll motion.

Consider the system

$$\ddot{x} + \mu\beta_1\dot{x} + \mu\beta_3\dot{x}^3 + x - \alpha x^3 = \mu z(t) + \mu u(t) \tag{7.1}$$

with positive parameters α, β_1 and β_3, a small perturbation parameter $\mu \in \mathbb{R}$, almost periodic excitations $z : \mathbb{R} \to \mathbb{R}$ and control functions $u : \mathbb{R} \to [-\rho, \rho]$ for a control radius $\rho > 0$. This model, proposed in Kreuzer and Sichermann [11], has been studied in Colonius, Kreuzer, Marquardt and Sichermann [4] without time-dependent excitation z. Note that in this application the terms $u(\cdot)$ are interpreted as time-dependent perturbations (not as controls) where only the range $[-\rho, \rho]$ is known. Here the control sets give information on the global stability behavior: An invariant control set around the origin indicates stability. If (for large perturbation amplitudes) it has merged with a variant control set and itself becomes variant, stability is lost. Hence it is of interest to compute all control sets.

System (7.1) is a special case of system (4.6). Hence, Proposition 4.9 shows that assumption (4.9) in Theorem 4.11 is satisfied for all $\rho_2 > \rho_1 \geq 0$. Thus every compact chain control set E^{ρ_1} is contained in the interior of a control set D^{ρ_2} and hence, for all up to countably many $\rho > 0$, Remark 4.12 shows that the compact chain control sets coincide with the closures of control sets.

Writing (7.1) as a first order system yields the two-dimensional perturbed Hamiltonian system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 + \mu\big(-\beta_1 x_2 - \beta_3 x_2^3 + z(t) + u(t)\big). \end{aligned} \tag{7.2}$$

Denote by $\varphi^\mu(t, x, z, u)$ the solution of this system and let

$$\psi^\mu(t, x, z, u) := \big(\varphi^\mu(t, x, z, u), \theta_t z\big).$$

In the unperturbed and uncontrolled case $\mu = 0$ system (7.2) has a fixed point in the origin and two hyperbolic fixed points at $(\pm 1/\sqrt{\alpha}, 0)$. The hyperbolic fixed points are connected by two heteroclinic orbits given by $x_\pm^h(t) := \pm\big(x_1(t), x_2(t)\big)$, where

$$x_1(t) := \frac{1}{\sqrt{\alpha}} \tanh \frac{t}{\sqrt{2}},\ \ x_2(t) := \frac{1}{\sqrt{2\alpha}} \operatorname{sech}^2 \frac{t}{\sqrt{2}},\ \ t \in \mathbb{R},$$

cp. Simiu [19, p. 131]. In the perturbed, uncontrolled case $u \equiv 0$ denote by $\Delta_\pm$ the Melnikov functions of system (7.2) with respect to $x_\pm^h$ and denote by $\zeta_\pm^\mu$ the almost periodic solutions near $(\pm 1/\sqrt{\alpha}, 0)$, which exist for sufficiently small μ (see Proposition 5.4). Let $z_0 \in \mathcal{Z}$ be the corresponding excitation and $\xi_\pm^\mu(t) := \big(\zeta_\pm^\mu(t), \theta_t z_0\big)$.

PROPOSITION 7.1. *Assume that the almost periodic excitation z is continuously differentiable with bounded derivative. If the functions $\Delta_\pm$ have simple zeros and μ is small enough, then system* (7.2) *has a control set D containing $\xi_\pm^\mu(\mathbb{R})$. Then D will be called a heteroclinic control set.*

Proof. This essentially follows from Proposition 6.1. To be precise, system (7.2) satisfies assumptions (i) to (v) of Theorem 5.5 for $u = 0$: Assumption (i) is satisfied for every bounded open set $V \subset \mathbb{R}^d$ and every $\bar{\mu} > 0$. Property (ii) is clearly satisfied, because z does not depend on x and μ. Assumptions (iii) and (iv) are true for a suitable bounded and open set $V \subset \mathbb{R}^d$. Property (v) holds by assumption.

