Controllability Properties of Nonlinear Behaviors

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Abstract

This paper proposes a topological framework for the analysis of the time shift on behaviors. It is shown that controllability is not a property of the time shift, while chain controllability is. This also leads to a global decomposition.

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1 Introduction

The analysis of differential equations and, more generally, dynamical systems, via the time shift on a space of trajectories is a classical approach going at least back to the work of Bebutov [1] in 1940 and has fostered the development of topological dynamics, compare Sell [10]. In control theory, the analysis of input and output functions has a long tradition. A new paradigm, called the behavioral approach to control, has been introduced by Willems [12] considering systems interacting with the environment without making a difference between inputs and outputs. However, so far, this latter theory has essentially been restricted to an algebraic framework. The present paper aims at the analysis of behaviors via topological dynamics of the time shift. It turns out that the basic notion of controllability does not directly lend itself to such an analysis. However, a weakened version, chain controllability, is intimately related to the classical notion of chain transitivity in topological dynamics (see, e.g., Easton [6], Robinson [9]). Additional assumptions allow us are to infer controllability from chain controllability.

Our initial goal[1] was to generalize the state space theory of control sets (i.e., of maximal controllable subsets) to input-output systems. This entails that instead of control flows (on the state space of the control system together with the input functions [2], [3]) pairs of input and output functions have to be considered. Now, in this setting, the difference between inputs and outputs turns out to be irrelevant. Thus it appears to us that behaviors provide an appropriate point of view. Clearly, this framework is much more general, and includes, in particular, many implicit systems. However, we will not pursue this direction in the present paper.

The contents are as follows: In Section 2, we consider shift invariant subsets of L_∞ endowed with the weak* topology. Restricted to compact subsets, the shift is continuous and we define and analyze controllable and chain controllable subsets in this context. In particular, maximal chain controllable sets are characterized as maximal chain transitive sets. In Section 3 topological behaviors are defined via filtrations in L_∞. A regular growth condition is used to show that, generically, controllability is obtained from chain controllability.

2 Control Sets and Chain Control Sets for the Time Shift

In this section, we study controllability properties for the time shift on subsets of $L_\infty-$spaces.

Let the time domain $\mathbb{T}$ be equal to $\mathbb{R}$ or $\mathbb{Z}$. Fix a $\sigma-$finite measure μ on the Lebesgue $\sigma-$algebra in $\mathbb{R}$ and define the time shift Θ by

$$\Theta : \mathbb{T} \times L_\infty(\mathbb{T}, \mathbb{R}^d, \mu) \to L_\infty(\mathbb{T}, \mathbb{R}^d, \mu),\ (t, w) \mapsto (\Theta_t w)(s) = w(t + s),\ s \in \mathbb{T}.$$

[1] A preliminary version of this paper appeared in [4]

A set $\mathcal{B} \subset L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ is $\Theta-$invariant if $w \in \mathcal{B}$ implies $\Theta_t w \in \mathcal{B}$ for every $t \in \mathbb{T}$. Recall that $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ is dual to $L_1(\mathbb{T}, \mathbb{R}^d, \mu)$ and that the weak* topology on $L_\infty(\mathbb{T}, \mathbb{R}^d \mu)$ is the weakest topology such that for all $\alpha \in L_1(\mathbb{T}, \mathbb{R}^d)$ the maps

$$L_\infty(\mathbb{T}, \mathbb{R}^d, \mu) \to \mathbb{R},\ w \mapsto \int_{\mathbb{T}} w(t)^T \alpha(t)\ \mu(dt)$$

are continuous. Fixing a countable dense subset $(\alpha_i) \subset L_1(\mathbb{T}, \mathbb{R}^d, \mu)$, the restriction of the metric

$$\mathrm{d}(v, w) = \sum_{i=1}^{\infty} 2^{-i} \frac{\left|\int_{\mathbb{T}} [v(t) - w(t)]^T \alpha_i(t)\ \mu(dt)\right|}{1 + \left|\int_{\mathbb{T}} [v(t) - w(t)]^T \alpha_i(t)\ \mu(dt)\right|}. \tag{1}$$

to a norm-bounded subset of $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ induces the weak* topology. Note also that every weak* compact subset of $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ is bounded.

Obviously, the time shift Θ on $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ defines a dynamical system, since $\Theta_{t+s} = \Theta_t \circ \Theta_s$ for all $s, t \in \mathbb{T}$ and $\Theta_0 = \mathrm{id}$. The following proposition shows that the restriction to compact subsets is continuous. It follows from a minor modification of the proof of Lemma 4.2.4 in [3].

Proposition 2.1 *Let $\mathcal{K}$ be a weak* compact subset of $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$. Then the restriction of the time shift $\Theta : \mathbb{T} \times \mathcal{K} \to L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ is weak* continuous.*

In the following we suppose that a $\Theta-$invariant subset $\mathcal{B} \subset L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ is given. We define controllability for the time shift by adapting a proposal by Jan Willems to our situation.

Definition 2.2 *For $v \in \mathcal{B}$ the positive orbit at time $T > 0$ is defined as*

$$\mathcal{O}_T^+(v) = \left\{ w \in \mathcal{B},\ \text{there is } w_1 \in \mathcal{B} \text{ with } w_1(t) = \left\{ \begin{array}{ccc} v(t) & \text{for} & t \leq 0 \\ w(t-T) & \text{for} & t \geq T \end{array} \right\} \right.,$$

and the positive orbit is defined by

$$\mathcal{O}^+(v) = \bigcup_{T>0} \mathcal{O}_T^+(v).$$

Of particular interest are subsets of complete controllability defined as follows.

Definition 2.3 *A control set is a nonvoid maximal* $\Theta-$*invariant subset* $\mathcal{D} \subset \mathcal{B}$ *with* $\mathcal{D} \subset \mathcal{O}^+(v)$ *for all* $v \in \mathcal{D}$.

