SUPPORTS OF INVARIANT MEASURES FOR PIECEWISE DETERMINISTIC MARKOV PROCESSES

M. BENAÏM
INSTITUT DE MATHÉMATIQUES, UNIVERSITÉ DE NEUCHÂTEL, NEUCHÂTEL, SUISSE AND F.COLONIUS AND R. LETTAU
INSTITUT FÜR MATHEMATIK, UNIVERSITÄT AUGSBURG, AUGSBURG, GERMANY

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Abstract. For a class of piecewise deterministic Markov processes, the supports of the invariant measures are characterized. This is based on the analysis of controllability properties of an associated deterministic control system. Its invariant control sets determine the supports.

1. Introduction. In this paper we determine the supports of invariant measures for certain Piecewise Deterministic Markov Processes (PDMP) using controllability properties of an associated deterministic control system. We refer to the monograph Davis [10] for background on PDMP. The results extend some of those given by Bakhtin and Hurth [4] and Benaïm, Le Borgne, Malrieu and Zitt [5], where the ergodic case is treated. We will show that, under appropriate assumptions, the supports of the invariant measures are determined by the invariant control sets. In particular, in the ergodic case this reduces to one of the main results in [5] (in particular Proposition 3.17).

A technical difference to the papers mentioned above is that, on the deterministic side, we use control systems instead of differential inclusions. This is due to the fact, that we bring to bear the theory of control sets (maximal sets of complete approximate controllability) for control systems (cf. Colonius and Kliemann [9]). This allows us to develop many results in analogy to the theory for degenerate Markov diffusion processes (cf. Arnold and Kliemann [1, 2], Kliemann [11], Colonius, Gayer, Kliemann [7]) and for certain random diffeomorphisms (Colonius, Homburg, Kliemann [8]).

The contents of this paper is as follows: In Section 2 we recall and partially strengthen some results on invariant control sets. Section 3 clarifies the relations between PDMP and the associated control system, and Section 4 establishes the relation between the supports of invariant measures for PDMP and invariant control sets.

Notation. For a subset $V \subset \mathbb{R}^n$ the convex hull is denoted by co$V$. The topological closure and the interior of V are denoted by cl$V$ and int$V$, respectively. We write $L^\infty(\mathbb{R}_+, V)$ for the set of $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $v(t) \in V$ for all $t \geq 0$.

2. Controllability properties. In this section we associate deterministic control systems to PDMP and discuss their controllability properties.

As in [5] we consider PDMP of the following form: Let E be a finite set with cardinality $m + 1 = \#E$, say $E := \{0, 1, ..., m\}$, and for any $i \in E$ let $F^i : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth (C^∞) vector field. We assume throughout that each F^i is bounded. Let $\Phi^i(t, x), t \geq 0$, be the corresponding solution map; this (semi-)flow exists globally, since we assume that the F^i are bounded. Frequently, we will also suppose that there exists a compact set $M \subset \mathbb{R}^d$ that is positively invariant under each Φ^i, meaning that $\Phi^i(t, x) \in M$ for all $t \geq 0$ and all $x \in M$.

We will consider a continuous-time piecewise deterministic Markov process $Z_t = (X_t, Y_t)$ living on $\mathbb{R}^d \times E$. This process will be described explicitly below, here we only remark that the continuous component X_t evolves according to the flows Φ^i; the component on E determines which of the flows Φ^i is active (with random switching times). Already here it is clear that, in a natural way, one may associate the following deterministic control system to the PDMP,

$$\dot{x}(t) = \sum_{i=0}^{m} v_i(t) F^i(x), \tag{2.1}$$

where the (piecewise constant or measurable) control functions v lie in $L^\infty(\mathbb{R}_+, S)$ and

$$S := \left\{ v = (v_i) \in \mathbb{R}^{m+1} \,\middle|\, \sum\nolimits_{i=0}^{m} v_i = 1 \text{ and } v_i \in \{0,1\} \text{ for all } i \right\} \tag{2.2}$$

stands for the canonical basis of $\mathbb{R}^{m+1}$. Thus only one vector field F^i is active at any time.

We also consider system (2.1) with convexified right hand side where the control range is the unit m-simplex

$$\mathrm{co}S := \left\{ v \in \mathbb{R}^{m+1} \,\middle|\, \sum\nolimits_{i=0}^{m} v_i = 1 \text{ and } v_i \in [0,1] \right\}. \tag{2.3}$$

The corresponding set of control functions is $L^\infty(\mathbb{R}_+, \mathrm{co}S)$.

REMARK 2.1. *We may write $v_0(t) = 1 - v_1(t) - \ldots - v_m(t)$. Then system (2.1) with control range S is equivalent to the following control system*

$$\dot{x} = F^0(x) + \sum_{i=1}^{m} v_i(t) \left[F^i(x) - F^0(x) \right] \tag{2.4}$$

with controls having range in

$$\left\{ v \in \mathbb{R}^m \,\middle|\, \sum\nolimits_{i=1}^{m} v_i \leq 1 \textit{ and } v_i \in \{0,1\} \textit{ for all } i \right\}. \tag{2.5}$$

Analogously, (2.1) with control range co$S$ *is equivalent to (2.4) with convex control range*

$$\left\{ v \in \mathbb{R}^m \,\middle|\, \sum\nolimits_{i=1}^{m} v_i \leq 1 \textit{ and } v_i \in [0,1] \textit{ for all } i \right\}. \tag{2.6}$$

System (2.4) (and hence (2.1)) is a special case of control-affine systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} v_i(t) f_i(x), (v_i) \in \mathcal{V} := L^\infty(\mathbb{R}_+, V) \tag{2.7}$$

with Lipschitz continuous vector fields f_i on $\mathbb{R}^d$ and compact control range $V \subset \mathbb{R}^m$. Next we discuss some properties of the general class of systems of the form (2.7). We assume that (unique) global solutions $\varphi(t, x, v), t \geq 0$, exist for controls v and initial condition $\varphi(0, x, v) = x$. This is certainly satisfied in a compact set M which is positively invariant, i.e., satisfying

$$\varphi(t, x, v) \in M \text{ for all } x \in M, v \in \mathcal{V} \text{ and } t > 0.$$

Define for $x \in \mathbb{R}^d$ and $T > 0$ the reachable and controllable sets of (2.7) up to time T by

$$\mathcal{O}^+_{\leq T}(x) := \{\varphi(t, x, v) \,|t \in [0, T] \text{ and } v \in \mathcal{V}\},$$
$$\mathcal{O}^-_{\leq T}(x) := \{y \,|x = \varphi(t, y, v) \text{ for some } t \in [0, T] \text{ and } v \in \mathcal{V}\}, \tag{2.8}$$

and the reachable and controllable sets by

$$\mathcal{O}^+(x) := \bigcup\nolimits_{T>0} \mathcal{O}^+_{\leq T}(x), \mathcal{O}^-(x) := \bigcup\nolimits_{T>0} \mathcal{O}^-_{\leq T}(x).$$

Similarly let $\mathcal{O}^+_{pc}(x)$ denote the subset of $\mathcal{O}^+(x)$ which can be reached by piecewise constant control functions (i.e., having only finitely many discontinuities on every bounded interval).