Furthermore, property (6.2) is satisfied, as can be shown by Proposition 4.9. So for sufficiently small μ Proposition 6.1 implies the existence of points $x^\mu_\pm \in \mathbb{R}^2$, control functions $u^\mu_\pm \in \mathcal{U}$ and times $s^\mu_\pm < t^\mu_\pm$ such that

$$\varphi^\mu(t, x^\mu_-, z_0, u^\mu_-) = \begin{cases} \zeta^\mu_+(t) & \text{if } t \le s^\mu_-, \\ \zeta^\mu_-(t) & \text{if } t \ge t^\mu_- \end{cases}$$

and

$$\varphi^\mu(t, x^\mu_+, z_0, u^\mu_+) = \begin{cases} \zeta^\mu_-(t) & \text{if } t \le s^\mu_+, \\ \zeta^\mu_+(t) & \text{if } t \ge t^\mu_+. \end{cases}$$

The set $\tilde{D} := \psi^\mu(\mathbb{R}, x^\mu_-, z_0, u^\mu_-) \cup \psi^\mu(\mathbb{R}, x^\mu_+, z_0, u^\mu_+) \cup \xi^\mu_-(\mathbb{R}) \cup \xi^\mu_+(\mathbb{R})$ satisfies properties (i) and (ii) of control sets and is thus contained in a control set D. This implies $\xi^\mu_\pm(\mathbb{R}) \subset \tilde{D} \subset D$. ☐

First we study the periodic case and choose $z(t) := F\cos\omega t$ for positive parameters F and ω, which leads to the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 + \mu\big(-\beta_1 x_2 - \beta_3 x_2^3 + F\cos\omega t + u(t)\big). \end{aligned} \tag{7.3}$$

The excitation z is C^1 and its derivative is bounded, so Proposition 7.1 is applicable. The Melnikov functions $\Delta_\pm$ of system (7.3) can easily be computed using the residue theorem:

$$\Delta_\pm(t_0) = -\frac{2\sqrt{2}\beta_1}{3\alpha} - \frac{8\sqrt{2}\beta_3}{35\alpha^2} \pm \frac{\sqrt{2}\pi\omega F}{\sqrt{\alpha}\sinh\frac{\pi\omega}{\sqrt{2}}} \cdot \cos\omega t_0.$$

The Melnikov functions $\Delta_\pm$ have simple zeros if and only if F exceeds a certain *critical amplitude* F_c, i. e. if $F > F_c := A^{-1}B$ for

$$A := \frac{\sqrt{2}\pi\omega}{\sqrt{\alpha}\sinh\frac{\pi\omega}{\sqrt{2}}} \text{ and } B := \frac{2\sqrt{2}\beta_1}{3\alpha} + \frac{8\sqrt{2}\beta_3}{35\alpha^2}.$$

COROLLARY 7.2. *If $F > F_c$, system* (7.3) *has a heteroclinic control set for sufficiently small μ.*

Proof. This follows from Proposition 7.1. ☐

As the excitation is T-periodic for $T := 2\pi/\omega$, it is possible to compute fibers of control sets by looking at the discrete control system given by the time-T map. For the following computations we restrict our view to the parameter values $\alpha = 0.674$, $\beta_1 = 0.231$ and $\beta_3 = 0.375$ (see [11] for a discussion of these parameters and this choice) and choose $\omega = 2.5$ and $\rho = 1.0$. Then $F_c \approx 5.62880$, so let $F := 6 > F_c$. Figure 7.1 shows the fiber in phase 0 for $\varepsilon = 0.1$. The control sets were approximated with the *graph algorithm* (see Dellnitz/Junge [6], Szolnoki [20]) using the implementation in GAIO[1]. For a spatial discretization into boxes, this algorithm computes strongly connected components of an associated graph whose nodes are given by the boxes and whose edges indicate reachability. The union of the resulting boxes is an approximation to a chain control set; as noted above, for system (7.1) the chain control sets typically coincide with the closures of control sets. Note that this figure shows the fiber of two control sets: an invariant control set around the origin (black) and the heteroclinic