More explicitly, a nonvoid $\Theta-$invariant subset $\mathcal{D}$ is a control set if for all $v, w \in \mathcal{D}$ there are $w_1 \in \mathcal{B}$ and a time $T > 0$ satisfying

$$w_1(t) = \begin{cases} v(t) & \text{for} \quad t \leq 0 \\ w(t-T) & \text{for} \quad t \geq T \end{cases},$$

and every set $\mathcal{D}'$ with $\mathcal{D} \subset \mathcal{D}' \subset \mathcal{B}$ with this property satisfies $\mathcal{D}' = \mathcal{D}$.

Remark 2.4 *Using maximality, it is easy to show that* w_1 *with the property above is also in* $\mathcal{D}$. *Note also that the controllability property is required within the set* $\mathcal{B} \subset L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$.

As an example we consider input-output pairs of continuous-time, control-affine systems in $\mathbb{R}^n$ described by

$$\begin{aligned} \dot{x}(t) &= f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \\ y(t) &= h(x(t), u(t)) = h_0(x(t)) + \sum_{i=1}^m u_i(t) h_i(x(t)), \end{aligned} \tag{2}$$

with inputs (u_i) taking values in $\mathbb{R}^m$; furthermore, the f_i are smooth ($C^\infty-$) vector fields and the output functions $h_i : \mathbb{R}^n \to \mathbb{R}^k$ are also smooth. Assume that for every $x \in \mathbb{R}^n$ and every input $u \in L_\infty(\mathbb{R}, \mathbb{R}^m)$ (here we take the Lebesgue measure λ on $\mathbb{R}$) there exists a unique absolutely continuous global solution $\varphi(t, x, u),\ t \in \mathbb{R}$. Also denote

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}$$

where U is a subset of $\mathbb{R}^m$.

Proposition 2.5 *Consider the input-output system (2). For every compact set* $K \subset \mathbb{R}^n$ *and every compact convex set* $U \subset \mathbb{R}^m$ *the sets*

$$\mathcal{B}_{K \times U} = \left\{ (u, y) \in L_\infty(\mathbb{R}, \mathbb{R}^{m+k}), \begin{array}{c} u \in \mathcal{U} \textit{ and there is } x \in \mathbb{R}^n \textit{ s.t. for } t \in \mathbb{R} \\ \varphi(t, x, u) \in K \textit{ and } y(t) = h(\varphi(t, x, u), u(t)) \end{array} \right\}$$

are weak compact and shift invariant.*

Proof. Shift invariance is clear by definition. The set $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ is weak* compact (compare [3, Lemma 4.2.1]) and the map

$$(t, x, u) \mapsto (\varphi(t, x, u), u(t + \cdot)) : \mathbb{R} \times K \times \mathcal{U} \to K \times \mathcal{U}$$

is continuous, uniformly on bounded intervals in $\mathbb{R}$, by [3, Lemma 4.3.2]. Then the desired compactness follows, since K is compact and h is control affine. ■

Remark 2.6 *For control-affine systems (2), a control set D with nonvoid interior in the state space $\mathbb{R}^n$ is defined a maximal set of approximate controllability [3]. If the system is locally accessible, exact controllability in the interior follows and one easily sees that the following set is contained in a control set $\mathcal{D}$ in the sense of Definition 2.3:*

$$\left\{ (u, y) \in \mathcal{B}, \begin{array}{c} \textit{there is } x_0 \in \operatorname{int} D \textit{ with } \varphi(t, x_0, u) \in \operatorname{int} D \\ \textit{and } y(t) = h(\varphi(t, x_0, u), u(t)) \textit{ for all } t \in \mathbb{R} \end{array} \right\}.$$

In fact, the controllability property is immediate. For input-state systems (i.e., $y = x$) note the difference to the lift of a control set as defined in [3] where, instead of input-trajectory pairs, the closure of the set of pairs $(u, x_0) \in \mathcal{U} \times \operatorname{int} D$ with $\varphi(t, x_0, u) \in \operatorname{int} D$ for all t is considered.

Example 2.7 *Consider a linear control systems $\dot{x} = Ax + Bu$ with $u(t) \in U$, for a compact and convex subset $U \subset \mathbb{R}^m$ containing the origin in its interior. Here it is known (see Colonius and Spadini [5]) that there exists a unique bounded control set in $\mathbb{R}^n$ if the pair (A, B) is controllable and A is hyperbolic. This implies that for $K \subset \mathbb{R}^n$ large enough, system (9) has a control set obtained as in Remark 2.6.*

For control-affine state space systems, control sets can be characterized [3] as maximal topologically transitive sets of an associated dynamical system on $\mathcal{U} \times \mathbb{R}^n$ provided that local accessibility holds (a topologically transitive set is the $\omega-$limit set of one of its elements). The following simple example shows that one cannot, in general, expect that control sets, as defined above, coincide with the maximal topologically transitive sets.

Example 2.8 *Consider a scalar control system given by*

$$\dot{x} = u(t) f(x)$$

with

$$f(x) = \begin{cases} 0 & \text{for} \quad |x| \geq 1 \\ > 0 & \text{for} \quad |x| < 1 \end{cases} \quad \text{and } u(t) \in U = [-1, 1].$$

There exists a control $u_0 \in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}),\ u(t) \in [-1, 1]$ *for all* $t\}$ *such that for the corresponding trajectory starting in the origin there are* $t_k, s_k \to \infty$ *with*

$$\varphi(t_k, 0, u_0) = 0 \text{ and } \varphi(s_k, 0, u_0) \to 1.$$

Then one may choose u_0 *such that*

$$\omega(u_0, \varphi(\cdot, 0, u_0)) \subset L_\infty(\mathbb{R}, \mathbb{R} \times \mathbb{R})$$

is topologically transitive. Clearly, it not contained in a control set. Note also that it has nonvoid intersection with the control set

$$\{(u, x_0) \in L_\infty(\mathbb{R}, \mathbb{R} \times \mathbb{R}),\ u(t) \in [-1, 1] \text{ for all } t \text{ and } x_0 \equiv 1\}.$$

The definition of a behavioral control set requires that one can precisely 'hit' the function w after some time. It may appear natural to introduce the following weaker concept, in analogy to chain controllability in the state space. Hopefully, also in the present situation this will lead to sets, which are better behaved. Observe that again this definition is *not* given in the flow context; it is strictly analogous to the definition of control sets. First observe that a notion of approximate controllability is obtained by requiring that the following semi-distance on $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$

$$\mathrm{d}^+(v, w) = \mathrm{d}(\chi_{[0,\infty)} \cdot v, \chi_{[0,\infty)} \cdot w)$$

is small; here $\chi_{[0,\infty)}$ is the characteristic function of $[0, \infty)$. Thus this only takes into account the future. Then we define sets of chain controllability in the following way.