We note the following standard properties of control systems.

THEOREM 2.2. *Consider a control system of the form (2.7). Then the following holds:*

(i) For every trajectory $\varphi(t, x, v), t \in [0, T]$, of (2.7) there exists a sequence (v_n) of piecewise constant controls with $\varphi(t, x, v_n) \to \varphi(t, x, v)$ uniformly for $t \in [0, T]$.

(ii) For every trajectory $\varphi(t, x, v), t \in [0, T]$, of (2.7) with control values in co$V$ *there exists a sequence (v_n) of controls with values in V such that $\varphi(t, x, v_n) \to \varphi(t, x, v)$, uniformly for $t \in [0, T]$.*

(iii) The trajectories $\varphi(t, x, v)$ of (2.7) with control values in co$V$ *coincide with the (absolutely continuous) solutions of the differential inclusion*

$$\dot{x} \in \left\{ f_0(x) + \sum\nolimits_{i=1}^m v_i f_i(x), (v_i) \in \mathrm{co}V \right\}. \tag{2.9}$$

Proof. For assertion (i) see Sontag [14, Lemma 2.8.2]. For assertion (ii) see, e.g., Berkovitz and Medhin [6, Theorem IV.2.6]. In assertion (iii) it is clear, that every trajectory of (2.7) with control values in co$V$ is a solution of the differential inclusion above, which has compact convex velocity sets depending continuously (in the Hausdorff metric) on x. The converse follows by a measurable selection theorem, cp., e.g., Aubin and Frankowska [3, Theorem 8.1.3]. □

REMARK 2.3. *A consequence of this theorem is that the points in the reachable and controllable sets defined in (2.8) can be approximated using only piecewise constant controls.*

We proceed to define maximal subsets of complete approximate controllability (for some background see Colonius and Kliemann [9] and Kawan [12]).

DEFINITION 2.4. *A nonempty set $D \subset \mathbb{R}^d$ is a control set of system (2.7) if (i) it is controlled invariant, i.e., for every $x \in D$ there is $v \in \mathcal{V}$ with $\varphi(t, x, v) \in D$ for all $t \geq 0$ (ii) for every $x \in D$ one has $D \subset \mathrm{cl}\mathcal{O}^+(x)$ and (iii) D is maximal with these properties. A control set C is called invariant, if $\mathrm{cl}C = \mathrm{cl}\mathcal{O}^+(x)$ for all $x \in C$.*

Invariant control sets need not be closed, as seen by the simple example

$$\dot{x} = x(1 - x)v(t), v(t) \in [-1, 1].$$

Here $x = 0$ and $x = 1$ are fixed points for every $v \in [-1, 1]$. Thus the sets $\{0\}, \{1\}$ and also the open interval $C := (0, 1)$ are invariant control sets.

We call a subset M of $\mathbb{R}^d$ invariant if

$$\{\varphi(t, x, v) \,|x \in M \text{ and } v \in \mathcal{V}\} = M \text{ for all } t \geq 0.$$

Proposition 2.5. *(i) An invariant control set C is closed, if for every $x \in \partial C$ the set $\mathcal{O}^+(x)$ has nonvoid interior.*

(ii) The compact invariant control sets coincide with the minimal compact invariant sets, i.e., the compact invariant subsets $M \subset \mathbb{R}^d$ which do not contain a proper compact invariant subset.

Proof. (i) We use repeatedly that on every bounded time interval the solution of a differential equation depends continuously on the initial value: One finds that $\mathrm{cl}\mathcal{O}^+(x) \subset \mathrm{cl}C$ for all $x \in \mathrm{cl}C$. Hence, for $x \in \partial C$ with $\mathrm{int}O^+(x) \neq \emptyset$ it follows that there is $y \in O^+(x) \cap C$. Then every $z \in C$ is in $\mathrm{cl}\mathcal{O}^+(x)$. Since $x \in \mathrm{cl}\mathcal{O}^+(z)$ for all $z \in C$, the maximality property of C implies that $x \in C$.

(ii) Let C be a compact invariant control set. Then $\mathrm{cl}\left(\bigcup_{x\in C} \mathcal{O}^+(x)\right) \subset C$, and hence Lamb, Rasmussen and Rodrigues [13, Proposition 3.2] implies that C contains a minimal compact invariant set M. We want to show now that $M = C$. If $M \subsetneq C$, then, since both sets are compact, there exists a $y \in C \setminus M$ with $\mathrm{dist}(y, M) > 0$. Since for all $x \in M$, one has $C = \mathrm{cl}\mathcal{O}^+(x)$, in particular $y \in \mathrm{cl}\mathcal{O}^+(x)$, the set M is not invariant, which is a contradiction, so C is a minimal compact invariant set. Conversely, every minimal compact invariant subset M satisfies $\mathrm{cl}\mathcal{O}^+(x) \subset M$ for all $x \in M$. By continuous dependence on the initial value, the set $\mathrm{cl}\mathcal{O}^+(x)$ is positively invariant, hence it contains a minimal compact invariant subset which by minimality of M coincides with M. □

Remark 2.6. *The result from [13, Proposition 3.2] used above is formulated for set valued dynamical systems. For control systems it means that in a compact positively invariant set M there is a compact subset N with*

$$\{\varphi(t,x,v) \,|t \geq 0, x \in N, v \in \mathcal{V}\} = N,$$

containing no proper subset with this property. This is proved as follows: Consider the collection

$$\mathcal{K} := \{A \subset M \,|A \text{ is compact with } \varphi(t,x,v) \in A \text{ for all } t \geq 0, x \in A, v \in \mathcal{V}\}.$$

This collection is partially ordered by set inclusion and every totally ordered subcollection has a lower bound in $\mathcal{K}$ given by the intersection of its elements. Then Zorn's Lemma implies that there exists at least one minimal element in $\mathcal{K}$ which turns out to be a minimal compact invariant set.

Control system (2.7) is called locally accessible in $x \in \mathbb{R}^d$ if for all $T > 0$ and all neighborhoods N of x

$$\mathrm{int}\mathcal{O}^+_{\leq T}(x) \cap N \neq \emptyset \text{ and } \mathrm{int}\mathcal{O}^-_{\leq T}(x) \cap N \neq \emptyset. \tag{2.10}$$

It is called locally accessible on a subset $M \subset \mathbb{R}^d$ (or M is locally accessible) if it is locally accessible in every point $x \in M$. Recall that the Lie algebra $\mathcal{LA}(\mathcal{F})$ generated by a family $\mathcal{F}$ of vector fields is the smallest vector space containing $\mathcal{F}$ that is closed under Lie brackets

$$[f,g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$

The analysis of controllability properties is simplified in the following situation.

Theorem 2.7. *Consider a control system of the form (2.7) and suppose that the Lie algebra $\mathcal{LA} = \mathcal{LA}(f_0 + \sum_{i=1}^m v_i f_i \,|v \in V\}$ satisfies for some $x \in \mathbb{R}^d$*

$$\{g(x) \,|g \in \mathcal{LA}\} = \mathbb{R}^d. \tag{2.11}$$

Then the system is locally accessible in x. Furthermore, it suffices to consider piecewise constant controls in (2.10).