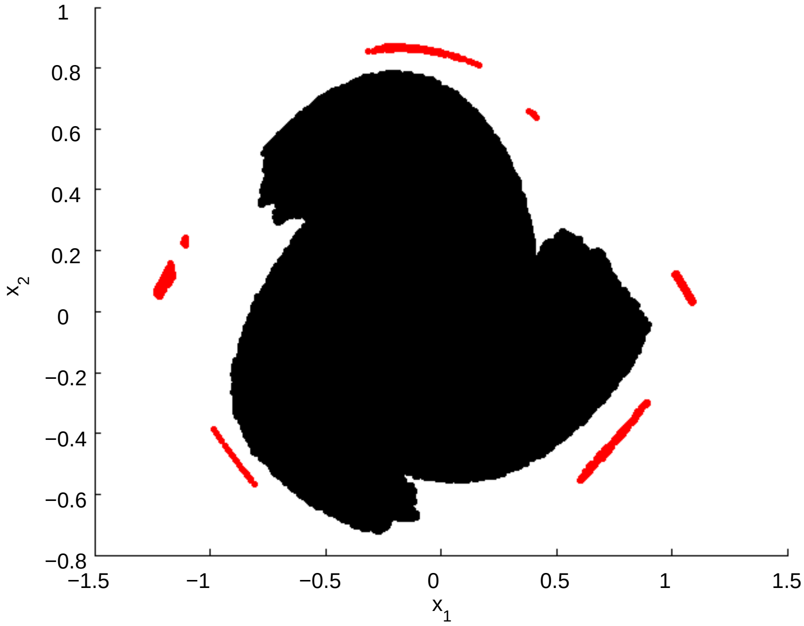

FIGURE 7.1. *Fiber of control sets for the periodically excited system* (7.3). *Computed in phase 0 for* $\alpha = 0.674$, $\beta_1 = 0.231$, $\beta_3 = 0.375$, $\omega = 2.5$, $\rho = 1.0$, $F = 6$ *and* $\varepsilon = 0.1$.

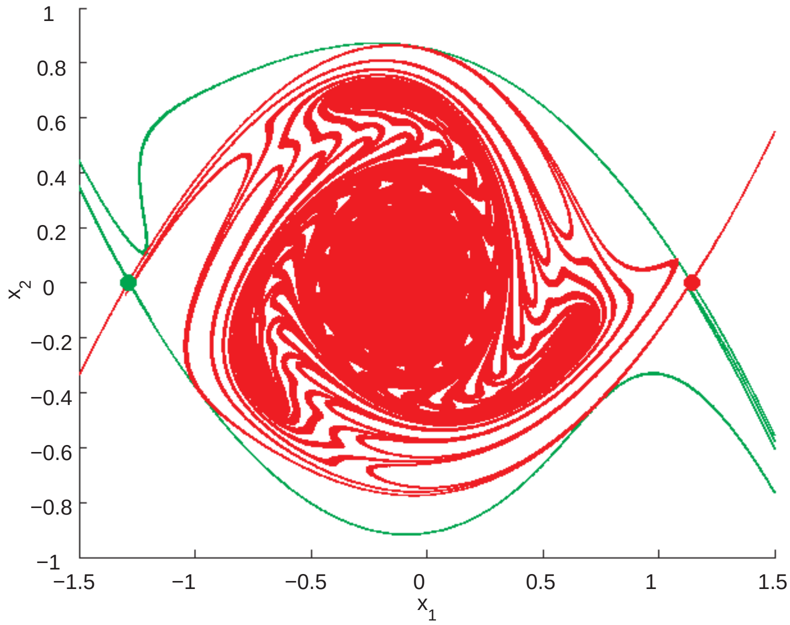

FIGURE 7.2. *Stable and unstable manifolds for the uncontrolled periodically excited system* (7.3). *Computed in phase 0 for* $\alpha = 0.674$, $\beta_1 = 0.231$, $\beta_3 = 0.375$, $\omega = 2.5$, $F = 6$ *and* $\varepsilon = 0.1$.

control set (red). Compare this to Figure 7.2, where the stable and unstable manifolds for these parameter values are shown, again for $\varepsilon = 0.1$ and in phase 0.

Next we examine quasi-periodic excitations of the form $z(t) := F\cos\omega_1 t + F\sin\omega_2 t$ for positive parameters F, ω_1, ω_2, which leads to the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 + \mu\big(-\beta_1 x_2 - \beta_3 x_2^3 + F\cos\omega_1 t + F\sin\omega_2 t + u(t)\big). \end{aligned} \tag{7.4}$$

The excitation z again is C^1 and its derivative is bounded. The Melnikov functions $\Delta_\pm$ of system (7.4) are

$$\Delta_\pm(t_0) = -\frac{2\sqrt{2}\beta_1}{3\alpha} - \frac{8\sqrt{2}\beta_3}{35\alpha^2} \pm \frac{\sqrt{2}\pi F}{\sqrt{\alpha}} \left(\frac{\omega_1 \cos\omega_1 t_0}{\sinh\frac{\pi\omega_1}{\sqrt{2}}} + \frac{\omega_2 \sin\omega_2 t_0}{\sinh\frac{\pi\omega_2}{\sqrt{2}}} \right).$$

The Melnikov function $\Delta_\pm$ has a simple zero if $F > F_c := A^{-1}(S_1 + S_2)^{-1}B$ for

$$A := \frac{\sqrt{2}\pi}{\sqrt{\alpha}}, S_i := \frac{\omega_i}{\sinh\frac{\pi\omega_i}{\sqrt{2}}}, i = 1, 2, \text{ and } B := \frac{2\sqrt{2}\beta_1}{3\alpha} + \frac{8\sqrt{2}\beta_3}{35\alpha^2}.$$

COROLLARY 7.3. *If $F > F_c$, system (7.4) has a heteroclinic control set for sufficiently small μ.*

Proof. This follows from Proposition 7.1. ∎

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