Definition 2.9 *For* $\varepsilon, T > 0$ *an* $(\varepsilon, T)^+-$*chain from* $v \in \mathcal{B}$ *to* $w \in \mathcal{B}$ *is given by*

$$n \in \mathbb{N}, w_0 = v, w_1, ..., w_n = w \in \mathcal{B}, T_0, ..., T_{n-1} \geq T,$$

such that

$$\mathrm{d}^+(\Theta_{T_i}(w_i), w_{i+1}) < \varepsilon \text{ for all } i.$$

If for all $\varepsilon, T > 0$ *there is an* $(\varepsilon, T)^+-$*chain from* $v \in \mathcal{B}$ *to* $w \in \mathcal{B}$*, we say that* v *is chain controllable to* w*. The chain orbit of* $v \in \mathcal{B}$ *is*

$$\mathcal{O}_c^+(v) := \left\{ w \in \mathcal{B},\ \text{for all } \varepsilon > 0 \text{ there is an } (\varepsilon, 1)^+ - \text{chain from } v \text{ to } w \right\}.$$

We will consider maximal subsets which are chain controllable.

Definition 2.10 *A nonvoid invariant subset $\mathcal{E} \subset \mathcal{B}$ is a chain control set if it is a maximal set such that for all $v, w \in \mathcal{E}$ and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)^+-$chain in $\mathcal{E}$ from v to w.*

For these sets, contrary to control sets, we will be able to provide a complete characterization in terms of the flow. Recall from the theory of dynamical systems (see [9]) that an $(\varepsilon, T)-$chain for a continuous flow is defined as in Definition 2.9, but with the semidistance d^+ replaced by the distance d in the metric space. They give rise to chain transitive sets in analogy to Definition 2.10. Furthermore the restriction of the time shift to a compact shift invariant subset of L_∞ defines a continuous flow on a compact metrizable space.

Theorem 2.11 *Let $\mathcal{B} \subset L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ be given. A nonempty compact and shift invariant set $\mathcal{E} \subset \mathcal{B}$ is a chain control set if and only if the restriction of the shift to $\mathcal{E}$ is chain transitive and $\mathcal{E}$ is a maximal set with this property, i.e., if $\mathcal{E} \subset \mathcal{E}' \subset \mathcal{B}$ and $\mathcal{E}'$ is compact and invariant such that the shift restricted to $\mathcal{E}'$ is chain transitive, then $\mathcal{E} = \mathcal{E}'$.*

Proof. Suppose that $\mathcal{E}$ is a chain control set. Let $v, w \in \mathcal{E}$ and pick ε, $T > 0$. Recall the definition of the metric d on $\mathcal{E}$ and choose $k \in \mathbb{N}$ large enough such that

$$\sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon. \tag{3}$$

For the finitely many $\alpha_1, ..., \alpha_k \in L_1(\mathbb{T}, \mathbb{R}^d, \mu)$ there is $S > 0$ such that for all i

$$\int_{\mathbb{T}\setminus[-S,S]} |\alpha_i(\tau)| \ \mu(d\tau) < \frac{\varepsilon}{\mathrm{diam}\mathcal{E}}. \tag{4}$$

We may assume without loss of generality that $T > S$. Chain controllability from v to $w(-S + \cdot)$ yields the existence of $n \in \mathbb{N}$ and $v_0, ..., v_n \in \mathcal{E}$, $T_0, ..., T_{n-1} > T + S$ with $v_0 = v$, $v_n = w(-S + \cdot)$ and

$$\mathrm{d}^+(\Theta_{T_i} v_i, v_{i+1}) < \varepsilon \text{ for } j = 0, ..., n-1. \tag{5}$$

Now construct an (ε, T)-chain from v to w in the following way ('we jump later'). Define

$$w_0 = v, \ w_j = \Theta_S v_j \text{ for } j = 1, ..., n-1, \ w_n = \Theta_S v_n = w,$$

and let the jump times be $t_j = T_j + S$. Then

$$\begin{aligned}
\mathrm{d}(\Theta_{t_0} w_0, w_1) &= \mathrm{d}(\Theta_{T_0+S} v, \Theta_S v_1) \\
&= \sum_{i=1}^{\infty} 2^{-i} \frac{\left| \int_{\mathbb{T}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right|}{1 + \left| \int_{\mathbb{T}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right|} \\
&\leq \sum_{i=1}^{k} 2^{-i} \frac{\left| \int_{\mathbb{T}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right|}{1 + \left| \int_{\mathbb{T}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right|} + \varepsilon.
\end{aligned}$$

Now for $i = 1, ..., k$

$$\begin{aligned}
&\left| \int_{\mathbb{T}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right| \\
&\leq \int_{\mathbb{T}\setminus[-S,S]} |\alpha_i(t)|\ \mu(dt)\ 2\mathrm{diam}\mathcal{E}\ + \left| \int_{-S}^{S} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t)\ \mu(dt) \right| \\
&< 2\varepsilon + \left| \int_0^{2S} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \mu(dt) \right| < 5\varepsilon,
\end{aligned}$$

since by (5) and (4)

$$\begin{aligned}
&\left| \int_0^{2S} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| \\
&= \left| \int_0^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) - \int_{2S}^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| \\
&\leq \left| \int_0^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

Thus

$$\mathrm{d}(\Theta_{t_0} w_0, w_1) < 6\varepsilon.$$

Analogously, one shows that $\mathrm{d}(\Theta_{t_j} w_j, w_{j+1}) < 6\varepsilon$ for all $j = 1, ..., n-1$. This proves that the restriction of Θ to the chain control set $\mathcal{E}$ is chain transitive.