We note that condition (2.11) is also necessary for local accessibility, if the involved vector fields are real analytic; cf. Sontag [14, Theorems 9 and 12] for a proof of Theorem 2.7 and the necessity statement.

REMARK 2.8. *For control system (2.1) (or (2.4) with control range (2.5) or (2.6)) the Lie algebra $\mathcal{LA}$ from Theorem 2.7 coincides with the Lie algebra generated by the vector fields $F^0, ..., F^m$.*

In order to derive some further properties of control sets we adapt the following lemma from Colonius and Kliemann [9, Lemma 4.5.4].

LEMMA 2.9. *Let $x \in \mathbb{R}^d$ and $v \in \mathcal{V}$ with $\varphi(T, x, v) \in \text{int}\mathcal{O}^+_{\le T+S}(x)$ for some $T, S > 0$ and assume that the system is locally accessible at $\varphi(T, x, v)$. Then*

$$x \in \text{int}\mathcal{O}^-_{\le T+2S}(\varphi(T, x, v)).$$

Proof. We find an open neighborhood $N(y) \subset \text{int}\mathcal{O}^+_{\le T+S}(x)$ of $y := \varphi(T, x, v)$. Local accessibility at y implies that there is $z \in N(y) \cap \text{int}\,\mathcal{O}^-_{\le t_0}(y)$ for every t_0 with $0 < t_0 \le S$. Then there are a control v and a neighborhood $N(x)$ of x such that $N(x)$ is mapped in a time $T_1 \le T + S$ via the solution map corresponding to v onto a neighborhood $N(z)$ of z contained in $N(y) \cap \mathcal{O}^-_{\le t_0}(y)$. We obtain

$$x \in N(x) \subset \mathcal{O}^-_{\le T_1+t_0}(y) \subset \mathcal{O}^-_{\le T+2S}(\varphi(T, x, v)).$$

□

THEOREM 2.10. *Consider system (2.7) and assume that it is locally accessible on a positively invariant subset $W \subset \mathbb{R}^d$.*

(i) There are at most countably many invariant control sets $C_r, r \in I \subset \mathbb{N}$ in W. They are closed and have nonvoid interiors which are positively invariant. Furthermore, cl int$C = C$ *and* int$C \subset \mathcal{O}^+(x)$ *for all $x \in C$.*

(ii) Let $K \subset W$ be compact. Then at most finitely many invariant control sets have nonvoid intersection with K.

Proof. (i) Closedness follows by Proposition 2.5(i). For every invariant control set C, the interior of C is nonvoid, since $\mathcal{O}^+(x) \subset C$ has nonvoid interior. This implies that there are at most countably many invariant control sets, since the topology of $\mathbb{R}^d$ has a countable base. Furthermore, suppose that there are $x \in \text{int}C, t > 0$ and $v \in \mathcal{V}$ with $\varphi(t, x, v) \notin C$. Let $T := \max\{t > 0 \,|\varphi(t, x, v) \in C\}$ and consider $y := \varphi(T, x, v) \in \partial C \subset C$. Hence there are $s > 0$ and $v \in \mathcal{V}$ with $\varphi(s, y, v) \in \text{int}C$. By continuous dependence on the initial value a neighborhood of y is mapped into the interior of C and hence, by the maximality property of controls sets, there is $S > T$ with $\varphi(S, x, v) \in C$. This contradiction proves that the interior of C is positively invariant. Furthermore, it is clear that cl int$C \subset C$. In order to see the converse inclusion, consider a nonvoid open subset $\mathcal{U} \subset \text{int}C$. Then for every $t > 0$ and $v \in \mathcal{V}$ the set $\{\varphi(t, x, v) \,|x \in \mathcal{U}\}$ is open, and the assertion follows, since $\mathcal{O}^+(x)$ is dense in C. Note that this argument does not use local accessibility, hence it shows that for any invariant control set either the interior is void or dense.

Finally, let $x \in C$ and $y \in \text{int}C$. Then there is $T > 0$ with $\text{int}\mathcal{O}^-_{\le T}(y) \subset C$. Since one can steer the point x to some point $z \in \text{int}\mathcal{O}^-_{\le T}(y)$. Concatenating the corresponding control with a control steering z to y one finds that $y \in \mathcal{O}^+(x)$, as claimed.

(ii) If the assertion is false, one finds countably many invariant control sets $C_n, n \in \mathbb{N}$ and points $x_n \in C_n \cap K$. This sequence has a cluster point and every cluster point x is in K. By local accessibility there are $T, S > 0$ and a control v with $\varphi(T,x,v) \in \mathrm{int}\mathcal{O}^+_{\leq T+S}(x)$. By Lemma 2.9 we find that $x \in \mathrm{int}\mathcal{O}^-_{\leq T+2S}(\varphi(T,x,v))$. Hence for $n \in \mathbb{N}$ large enough one has

$$x_n \in \mathcal{O}^-_{\leq T+2S}(\varphi(T,x,v)).$$

This shows that for all n the points $x_n \in C_n$ can be steered to the single point $\varphi(T,x,v)$. This contradicts invariance of the pairwise different invariant control sets C_n. □

The next theorem does not assume that the system is locally accessible on W.

THEOREM 2.11. *Consider system (2.7) on a positively invariant subset* $W \subset \mathbb{R}^d$*.*

(i) If there exists a compact subset $K \subset W$ *such that for every* $x \in W$ *one has* $\mathrm{cl}\mathcal{O}^+(x) \cap K \neq \emptyset$*, then for every* $x \in W$ *there exists an invariant control set* $C \subset \mathrm{cl}\mathcal{O}^+(x)$*. If the system is also locally accessible on* W*, then there are only finitely many invariant control sets in* W*.*

(ii) Conversely, if for every $x \in W$ *there exists an invariant control set* $C \subset \mathrm{cl}\mathcal{O}^+(x)$ *and there are only finitely many invariant control sets in* W*, then there exists a compact subset* $K \subset W$ *such that for every* $x \in W$ *one has* $\mathrm{cl}\mathcal{O}^+(x) \cap K \neq \emptyset$*,*

Proof. (i) In order to show that for every $x \in W$ there is an invariant control set $C \subset \mathrm{cl}\mathcal{O}^+(x)$ define $K(y) := \mathrm{cl}\mathcal{O}^+(y) \cap K$ for $y \in \mathrm{cl}\mathcal{O}^+(x)$. Consider the family $\mathcal{K}$ of nonvoid and compact subsets of W given by $\mathcal{K} = \{K(y) \,|y \in \mathrm{cl}\mathcal{O}^+(x)\}$. Then $\mathcal{K}$ is ordered via

$$K(z) \prec K(y) \text{ if } y \in \mathrm{cl}\mathcal{O}^+(z).$$

In fact, this is an order: If $K(z) \prec K(y)$ and $K(y) \prec K(z)$, then $y \in \mathrm{cl}\mathcal{O}^+(z)$ and $z \in \mathrm{cl}\mathcal{O}^+(y)$ implying $\mathrm{cl}\mathcal{O}^+(y) = \mathrm{cl}\mathcal{O}^+(z)$ and hence $K(y) = K(x)$. If $K(y_1) \prec K(y_2)$ and $K(y_2) \prec K(y_3)$, then $y_2 \in \mathrm{cl}\mathcal{O}^+(y_1)$ and $y_3 \in \mathrm{cl}\mathcal{O}^+(y_2)$, hence $y_3 \in \mathrm{cl}\mathcal{O}^+(y_1)$ implying $K(y_1) \prec K(y_3)$.