Conversely, suppose that $\mathcal{E}$ is a chain transitive set, and let $v, w \in \mathcal{E}$. By assumption one finds for all $\varepsilon, T > 0$ an $(\varepsilon, T)-$chain given by $v_0 = v, v_1, ..., v_n = \Theta_S w$ in $\mathcal{E}$ and $T_0, ..., T_{n-1} > T$ from v to w with

$$\mathrm{d}(\Theta_{T_i}(v_i, v_{i+1}) < \varepsilon. \tag{6}$$

We may assume that conditions (3) and (4) are satisfied and that $T_j - S > T$. This gives rise to an $(\varepsilon, T)^+-$chain in the following way ('we jump earlier').

Define

$$w_0 = v,\ w_j = \Theta_{-S} v_j \text{ for } j = 1, ..., n-1,\ w_n = \Theta_{-S} v_n = w,$$

and let the jump times be $t_j = T_j - S$. Then

$$\begin{aligned}
&\mathrm{d}^+(\Theta_{t_0} w_0, w_1) = \mathrm{d}^+(\Theta_{T_0 - S} v, \Theta_{-S} v_1) \\
&= \sum_{i=1}^{\infty} 2^{-i} \frac{\left| \int_0^\infty [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right|}{1 + \left| \int_0^\infty [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right|} \\
&\le \sum_{i=1}^{k} 2^{-i} \frac{\left| \int_0^\infty [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right|}{1 + \left| \int_0^\infty [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right|} + \varepsilon.
\end{aligned}$$

Now for $i = 1, ..., k$

$$\begin{aligned}
&\left| \int_0^\infty [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right| \\
&\le \int_{2S}^\infty |\alpha_i(t)|\ \mu(dt)\ 2\mathrm{diam}\mathcal{E}\ + \left| \int_0^{2S} [v(t+T_0-S) - v_1(t-S)]^T \alpha_i(t)\ \mu(dt) \right| \\
&< 2\varepsilon + \left| \int_{-S}^{S} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| < 5\varepsilon,
\end{aligned}$$

since by (6) and (4)

$$\begin{aligned}
&\left| \int_{-S}^{S} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| \\
&= \left| \int_{\mathbb{T}} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) - \int_{\mathbb{T}\setminus[-S,S]} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| \\
&\le \left| \int_{\mathbb{T}} [v(t+T_0) - v_1(t)]^T \alpha_i(t)\ \mu(dt) \right| + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

Thus

$$\mathrm{d}^+(\Theta_{t_0} w_0, w_1) < 6\varepsilon.$$

Analogously, one shows that $\mathrm{d}^+(\Theta_{t_j} w_j, w_{j+1}) < 6\varepsilon$ for all $j = 1, ..., n-1$.

It only remains to show the maximality properties. A chain control set $\mathcal{E}$ is a maximal chain transitive set: In fact, suppose that the restriction of Θ to $\mathcal{E}' \supset \mathcal{E}$ is chain transitive. Then it follows that $\mathcal{E}' = \mathcal{E}$, since chain transitivity of $\mathcal{E}'$ implies, as just proven, that $\mathcal{E}'$ is chain controllable and $\mathcal{E}$ is a maximal chain controllable set. In the same way, one sees that a chain control set $\mathcal{E}$ is a maximal set with the property that the restriction of Θ to $\mathcal{E}$ is chain transitive. ■

Remark 2.12 *(On connections) Let $\mathcal{B}^1$ and $\mathcal{B}^2$ be $\Theta-$invariant subsets of $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ given by input-output pairs (u_i, y_i). Suppose that $\mathcal{E}^1$ and $\mathcal{E}^2$ are chain control sets of $\mathcal{B}^1$ and $\mathcal{B}^2$, respectively, and the intersection*

$$\{y_1,\ \textit{there is } u_1 \textit{ with } (u_1, y_1) \in \mathcal{E}^1\} \cap \{u_2,\ \textit{there is } y_2 \textit{ with } (u_2, y_2) \in \mathcal{E}^2\} \quad (7)$$

is nonvoid and compact. Then this set has nonvoid intersection with a chain control set of the new behavior

$$\mathcal{B}^1 * \mathcal{B}^2 := \{(u_1, y_2),\ \textit{there is } y_1 = u_2 \textit{ with } (u_1, y_1\} \in \mathcal{B}^1 \textit{ and } (u_2, y_2) \in \mathcal{B}^2\}.$$

This follows since the set in (7) is compact and $\Theta-$invariant (but not necessarily chain transitive). Hence it contains a chain transitive subset.

Having identified the chain control sets as the maximal chain transitive sets, one obtains the following result on global decompositions from Conley's Fundamental Theorem. The stable set of a closed $\Theta-$invariant set Y is defined as

$$W^s(Y) = \{x \in X,\ \mathrm{d}(\Theta_t w, Y) \to 0 \text{ for } t \to \infty\}.$$

Corollary 2.13 *Let $\mathcal{K}$ be a compact subset of a $\Theta-$invariant set $\mathcal{B} \subset L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$. Then $\mathcal{K}$ is the disjoint union of the stable sets of the chain control sets in $\mathcal{K}$ together with the set $\mathcal{L}$ of the points in $\mathcal{K}$ leaving $\mathcal{K}$ in positive time.*

Proof. For discrete time systems, i.e., $\mathbb{T} = \mathbb{Z}$, this is proved in Easton [6] (with slightly different, but equivalent notions). For $\mathbb{T} = \mathbb{R}$, one has to observe that in a maximal chain transitive set one may take all jump times equal to 1 (the relevant arguments are given e.g. in Szolnoki [11]). Thus the continuous time case can be reduced to the discrete time case. ■

3 Topological Behaviors and Regular Growth

In this section, we define topological behaviors, and show that under a regular growth condition chain control sets and control sets generically coincide.

Recall that a *quasi-order* on a set A is a relation $\leq$ that is reflexive and transitive. As usual, we write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. The following examples where the order is given by inclusion will be relevant.

Example 3.1 *Let* $\mathrm{co}(\mathbb{R}^d)$ *denote the family of all compact subsets of* $\mathbb{R}^d$*. For* $K, L \in \mathrm{co}(\mathbb{R}^d)$*, define a quasi-order by* $L \leq K$ *if* $L \subset K$*. Other examples are the family* $\mathrm{Co}_0(\mathbb{R}^d)$ *of all compact convex subsets of* $\mathbb{R}^d$ *that contain the origin in their interior, or the set of all compact subsets of* $\mathbb{R}^d$*. If we fix* $K \in \mathrm{Co}_0(\mathbb{R}^d)$*, the family of sets* $K^\rho := \rho \cdot K, \rho > 0$*, is quasi-ordered. In product spaces also combinations of these quasi-ordered sets yield quasi-ordered sets.*

Now we define the central notion of this paper.

Definition 3.2 *A topological behavior* $\mathcal{B}$ *is a* $\Theta-$*invariant subset of* $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ *together with a filtration* $\{\mathcal{B}_\alpha\}_{\alpha \in A}$ *in the following sense: The sets* $\mathcal{B}_\alpha$ *are weak* compact in* $L_\infty(\mathbb{T}, \mathbb{R}^d, \mu)$ *and* A *is quasi-ordered with*

$$\mathcal{B} = \bigcup_{\alpha \in A} \mathcal{B}_\alpha \text{ and } \alpha \leq \beta \text{ in } A \text{ implies } \mathcal{B}_\alpha \subset \mathcal{B}_\beta.$$

The following proposition shows how topological behaviors arise from control-affine input-output systems.

Proposition 3.3 *Consider system (2) and assume that there is a real function* $\alpha(t) \geq 1,\ t \in \mathbb{R}$*, which is locally Lebesgue integrable with the following property: for every compact set* $K \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ *there is* $\alpha_{K,U} > 0$ *such that for all* $x \in K$ *and* $u \in \mathcal{U}$

$$|h(\varphi(t, x, u), u(t))| \leq \alpha(t)\alpha_{K,U} \text{ for almost all } t \in \mathbb{R}. \tag{8}$$

Define a measure μ *by* $\mu = \alpha(\cdot)^{-1}\lambda$*. Then*

$$\mathcal{B} = \left\{ (u, y) \in L_\infty(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^k, \mu),\ \begin{array}{c} u \in L_\infty(\mathbb{R}, \mathbb{R}^m) \text{ and there is } x \in \mathbb{R}^n \\ \text{with } y(t) = h(\varphi(t, x, u), u(t)) \text{ for } t \in \mathbb{R} \end{array} \right\}$$

is a topological behavior with filtration given by the sets $\mathcal{B}_{K \times U}$ *defined in Proposition 2.5 and* $A = \{K \times U,\ K \in \mathrm{co}(\mathbb{R}^n) \text{ and } U \in \mathrm{Co}_0(\mathbb{R}^m)\}$ *quasi-ordered by inclusion.*

Proof. Observe that the assumptions on α guarantee that $0 < \alpha(t)^{-1} \leq 1$ and hence $L_p(\mathbb{R},\mathbb{R}^m) \subset L_p(\mathbb{R},\mathbb{R}^m,\alpha(\cdot)^{-1}\lambda), p = 1,\infty$. Furthermore note that for every $x \in \mathbb{R}^n$ and $u \in L_\infty(\mathbb{R},\mathbb{R}^m)$ the output $y \in L_\infty(\mathbb{R},\mathbb{R}^k,\mu)$, since

$$|y(t)|\,\alpha(t)^{-1} = |h(\varphi(t,x,u),u(t)|\,\alpha(t)^{-1} \leq \alpha_{K,U}$$

where U is taken as the ball around the origin with radius $\|u\|_\infty$. Invariance of $\mathcal{B}$ is obvious by definition. For a compact and convex subset U the set $\mathcal{U}$ is also a weak* compact subset of $L_\infty(\mathbb{R},\mathbb{R}^m,\mu)$ (compare [3, Lemma 4.2.1]). Then the other assertions follow from Proposition 2.5. ■

The following example shows that observed linear control systems define a behavior in the sense above.

Example 3.4 *Consider*

$$\dot{x} = Ax + Bu,\;\; y = Cx + Du \tag{9}$$

with matrices A,B,C,D *of appropriate dimensions. Let* $\lambda_{\max} := \max\{|\mathrm{Re}\,\nu|\}$, *where the maximum is taken over the eigenvalues* ν *of* A. *There is a constant* $c_0 > 0$ *such that, by the variations-of-constants formula,*

$$\|\varphi(t,x_0,u)\| \leq c_0 e^{\lambda_{\max}|t|}\left[\|x_0\| + \|B\|\,\|u\|_\infty\right] + \|B\|\,\|u\|_\infty\,,\;\; t \in \mathbb{R}.$$

Since

$$|h(\varphi(t,x,u),u(t))| \leq \|C\|\,\|\varphi(t,x_0,u)\| + \|D\|\,\|u(t)\|\,,$$

this furnishes the desired estimate (8) with $\alpha(t) = \max\{1, e^{\lambda_{\max}|t|}\}$ *and*

$$\alpha_{K,U} = \max\{1, c_0\,\|C\|\,[\|x_0\| + \|B\|\,\|u\|] + \|D\|\,\|u\|\,,\;\; x_0 \in K \text{ and } u \in U\}.$$

Thus Proposition 3.3 describes the input-output behavior of system (9).

For a topological behavior $\mathcal{B}$, one can study the controllability properties of each of the (not-invariant) sets $\mathcal{B}_\alpha$. We denote the corresponding objects by an index α; e.g. $\mathcal{O}^{\alpha,+}(v)$ denotes the positive orbit in $\mathcal{B}_\alpha$. The condition of regular growth formulated below allows us to show that, generically, control sets and chain control sets coincide.