Every linearly ordered set $\{K(y_i) \,|i \in I\}$ has an upper bound $K(y)$ for some $y \in \bigcap_{i\in I} K(y_i)$, because the intersection of decreasing compact subsets of the compact set K is nonempty. Thus Zorn's lemma implies that the family $\mathcal{K}$ has a maximal element $K(y)$. Now we claim that the set

$$C := \mathrm{cl}\mathcal{O}^+(y)$$

is an invariant control set. It is clear that $C \subset \mathrm{cl}\mathcal{O}^+(x)$. Every $z \in C$ is approximately reachable from y, hence $K(y) \prec K(z)$ and maximality of $K(y)$ implies $K(y) = K(z)$, hence for every $z \in C$ one has $y \in \mathrm{cl}\mathcal{O}^+(z)$. Then it follows that $C = \mathrm{cl}\mathcal{O}^+(z)$ for every $z \in C$, hence C is an invariant control set.

If W is locally accessible, the invariant control sets are closed by Theorem 2.10(i). Thus, if x is in an invariant control set C, then $\mathrm{cl}\mathcal{O}^+(x) = C$ and hence $C \cap K \neq \emptyset$. By Theorem 2.10(ii) only finitely many invariant control sets have nonvoid intersection with K, hence only finitely many invariant control sets in W exist.

(ii) For each of the finitely many invariant control sets C_i pick $x_i \in C_i$ and define $K := \{x_i \,|1 \leq i \leq N\}$. Then continuous dependence on the initial value implies that for every $x \in K$ one has $\mathrm{cl}\mathcal{O}^+(x) \cap K \neq \emptyset$. □

The following corollary is a consequence of Theorems 2.10 and 2.11.

COROLLARY 2.12. *Suppose that* M *is a compact positively invariant set for system (2.7).*

(i) For every $x \in M$ there is an invariant control set $C \subset M$ with $C \subset \mathrm{cl}\mathcal{O}^+(x)$. If C is compact and $\mathrm{int}C \neq \emptyset$ *then* $C = \mathrm{cl}\ \mathrm{int}C$ *and* $\mathrm{int}C \cap \mathcal{O}^+_{pc}(x) \neq \emptyset$

(ii) Suppose that M, additionally, is locally accessible. Then M contains at least one and at most finitely many invariant control sets $C_r, r = 1, ..., l$. They are compact (hence characterized by Proposition 2.5 (ii)), have nonvoid interiors $\mathrm{int}C_r$ *and for every point $x \in M$ there is $r \in \{1, \dots, l\}$ with* $\mathrm{int}C_r \subset \mathcal{O}^+_{pc}(x)$.

Proof. (i) The first assertion follows from Theorem 2.11(i). Then the second assertion follows using $\mathrm{int}C \cap \mathcal{O}^+(y) \neq \emptyset$ for every $y \in C$; thus Theorem 2.2(i) and continuous dependence on the initial value show that there is $z \in \mathcal{O}^+_{pc}(x)$ with $\mathcal{O}^+_{pc}(z) \cap \mathrm{int}C \neq \emptyset$ and hence $\mathcal{O}^+_{pc}(x) \cap \mathrm{int}C \neq \emptyset$.

(ii) This is immediate from Theorems 2.10 and 2.11. □

Remark 2.13. *Consider a linear control system of the form $\dot{x}(t) = Ax(t) + Bv(t), v(t) \in V \subset \mathbb{R}^m$ with matrices $A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}$ and compact control range $V \subset \mathbb{R}^m$ with $0 \in \mathrm{int}V$. Suppose that the controllability rank condition* $\mathrm{rank}[B, AB, ..., A^{d-1}B] = d$ *holds (this condition, called Kalman's rank condition is equivalent to the fact that the system without control constraint satisfies $\mathcal{O}^+(x) = \mathbb{R}^d$ for all $x \in \mathbb{R}^d$). Then there exists a unique control set with nonvoid interior. It is a compact invariant control set if A is stable, i.e., all eigenvalues of A have negative real parts (cp. Colonius and Kliemann [9, Example 3.2.16]). The example in [5, Section 5.2] discusses for such a situation (with $d = 2$) the invariant measures.*

Benaïm, Le Borgne, Malrieu and Zitt in [5] define for the control system (2.1) with control range (2.2) and a positively invariant compact set $M \subset \mathbb{R}^d$ the accessible set

$$\Gamma := \bigcap_{x\in M} \mathrm{cl}\mathcal{O}^+_{pc}(x). \tag{2.12}$$

See also Proposition 3.11 in [5] for the relations with the associated differential inclusion.

In complete analogy, one can also define the accessible set Γ of a general control system of the form (2.7) and any $M \subset \mathbb{R}^d$. Its relation to invariant control sets is clarified in the following theorem.

Proposition 2.14. *Consider control system (2.7) and let $M \subset \mathbb{R}^d$.*

(i) If Γ is a nonempty subset of M, then Γ is a closed invariant control set.

(ii) Let M be compact and positively invariant and suppose that there is only one invariant control set C in M. Then $C = \Gamma$, in particular, C is closed.

(iii) If a compact positively invariant set M contains two closed invariant control sets, then Γ is empty.

Proof. (i) Suppose that $\Gamma = \bigcap_{x\in M} \mathrm{cl}\mathcal{O}^+(x) \neq \emptyset$. Continuous dependence on the initial value shows that Γ is positively invariant, hence $\mathrm{cl}\mathcal{O}^+(x) \subset \Gamma$ for every $x \in \Gamma$. For all $x, y \in \Gamma$ one has $y \in \mathrm{cl}\mathcal{O}^+(x)$ and $x \in \mathrm{cl}\mathcal{O}^+(y)$. Thus it is an invariant control set.

(ii) By Corollary 2.12, for every $x \in M$ there is an invariant control set in $\mathrm{cl}\mathcal{O}^+(x)$, hence the inclusion $C \subset \bigcap_{x\in M} \mathrm{cl}\mathcal{O}^+(x)$ holds. For the converse inclusion, note that this implies that Γ is a nonempty subset of M, and hence the assertion follows by (i).

(iii) If Γ is nonempty, it is by (i) a closed invariant control set. Let $C \neq \Gamma$ be another closed invariant control set. Then $\mathrm{cl}\mathcal{O}^+(x) \subset C$ for $x \in C$. Since, by the maximality property, the intersection of two control sets $C_1 \neq C_2$ is void, one obtains the contradiction $\Gamma = \emptyset$. □

Next we discuss the points which can be steered into an invariant control set.