Definition 3.5 *For a topological behavior* $\mathcal{B}$, *let* $\alpha \in A$. *A function* $v \in \mathcal{B}_\alpha$ *is inner if there is* $T > 0$ *such that for all* $\beta > \alpha$ *there is* $\varepsilon > 0$ *such that*

$$\mathrm{d}^+(w,\Theta_T v) < \varepsilon \text{ in } \mathcal{B}_\alpha \text{ implies } w \in \mathcal{O}^{\beta,+}_{\leq T}(v).$$

A behavior $\{\mathcal{B}_\alpha\}_{\alpha\in A}$ *has regular growth if for all* $\alpha \in A$ *the elements of* $\mathcal{B}_\alpha$ *are uniformly inner, i.e., one may choose* $\varepsilon = \varepsilon(\beta,T)$ *independent of* $v \in \mathcal{B}_\alpha$.

The following proposition shows that for control affine systems the regular growth condition reduces to a condition in the state space.

Proposition 3.6 *Consider a topological behavior associated to a control-affine system as in Proposition 3.3 with $h(x,u) = x$ and filtration given by*

$$K \times \{\rho U, \rho \geq 0\},$$

where $K \in \mathrm{co}(\mathbb{R}^n)$ and $U \in \mathrm{Co}_0(\mathbb{R}^m)$ are fixed (thus $A = \{\rho \geq 0\}$). Then for $\rho \geq 0$ and $u \in \rho \cdot \mathcal{U}, x \in \mathbb{R}^n$ the following conditions are equivalent:
(i) The input-state pair $v = (u, \varphi(\cdot, x, u))$ with $u \in \rho \cdot \mathcal{U}$ is inner for the behavior specified above.
(ii) There is $T > 0$ such that for all $\rho' > \rho$

$$\varphi(T, x, u) \in \mathrm{int}\{\varphi(t, x, u'),\ 0 \leq t \leq T \text{ and } u' \in \rho' \cdot \mathcal{U}\}.$$

Proof. Suppose that (ii) holds. Then there is $T > 0$ such that for $\rho' > \rho$

$$\varphi(T, x, u) \in \mathrm{int}\{\varphi(t, x, u'),\ 0 \leq t \leq T \text{ and } u' \in \rho' \cdot \mathcal{U}\}.$$

Consider $w_k = (\varphi(\cdot, x_k, u_k), u_k)$ with $u_k \in \rho \cdot \mathcal{U}$ and let

$$\mathrm{d}^+(w_k, \Theta_T v) < \varepsilon_k.$$

We claim that for $\varepsilon_k \searrow 0$ it follows that $x_k \to \varphi(T, x, u)$. For every subsequence there is $x^* \in K$ with $x_k \to x^*$. Now $\mathrm{d}^+(w_k, \Theta_T v) \to 0$ implies weak* convergence $u_k \to u(T + \cdot)$ on $[0, \infty)$ and hence uniform convergence on bounded intervals of $\varphi(t, x_k, u_k)$ to $\varphi(t, x^*, u(T + \cdot))$. This implies weak* convergence and hence it follows that

$$\mathrm{d}^+(w_k, v^*) \to 0 \text{ with } v^* := (x^*, u(T + \cdot)).$$

Since the limit is unique on $[0, \infty)$, we obtain $x^* = \varphi(T, x, u)$ as claimed.

We conclude, using (ii), that for $\varepsilon > 0$, small enough, $\mathrm{d}^+(w, \Theta_T v) < \varepsilon$ implies that $x_k = \varphi(t, x, u^*)$ for some $t \in [0, T]$ and some $u^* \in \rho' \cdot \mathcal{U}$. This shows that (i) holds.

Conversely, suppose that (i) holds and let $\rho' > \rho$. Then there is $\varepsilon > 0$ such that $\mathrm{d}^+(w, \Theta_T v) < \varepsilon$ in $\mathcal{B}_\rho$ implies $w \in \mathcal{O}^{\rho',+}_{\leq T}(v)$. There is $\delta > 0$ such that $|y - \varphi(T + \cdot, x, u)| < \delta$ implies for $w := (\varphi(\cdot, y, u), u(T + \cdot))$ that

$$\mathrm{d}^+(w, (\varphi(T + \cdot, x, u), u(T + \cdot)) = \mathrm{d}^+(w, \Theta_T v) < \varepsilon.$$

Thus $w \in \mathcal{O}^{\rho',+}(v)$ and hence $y \in \mathcal{O}^{\rho',+}(x)$ as claimed. ■

Remark 3.7 *Condition (ii) in Proposition 3.6 has been used in order to analyze the relation between control sets and chain control sets for control-affine systems [3]. Gayer [7] shows that it is satisfied for a large class of control systems.*

We return to general topological behaviors $\mathcal{B}$. Let $\rho \mapsto \alpha(\rho) : [\rho_*, \rho^*) \to A$, $\rho_* < \rho^*$, be an increasing map. Then for all $v \in \mathcal{B}_{\alpha_0}$ the maps $\alpha \mapsto \mathrm{cl}\mathcal{O}^{\alpha,+}(v)$ and $\alpha \mapsto \mathcal{O}_c^{\alpha,+}(v)$ defined for $\alpha \geq \alpha_0$ are increasing.

Lemma 3.8 *Let $\rho \mapsto \alpha(\rho) : [\rho_*, \rho^*) \to A$, $\rho_* < \rho^*$, be an increasing map, such that $\rho \mapsto \mathcal{B}_{\alpha(\rho)}$ is Hausdorff continuous. Then for all but at most countably many $\rho-$values the maps $\rho \mapsto \mathrm{cl}\mathcal{O}^{\alpha(\rho),+}(v)$ and $\rho \mapsto \mathcal{O}_c^{\alpha(\rho),+}(v)$ are increasing and continuous.*

Proof. Monotonicity is obvious. The continuity assertion follows from Scherbina's Lemma ([8] or [3, Proposition B.1.5]) which states that increasing, compact-valued mappings defined on $[0, \infty)$ are continuous with respect to the Hausdorff metric for all but at most countably many $\rho-$values. ■

Proposition 3.9 *Let $\alpha \in A$ and suppose that the elements of $\mathcal{B}_\alpha$ are uniformly inner. Then $\mathcal{O}_c^{\alpha,+}(v) \subset \mathcal{O}^{\beta,+}(v)$ for all $v \in \mathcal{B}_\alpha$ and for all $\beta > \alpha$.*

Proof. Let $\beta > \alpha$. Then there is $\varepsilon > 0$ such that for $v \in \mathcal{B}_\alpha$

$$\mathrm{d}^+(w, \Theta_T v) < \varepsilon \text{ in } \mathcal{B}_\alpha \text{ implies } w \in \mathcal{O}_{\leq T}^{\beta,+}(v).$$

Thus the assertion follows since $\varepsilon > 0$ does not depend on the considered element of $\mathcal{B}_\alpha$. ■

Remark 3.10 *For the state space theory, [3, Lemma 4.5.5] uses local accessibility in a crucial way in order to show that inner pairs in an invariant compact set are uniformly inner.*

If $\mathcal{E}^\alpha$ is a chain control set and $\beta > \alpha$, there exists a unique chain control set $\mathcal{E}^\beta$ of $\mathcal{B}_\beta$ containing $\mathcal{E}^\alpha$; similarly for control sets. With this notation, the following results hold.

Lemma 3.11 *Let $\rho \mapsto \alpha(\rho) : [\rho_*, \rho^*) \to A$, $\rho_* < \rho^*$, be a strictly increasing map such that $\rho \mapsto \mathcal{B}_{\alpha(\rho)}$ is Hausdorff continuous. Then the corresponding control sets and chain control sets satisfy for all but at most countably many $\rho-$values*

$$\mathcal{E}^{\alpha(\rho)} = \bigcap_{\rho' > \rho} \mathcal{E}^{\alpha(\rho')} \text{ and } \mathrm{cl}\mathcal{D}^{\rho} = \bigcap_{\rho' > \rho} \mathrm{cl}\mathcal{D}^{\alpha(\rho')}.$$

Proof. This again is Scherbina's Lemma. ■

The next theorem shows that under the assumption of regular growth, generically the chain orbits are the closures of orbits, and, similarly, the control sets are the closures of control sets.

Theorem 3.12 *Suppose that $\mathcal{B}$ is a topological behavior with regular growth and let $\rho \mapsto \alpha(\rho) : [\rho_*, \rho^*) \to A, \rho_* < \rho^*$, be a strictly increasing map such that $\rho \mapsto \mathcal{B}_{\alpha(\rho)}$ is Hausdorff continuous. Then, abbreviating $\mathcal{E}^\rho := \mathcal{E}^{\alpha(\rho)}$ and $\mathcal{D}^\rho := \mathcal{D}^{\alpha(\rho)}$ one has*

(i) for all chain control sets $\mathcal{E}^{\rho_*}$ there are chain control sets $\mathcal{E}^\rho$ with $\mathcal{E}^{\rho^*} \subset \mathcal{E}^\rho$ for all $\rho > \rho_*$;

(ii) for all but at most countably many $\rho-$values the chain orbits are the closures of orbits and the chain control sets $\mathcal{E}^\rho$ are the closures of control sets $\mathcal{D}^\rho$.

Proof. Assertion (i) is obvious. For assertion (ii) observe that $\rho < \rho'$ implies

$$\mathcal{E}^\rho \subset \mathcal{D}^{\rho'},$$

since $\mathcal{E}^\rho \subset \mathcal{O}_c^{\rho,+}(v) \subset \mathcal{O}^{\rho',+}(v)$ for all $v \in \mathcal{E}^\rho$. Furthermore for all but at most countably many $\rho-$values

$$\mathrm{cl}\mathcal{D}^\rho \subset \mathcal{E}^\rho \subset \bigcap_{\rho'>\rho} \mathcal{D}^{\rho'} \subset \bigcap_{\rho'>\rho} \mathcal{E}^{\rho'} \subset \bigcap_{\rho'>\rho} \mathcal{D}^{\rho'} = \mathrm{cl}\mathcal{D}^\rho;$$

here the fourth inclusion follows since one finds for $\rho' > \rho$ a ρ'' with $\rho' > \rho'' > \rho$ and hence $\alpha(\rho') > \alpha(\rho'') > \alpha(\rho)$. ■

Remark 3.13 *A linear behavior is given by a behavior on a vector bundle $\mathcal{V}$ for which the time shifts are linear flows (for each $\alpha \in A$). An example are bilinear control systems (without output): As a special case of (3.3), consider*

$$\dot{x} = [A_0 + \sum_{i=1}^m u_i(t) A_i]x, \ y = x$$

with matrices $A_i \in \mathbb{R}^{n\times n}$ Then the input-state pairs

$$\{(u,x) \in L_\infty(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^n, \mu), \text{ there is } x_0 \in \mathbb{R}^n \text{ with } x(t) = \varphi(t, x_0, u) \text{ for } t \in \mathbb{R}\}$$

may be considered as a vector bundle over the chain transitive base space $\mathcal{U}$ (here the measure μ has a density with respect to Lebesgue measure determined by the maximal and and minimal Lyapunov exponents). Hence Selgrade's Theorem (see e.g. [3, Theorem 5.2.5]) implies that there are at most n chain control sets in the projective bundle; they form a finest Morse decomposition.

Next we show that a control set of an input-output system (2) uniquely determines a control set in the state space, if the input-output pair determines the state trajectory.

Proposition 3.14 *Consider a set $\mathcal{B}_{K\times U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^k, \mu)$ of input-output pairs as in Proposition 2.5 with $K \in \mathrm{co}(\mathbb{R}^n)$ and $U \in \mathrm{Co}_0(\mathbb{R}^m)$ and assume that for all $(u,y) \in \mathcal{B}_{K\times U}$ and all $T > 0$ there is a unique point $x \in \mathbb{R}^n$ such that*

$$y(t) = h(\varphi(t,x,u), u(t)), t \in [0,T].$$

Let $\mathcal{D} \subset \mathcal{B}_{K\times U}$ be a control set. Then there is a unique control set $D \subset \mathbb{R}^n$ such that $(u,y) \in \mathcal{D}$ implies that there is $x \in D$ with $y(t) = h(\varphi(t,x,u), u(t))$ for all $t \in \mathbb{R}$.