Definition 2.15. *The domain of attraction of an invariant control set C is*

$$\mathcal{A}(C) := \{x \in \mathbb{R}^d \left| \mathrm{cl}\mathcal{O}^+(x) \cap C \neq \emptyset \right.\}$$

and its strict domain of attraction is

$$\mathcal{A}_{strict}(C) := \{x \in \mathbb{R}^d \left| \mathrm{cl}\mathcal{O}^+(x) \cap C \neq \emptyset \text{ and } \mathrm{cl}\mathcal{O}^+(x) \cap C' \neq \emptyset \text{ implies } C' = C \right.\},$$

where C' denotes any invariant control set.

It is easily seen that for an invariant control set C with nonvoid interior $x \in \mathcal{A}(D)$ if and only if $\mathcal{O}^+(x) \cap \mathrm{int}C \neq \emptyset$. In fact, for $y \in \mathrm{cl}\mathcal{O}^+(x) \cap C$ there are $v \in \mathcal{V}$ and $t > 0$ with $\varphi(t, x, v) \in \mathrm{int}D$ and hence, by continuous dependence on the initial value, it follows that $\mathcal{O}^+(x) \cap \mathrm{int}C \neq \emptyset$ (recall that for locally accessible systems, every invariant control set has nonvoid interior.) This also shows that the domain of attraction is open. The points in the strict domain of attraction can only be steered to a single invariant control set. If C is closed, the strict domain of attraction trivially contains C. It need not be open, since, e.g., a point of the boundary $C \cap \partial C$ may be in $\partial \left(\mathcal{A}_{strict}(C)\right)$.

If the system is locally accessible on a compact positively invariant set $M \subset \mathbb{R}^d$, Corollary 2.12 implies that every $x \in M$ is in the domain of attraction of at least one of the finitely many invariant control set. Furthermore, if M is also connected and contains at least two invariant control sets, it follows that for every invariant control set C_k the set $[\mathcal{A}(C_k) \cap M] \setminus \mathcal{A}_{strict}(C_k)$ is nonvoid. Otherwise, one would obtain a decomposition of M into two disjoint open sets

$$M = [\mathcal{A}(C_k) \cap M] \cup \textstyle\bigcup_{i \neq k} [\mathcal{A}(C_i) \cap M] \, .$$

Consequently, one finds for every invariant control set C_k an invariant control set $C_i \neq C_k$ with

$$\mathcal{A}(C_k) \cap \mathcal{A}(C_i) \neq \emptyset.$$

Thus there are points which can be steered into two different invariant control sets.

Finally, we show that one may consider an invariant control set C as an "accessible set" (similar to (2.12)) for a neighborhood of C, which, however, need not be positively invariant.

Proposition 2.16. *Let M be a compact positively invariant and locally accessible set. Then for every invariant control set $C \subset M$ there is a compact neighborhood $\mathcal{U}$ of C such that*

$$C = \bigcap_{x \in \mathcal{U}} \mathrm{cl}\mathcal{O}^+_{pc}(x).$$

Proof. There are only finitely many invariant control sets in M and they are compact. For every point $x \in \partial C$ there are $T > 0$ and a piecewise constant control v with $\varphi(T, x, v) \in \mathrm{int}C$. Using compactness of ∂C and continuous dependence on the initial value one finds a neighborhood $\mathcal{U}$ of C such that $C \subset \bigcap_{x \in \mathcal{U}} \mathrm{cl}\mathcal{O}^+_{pc}(x)$. Using the definition of invariant control sets one sees that here equality holds. □

3. Piecewise Deterministic Markov Processes. We will now, following Benaïm, Le Borgne, Malrieu and Zitt [5], define piecewise deterministic processes. With the notation introduced in Section 2, assume that there is a compact set $M \subset \mathbb{R}^d$, that is positively invariant for every flow Φ^i, i.e., $\Phi^i_t(M) \subset M$ for all $t \geq 0$ and all $i \in E$.

Let $x \mapsto Q(x) = (Q(x,i,j))_{ij} : \mathbb{R}^d \mapsto \mathbb{R}^{(m+1)\times(m+1)}$ be continuous with $Q(x)$ an irreducible, aperiodic Markov transition matrix, which means that for all x there is $n_x \in \mathbb{N}$ with $Q^{n_x}(x,i,j) > 0$ for all $i,j \in E$. Let $(N_t)_{t\geq 0}$ be a homogenous Poisson process with intensity λ, jumps times $(T_n)_{n\geq 0}$ and denote by $(U_n)_{n\geq 1}$ with $U_n = T_n - T_{n-1}$ the times between the jumps. Assume that $\tilde{Z}_0 \in M \times E$ is a random variable independent of $(N_t)_{t\geq 0}$.

We define the discrete-time process $(\tilde{Z}_n)_n = (\tilde{X}_n, \tilde{Y}_n)_n$ on $M \times E$ recursively by

$$\tilde{X}_{n+1} = \Phi^{\tilde{Y}_n}(U_{n+1}, \tilde{X}_n),$$
$$\mathbb{P}\left[\tilde{Y}_{n+1} = j \,\middle|\, \tilde{X}_{n+1}, \tilde{Y}_n = i\right] = Q(\tilde{X}_{n+1}, i, j).$$

and by interpolation its continuous time version $(Z_t)_{t\geq 0}$

$$Z_t = \left(\Phi^{\tilde{Y}_n}(t - T_n, \tilde{X}_n), \tilde{Y}_n\right) \text{ for } t \in [T_n, T_{n+1}). \tag{3.1}$$

We define for $n \in \mathbb{N}^*$

$$\mathbb{T}_n = \{(\mathbf{i}, \mathbf{u}) = ((i_0, i_1, \cdots i_n), (u_1, \cdots u_n)) \in E^{n+1} \times \mathbb{R}^n_+\}$$

and

$$\mathbb{T}^{ij}_n = \{(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_n | i_0 = i, i_n = j\}.$$

Then we can define the trajectory with initial value $x \in M$ induced by $(\mathbf{i}, \mathbf{u})$:

$$\eta_{x,\mathbf{i},\mathbf{u}}(t) = \begin{cases} x & t = 0 \\ \Phi^{i_{k-1}}_{t-t_{k-1}}(x_{k-1}) & t_{k-1} < t \leq t_k \\ \Phi^{i_n}_{t-t_n}(x_n) & t > t_n \end{cases}.$$

We can then denote

$$\Phi^{\mathbf{i}}_{\mathbf{u}}(x) = \eta_{x,\mathbf{i},\mathbf{u}}(t_n)$$

and note that $(x, \mathbf{u}) \mapsto \Phi^{\mathbf{i}}_{\mathbf{u}}(x)$ is continuous. In terms of the associated deterministic control system (2.1) this means that we consider a piecewise constant control function generating the trajectory $\varphi(t, x, v) = \eta_{x,\mathbf{i},\mathbf{u}}$.

We can define

$$p(x, \mathbf{i}, \mathbf{u}) := \prod_{j=1}^{n} Q(x_j, i_{j-1}, i_j) \tag{3.2}$$

and the set of adapted elements

$$\mathbb{T}_{n,ad(x)} = \{(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_n : p(x, \mathbf{i}, \mathbf{u}) > 0\}.$$

Note that $x \mapsto p(x, \mathbf{i}, \mathbf{u})$ is continuous.