Proof. For $i = 0, 2$ let $(u_i, y_i) \in \mathcal{D}$. Then there are unique $x_i \in \mathbb{R}^n$ such that for all t

$$y_i(t) = h(\varphi(t, x_i, u_i), u_i(t)).$$

One can control from (u_0, y_0) to (u_2, y_2), i.e., one finds $(u_1, y_1) \in \mathcal{D}$ and $T > 0$ with

$$(u_1(t), y_1(t)) = \begin{cases} (u_0(t), y_0(t)) & \text{for } \ t \le 0 \\ (u_2(t-T), y_2(t-T)) & \text{for } \ t \ge T \end{cases}.$$

There is a unique $x_1 \in \mathbb{R}^n$ with

$$y_1(t) = h(\varphi(t, x_1, u_1), u_1(t)) \text{ for all } t \in \mathbb{R}.$$

Since, by our assumption, the corresponding point in $\mathbb{R}^n$ is already determined by any time interval, this implies that

$$x_1 = x_0 \text{ and } \varphi(T, x_1, u_1) = x_2.$$

Reversing the roles of (u_0, y_0) and (u_2, y_2), one sees that x_0 and x_1 lie in a control set $D \subset \mathbb{R}^n$. Using shift invariance of $\mathcal{D}$, one also sees that $\varphi(t, x_0, u_0) \in D$ for all $t \in \mathbb{R}$. Hence the inclusion follows. Uniqueness is clear. ■

Remark 3.15 *If local accessibility holds, then one has exact controllability in the interior of a control set in the state space. Thus for all x in the interior of a control set D in $\mathbb{R}^n$ there is a control set $\mathcal{D}$ with*

$\varphi(t, x, u) \in \text{int}D$ *and* $y(t) = h(\varphi(t, x, u), u(t))$ *for all* $t \in \mathbb{R}$ *implies* $(u, y) \in \mathcal{D}$.

Remark 3.16 *The assumption of Proposition 3.14 holds, e.g., for systems of the type*

$$\ddot{x} + g_0(x)\dot{x} + g_1(x) = u(t), \;\; y = x$$

with smooth real functions g_0, g_1. Here on any nontrivial interval the function $y(t) = x(t)$ determines the state $(x(t), \dot{x}(t))$ and hence the initial state $x \in \mathbb{R}^2$.

We add a simple example, where the behavioral control set is not just the projected lift of a control set in the state space.

Example 3.17 *Consider an observed affine control system*

$$\dot{x} = Ax + Bu + d, u(t) \in U, \;\; y = Cx. \tag{10}$$

Suppose that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-k} \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} I_k & 0 \end{pmatrix}, \; d = \begin{pmatrix} 0 \\ d_2 \end{pmatrix}.$$

Assume that $(A_1, B_1) \in \mathbb{R}^{k \times k} \times \mathbb{R}^{k \times m}$ is controllable and $0 \neq d_2 \in \mathbb{R}^{n-k}$, and U is compact and convex with $0 \in \text{int}U$. The trajectories of (10) with initial condition $x(0) = (x_1^0, x_2^0)^T$ are given by

$$\begin{aligned} \varphi(t, x, u) &= \begin{pmatrix} e^{A_1 t} x_1^0 \\ e^t x_2^0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{A_1(t-s)} B_1 u(s) \\ e^{(t-s)} d_2 \end{pmatrix} ds \\ &= \begin{pmatrix} e^{A_1 t} x_1^0 + \displaystyle\int_0^t e^{A_1(t-s)} B_1 u(s) \; ds \\ e^t x_2^0 + (e^t - 1)\, d_2 \end{pmatrix}. \end{aligned}$$

This system has no control set, since at least one of the last $n-k$ components is strictly monotone. On the other hand, the linear system

$$\dot{x}_1 = A_1 x_1 + B_1 u,\ \ u(t) \in U$$

has a unique control set $D_1 \subset \mathbb{R}^k$ (cp. [3, Example 3.2.16]). Denote the corresponding trajectories by $\varphi_1(t, x_1, u)$ and consider

$$\mathcal{D} = \{(u, y), \text{there is } x_1 \in D_1 \text{ with } y(t) = \varphi_1(t, x_1, u) \text{ for } t \in \mathbb{R}\}.$$

Clearly, for all (x_1, x_2)

$$\varphi_1(t, x_1, u) = e^{A_1 t} x_1 \int_0^t e^{A_1(t-s)} B_1 u(s)\ ds = C\varphi(t, (x_1, x_2), u).$$

Thus it easily follows that the subset $\mathcal{D} \subset L_\infty(\mathbb{R}, \mathbb{R}^{m+k}, \mu)$ (with μ as in Example 3.4) defined by

$$\mathcal{D} := \{(u, y),\ \text{there are } x_1 \in D_1 \text{ and } x_2 \in \mathbb{R}^{n-k} \text{ with } y = C\varphi(t, (x_1, x_2), u) \text{ for } t \in \mathbb{R}\}$$

is a control set for system (10).

Remark 3.18 *System (10) is not locally accessible. However, it can be easily modified in the following way: Replace B by*

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & I_k \end{pmatrix},$$

and replace the control range by

$$U \times [0, 1]^k.$$

Then the resulting system is locally accessible, while it has no control set. The other arguments remain valid.

Remark 3.19 *Using the positive orbits introduced in Definition 2.2, one might introduce a notion of accessibility of behaviors, as the property that the interior of $\mathcal{O}^+(v)$ is nonvoid. The consequences of such a property remain to be explored.*

4 Conclusions

The present paper introduces a topological notion of behaviors by considering the time shift on $L_\infty-$spaces. Subsets of complete controllability and chain controllability are studied. While controllability is not, in general, a notion of the corresponding topological dynamical system, chain controllability turns out to be equivalent to chain transitivity. This is obtained by an appropriate generalization of results from the theory of continuous-time control-affine state space systems.

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