The relation between the trajectories of the Piecewise Deterministic Markov Process and control system (2.1) is clarified by the following results from [5], slightly reformulated for control systems instead of differential inclusions. The first result is, in the terminology of Arnold and Kliemann [1], a tube lemma. It shows that tubes around any (finite-time) trajectory of the control system have positive probability. It reformulates the development in [5, Section 3.1]. Recall that co(S) is the unit m-simplex.

LEMMA 3.1 (Tube lemma). *For all $T > 0$, $x \in M$, $i \in E, \delta > 0$ and every trajectory $\varphi(t, x, v)$ of system (2.1) with controls in $L^\infty(\mathbb{R}_+, \mathrm{co}(S))$ there is $\varepsilon_{x,i} > 0$ such that*

$$\mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \varphi(t, x, v)\| \leq \delta\right] \geq \varepsilon_{x,i} > 0.$$

Proof. Theorem 2.2(ii) shows that for every control $v \in L^\infty(\mathbb{R}_+, \mathrm{co}(S))$ there is a control $v_1 \in L^\infty(\mathbb{R}_+, S)$ such that

$$\sup_{t\in[0,T]} \|\varphi(t, x, v) - \varphi(t, x, v_1)\| \leq \frac{\delta}{3}.$$

By Theorem 2.2(i) we can find a piecewise constant control with values in S such that the corresponding trajectory approximates $\varphi(t, x, v_1)$ uniformly on $[0, T]$. Thus there are $n \in \mathbb{N}$ and $(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_n$ with

$$\sup_{t\in[0,T]} \|\varphi(t, x, v_1) - \eta_{x,\mathbf{i},\mathbf{u}}(t)\| \leq \frac{\delta}{3}.$$

By [5, Lemma 3.2] there is $\varepsilon_{x,i} > 0$ such that

$$\mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \eta_{x,\mathbf{i},\mathbf{u}}(t)\| \leq \frac{\delta}{3}\right] \geq \varepsilon_{x,i} > 0.$$

Taken together, this implies

$$\begin{aligned}
&\mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \varphi(t, x, v)\| \leq \delta\right] \\
&= \mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \eta_{x,\mathbf{i},\mathbf{u}}(t) + \eta_{x,\mathbf{i},\mathbf{u}}(t) - \varphi(t, x, v_1) + \varphi(t, x, v_1) - \varphi(t, x, v)\| \leq \delta\right] \\
&\geq \mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \eta_{x,\mathbf{i},\mathbf{u}}(t)\| \leq \frac{\delta}{3}\right] \geq \varepsilon_{x,i} > 0.
\end{aligned}$$

□

The next result establishes a relation between the law of the continuous-time process and the associated control system (with convexified control range).

THEOREM 3.2. *For control system (2.1) with controls in $L^\infty(\mathbb{R}_+, \mathrm{co}(S))$ and $x \in M$ let the set S^x of trajectories starting in x be*

$$S^x := \{\varphi(\cdot, x, v) \in C([0, \infty), \mathbb{R}^d)\, |v \in L^\infty(\mathbb{R}_+, \mathrm{co}(S))\,\}.$$

If $X_0 = x \in M$ then the support of the law of $(X_t)_{t\geq 0}$ equals S^x.

Proof. By [5, Theorem 3.4], the support of the law of $(X_t)_{t\geq 0}$ equals the set of trajectories $\eta(\cdot)$ with $\eta(0)=x$ of the differential inclusion

$$\dot{\eta}(t) \in \left\{ \sum_{i=0}^{m} v_i(t) F^i(x) \,\Big|\, \sum\nolimits_{i=0}^{m} v_i(t) = 1 \text{ and } v_i \in [0,1]\} \right\}.$$

By Theorem 2.2(iii) the trajectories of this differential inclusion coincide with the set of trajectories of the associated control system. □

A first consequence of the tube lemma is the following.

PROPOSITION 3.3. *Let $C \subset M$ be an invariant control set with nonvoid interior and $x \in \mathcal{A}(C)$. Then there are $T_x > 0$ and $\varepsilon_x > 0$ with*

$$\mathbb{P}_{x,i}\left[X_{T_x} \in \text{int}C\right] \geq \varepsilon_x$$

for all $i \in E$.

Proof. By the definition of C there are $T_x \geq 0$ and a control $v_x \in \mathcal{V}^{co}$ with $y := \varphi(T_x, x, v_x) \in \text{int}C$, hence $B_\delta(y) \subset \text{int}C$ for some $\delta > 0$. Hence the tube lemma, Lemma 3.1, implies

$$\begin{aligned}\mathbb{P}_{x,i}[X_{T_x} \in \text{int}C] &\geq \mathbb{P}_{x,i}[X_{T_x} \in B_\delta(y)] \geq \mathbb{P}_{x,i}\left[\|X_{T_x} - \varphi(T_x,x,v_x)\| \leq \delta\right] \\ &\geq \mathbb{P}_{x,i}\left[\sup_{t\in[0,T_x]} \|X_t - \varphi(t,x,v_x)\| \leq \delta\right] \geq \varepsilon_x,\end{aligned}$$

where $\varepsilon_x := \min_{i\in E} \varepsilon_{x,i}$. □

4. Invariant measures and their supports. The purpose of this section is to show that the support of every invariant measure is contained in the union of the invariant control sets and that, for an ergodic invariant measure, the support coincides with an invariant control set times E.

Recall that a point x_0 is in the support of a measure if every neighborhood of x_0 has positive measure and that the support is closed.

For the following results on existence of invariant measures for the discrete time process and the continuous time process compare Benaïm, Le Borgne, Malrieu and Zitt [5]. First one notes that the invariant measures of the discrete-time process and the continuous-time process are homeomorphic to each other, the homeomorphism preserves ergodicity and the supports coincide [5, Proposition 2.4 and Lemma 2.6].

In our context, [5, Lemma 3.16] is replaced by the following technical lemma.

LEMMA 4.1. *Let M be a compact positively invariant set for system (2.1) with controls in $L^\infty(\mathbb{R}_+, \text{co}(S))$ and suppose that there are only finitely many invariant control sets $C_r \subset M, r = 1,\ldots,l$, and $C_r = \text{cl int}C_r$ and $\text{int}C_r \subset \mathcal{O}^+(x)$ for every $x \in C_r$ and all r. Consider the process $(\tilde{Z}_n)_n = (\tilde{X}_n, \tilde{Y}_n)_n$. Pick $p_r \in \text{int}C_r$ and let $\mathcal{U}_r$ be an open neighborhood of p_r with $\text{cl}\mathcal{U}_r \subset \text{int}C_r$ and $\mathcal{U} := \bigcup_{r=1}^{l} \mathcal{U}_r$ and, finally, choose $i,j \in E$.*

Then there exist $m \in \mathbb{N}^*$ and $\varepsilon, \beta > 0$, finite sequences $(\mathbf{i}^1, \mathbf{u}^1), ..., (\mathbf{i}^N, \mathbf{u}^N) \in \mathbb{T}_m^{ij}$ and an open covering $\mathcal{O}^1, ..., \mathcal{O}^N$ of M such that for all $x \in M$ and $\mathbf{t} \in \mathbb{R}_+^m$,

$$x \in \mathcal{O}^k \text{ and } \left\|\mathbf{t} - \mathbf{u}^k\right\| < \varepsilon \text{ implies } \Phi_{\mathbf{t}}^{\mathbf{i}^k}(x) \in \mathcal{U} \text{ and } p(x, \mathbf{i}^k, \mathbf{t}) \geq \beta.$$

Furthermore, m, ε and β can be chosen independently of $i,j \in E$.

REMARK 4.2. *If M is a compact positively invariant and locally accessible set, then the assumptions of Lemma 4.1 hold.*

Proof. Fix i and j and consider open neighborhoods $\mathcal{W}_r$ of p_r with $\text{cl}\mathcal{W}_r \subset \mathcal{U}_r, r = 1, ..., l$. Let $\mathcal{W} := \bigcup_{r=1}^{l} \mathcal{W}_r$. For all $\beta > 0$ and all finite sequences $(\mathbf{i}, \mathbf{u})$ the sets

$$\mathcal{O}(\mathbf{i}, \mathbf{u}, \beta) := \left\{ x \in M \,\middle|\, \Phi_{\mathbf{u}}^{\mathbf{i}}(x) \in \mathcal{W}, p(x, \mathbf{i}, \mathbf{u}) > \beta \right\}$$

are open, since $\Phi_{\mathbf{u}}^{\mathbf{i}}$ and p are continuous with respect to x. Using Corollary 2.12(i) one finds for every point $x \in M$ an invariant control set $C_r, r \in \{1, ..., l\}$, with $\text{int}C_r \subset \mathcal{O}^+(x)$. Hence, for every $x \in M$ there are a time $T > 0$ and a control v such that $\varphi(T, x, v) \in \mathcal{U}$.

Then the tube lemma, Lemma 3.1, implies

$$M = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{\beta > 0} \bigcup_{(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_n^{ij}} \mathcal{O}(\mathbf{i}, \mathbf{u}, \beta) \right).$$

By adding 'false' jumps to a chain $(\mathbf{i}, \mathbf{u})$, we get:

$$\forall (\mathbf{u}, \mathbf{i}) \in \mathbb{T}_n^{ij} \; \forall n' \geq n \; \forall \beta \; \exists \beta' > 0 \; \exists (\mathbf{i}', \mathbf{u}') \in \mathbb{T}_{n'}^{ij} : \mathcal{O}(\mathbf{i}, \mathbf{u}, \beta) \subset \mathcal{O}(\mathbf{i}', \mathbf{u}', \beta').$$

Therefore the union over n is increasing, so by compactness of M, there is m_{ij} with

$$M \subset \bigcup_{\beta > 0} \left(\bigcup_{(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_{m_{ij}}^{ij}} \mathcal{O}(\mathbf{i}, \mathbf{u}, \beta) \right).$$

Since there only finitely many i and j, we can choose m independently of $i, j \in E$.

The inclusions

$$\mathcal{O}(\mathbf{i}, \mathbf{u}, \beta_1) \subset \mathcal{O}(\mathbf{i}, \mathbf{u}, \beta_2) \text{ for } \beta_1 \geq \beta_2$$

show that the union over the β is increasing with decreasing β. Thus, by compactness there is $\beta_0 > 0$ such that for all $i, j \in E$

$$M = \bigcup_{(\mathbf{i}, \mathbf{u}) \in \mathbb{T}_m^{ij}} \mathcal{O}(\mathbf{i}, \mathbf{u}, \beta_0).$$

Again compactness of M shows that there is $N \in \mathbb{N}$ with

$$M = \bigcup_{k=1}^{N} \mathcal{O}_k,$$

where $\mathcal{O}_k := \mathcal{O}\left(\mathbf{i}^k, \mathbf{u}^k, \beta_0\right)$ for some $(\mathbf{i}^k, \mathbf{u}^k) \in \mathbb{T}_m^{ij}$. Since $\text{cl}\mathcal{W}_r \subset \text{int}D_r$ for $r = 1, ..., l$, the distance between $\text{cl}\mathcal{W}$ and the complement of $\mathcal{U}$ is positive by compactness. So we can choose ε small enough such that for all $x \in \mathcal{O}_k$ and all $\mathbf{t} \in \mathbb{R}_+^m$ with $\left\| \mathbf{t} - \mathbf{u}^k \right\| \leq \varepsilon$

$$\Phi_{\mathbf{t}}^{\mathbf{i}^k}(x) \in \mathcal{U} \text{ and } p(x, \mathbf{i}^k, \mathbf{t}) \geq \beta.$$

Using compactness of M one can here choose $\varepsilon, \beta > 0$ independently of $x \in M$. □

We denote the n-step transition probability from (x, i) to a measurable set $A \subset M \times E$ by $P_n((x, i), A) = \mathbb{E}\left[\tilde{Z}_n \in A | \tilde{Z}_0 = (x, i)\right]$.

LEMMA 4.3. *(i) Let $y \in \mathcal{O}^+(x)$ and consider a neighborhood $\mathcal{W}(y)$ of y. Then there are a neighborhood $\mathcal{W}(x), n \in \mathbb{N}$ and $\delta > 0$ such that for all $z \in \mathcal{W}(x)$ and all $i, j \in E$*

$$P_n\left((z,i), \mathcal{W}(y) \times \{j\}\right) \geq \delta.$$

(ii) Under the assumptions of Lemma 4.1 the following holds:
There are $n \in \mathbb{N}$ and $\delta > 0$, such that for all $x \in M$ there is an open neighborhood $\mathcal{W}(x)$ of x with

$$P_n\left((y,i), \bigcup_r \operatorname{int} C_r \times E\right) \geq \delta \text{ for all } y \in \mathcal{W}(x) \text{ and } i \in E.$$

Proof. (ii) By Lemma 4.1 one finds $n \in \mathbb{N}$ and $\varepsilon, \beta > 0$ such that for every $x \in M$ and $i, j \in E$ there is an open neighborhood $\mathcal{W}_{ij}(x)$ of x and $(\mathbf{i}_{ij}, \mathbf{u}_{ij}) \in \mathbb{T}_n^{ij}$ with the following property:
For $\mathbf{t} \in \mathbb{R}_+^m$ with $\|\mathbf{t} - \mathbf{u}_{ij}\| \leq \varepsilon$ and $y \in \mathcal{W}_{ij}(x)$ it follows that $\Phi_{\mathbf{t}}^{\mathbf{i}_{ij}}(y) \in \bigcup_{r=1}^{l} \operatorname{int} C_r$ and $p(y, \mathbf{i}_{ij}, \mathbf{t}) \geq \beta$.
The set $\mathcal{W}(x) := \bigcap_{i,j \in E} \mathcal{W}_{ij}(x)$ is an open neighborhood of x and we have for all $y \in \mathcal{W}(x)$ and all $i, j \in E$

$$\begin{aligned} P_n((y,i), \bigcup_r \operatorname{int} C_r) &\geq \mathbb{P}_{(y,i)}\left(\|\mathbf{u} - \mathbf{u}_{ij}\| \leq \varepsilon, \mathbf{i} = \mathbf{i}_{ij}\right) \\ &= \mathbb{P}_{(y,i)}\left(\|\mathbf{u} - \mathbf{u}_{ij}\| \leq \varepsilon\right) \cdot \mathbb{P}_{(y,i)}\left(\mathbf{i} = \mathbf{i}_{ij} \,|\, \|\mathbf{u} - \mathbf{u}_{ij}\| \leq \varepsilon\right) \\ &\geq \mathbb{P}_{(y,i)}\left(\|\mathbf{u} - \mathbf{u}_{ij}\| \leq \varepsilon\right) \cdot \beta. \end{aligned}$$

Since the components of $\mathbf{u}$ are identically and independently distributed and the distribution is exponential, there is $\gamma_{ij} > 0$ such that $\mathbb{P}_{(y,i)}\left(\|\mathbf{u} - \mathbf{u}_{ij}\| \leq \varepsilon\right) \geq \gamma_{ij}$, and with $\gamma_i := \min_{j \in E} \gamma_{ij}$

$$P_n((y,i), \bigcup_r \operatorname{int} C_r) \geq \beta \gamma_i.$$

Setting $\delta := \beta \min_i \gamma_i$ finishes the proof.
(i) This follows by the same arguments. □
Similar arguments also lead to the following extension of Lemma 4.1.
LEMMA 4.4. *Let the assumptions and notation of Lemma 4.1 be satisfied. Then there exist an open subset $\mathcal{U}$ of $\bigcup_r \operatorname{int} C_r$, a natural number $m \in \mathbb{N}^*$, a constant $\delta > 0$ and an open covering $\mathcal{O}^1, ..., \mathcal{O}^N$ of M such that for all $x \in M$ and all $i, j \in E$*

$$x \in \mathcal{O}^k \text{ implies } P_n\left((x,i), \mathcal{U} \times \{j\}\right) \geq \delta.$$

Here m and δ can be chosen independently of $i, j \in E$.
For an invariant measure μ of $\left(\tilde{Z}_n\right)$, a simple calculation shows for every measurable set $A \subset M \times E$

$$\mu(A) = \int_{M \times E} P_n\left((x,i), A\right) d\mu \text{ for } n \in \mathbb{N}. \tag{4.1}$$

The following theorem shows that the supports of the invariant measures of the discrete-time process are determined on the invariant control sets.
THEOREM 4.5. *Let M be a compact positively invariant set for system (2.1) with controls in $L^\infty(\mathbb{R}_+, \operatorname{co}(S))$, and suppose that there are only finitely many invariant*

control sets $C_r \subset M, r = 1, \ldots, l$, *and* $C_r = \text{cl int} C_r$ *and* $\text{int} C_r \subset \mathcal{O}^+(x)$ *for every* $x \in C_r$ *and all* r. *(this holds in particular, if* M *is also locally accessible). Then for every invariant measure* μ *of the discrete-time process* $(\tilde{Z}_n)_n = (\tilde{X}_n, \tilde{Y}_n)_n$

$$\text{supp}\mu \subset \bigcup_{r=1}^{l} C_r \times E.$$

Proof. Suppose, by way of contradiction, that there is $(x_0, i_0) \in A := \text{supp}\mu \setminus (\bigcup_r C_r \times E)$. Since the invariant control sets C_r are closed, there is an open neighborhood $\mathcal{W}(x_0)$ of x_0 with $(\mathcal{W}(x_0) \times E) \cap (\bigcup_r C_r \times E) = \emptyset$. Then

$$\mu\left(M \setminus \left(\bigcup\nolimits_r C_r \times E\right)\right) \geq \mu(\mathcal{W}(x_0) \times E) > 0$$

and

$$\begin{aligned} 1 = \mu(M \times E) &= \mu\left(M \setminus \left(\bigcup\nolimits_r C_r \times E\right)\right) + \mu\left(\bigcup\nolimits_r C_r \times E\right) = \mu\left(\text{supp}\mu\right) \\ &= \mu\left(\text{supp}\mu \setminus \left(\bigcup\nolimits_r C_r \times E\right)\right) + \mu\left(\bigcup\nolimits_r C_r \times E\right). \end{aligned}$$

It follows that

$$0 < \mu\left(M \setminus \left(\bigcup\nolimits_r C_r \times E\right)\right) = \mu\left(\text{supp}\mu \setminus \left(\bigcup\nolimits_r C_r \times E\right)\right) = \mu(A).$$

Lemma 4.4 shows that there is $\delta > 0$ with

$$P_n((x,i), \bigcup_r \text{int} C_r \times E) \geq \delta \text{ for all } (x,i) \in A.$$

By invariance of μ and positive invariance of the invariant control sets C_r we find

$$\begin{aligned} \mu(A) &= \int_{\text{supp}\mu} P_{n+1}\left((x,i), A\right) \mu(d(x,i)) \\ &= \underbrace{\int_{\text{supp}\mu \cap (\bigcup C_r \times E)} P_{n+1}\left((x,i), A\right) d\mu}_{=0} + \int_{\text{supp}\mu \setminus (\bigcup C_r \times E)} P_{n+1}\left((x,i), A\right) d\mu \\ &= \int_A P_{n+1}\left((x,i), A\right) \mu(d(x,i)). \end{aligned}$$

Using the Chapman-Kolmogorov equation and again positive invariance of the invariant control sets C_r, we can estimate for all $(x,i) \in A$

$$\begin{aligned}
P_{n+1}\left((x,i),A\right) &= \int\limits_{M\times E} P_n\left((x,i),(y,j)\right) P_1\left((y,j),A\right)\mu(d(y,j)) \\
&= \int\limits_{(M\times E)\setminus(\bigcup C_r\times E)} P_n\left((x,i),(y,j)\right) \underbrace{P_1\left((y,j),A\right)}_{\leq 1}\mu(d(y,j)) \\
&\qquad \underbrace{+ \int\limits_{\bigcup C_r\times E} P_n\left((x,i),(y,j)\right) P_1\left((y,j),A\right)\mu(d(y,j))}_{=0} \\
&\leq \int\limits_{(M\times E)\setminus(\bigcup C_r\times E)} P_n\left((x,i),(y,j)\right)\mu(d(y,j)) \\
&= P_n\left((x,i),(M\times E)\setminus\left(\bigcup C_r\times E\right)\right) \\
&= \underbrace{P_n\left((x,i),M\times E\right)}_{=1} - \underbrace{P_n\left((x,i),\bigcup C_r\times E\right)}_{\geq\delta} \leq 1-\delta < 1.
\end{aligned}$$

Taken together this yields the contradiction

$$\mu(A) = \int\limits_{A} P_{n+1}\left((x,i),A\right)d\mu \leq (1-\delta)\mu(A) < \mu(A).$$

□

We note the following property.

Proposition 4.6. *Let μ be an invariant measure for the discrete-time process $(\tilde{Z}_n)_n$. If $\mathrm{supp}\mu \cap (C\times E) \neq \emptyset$ for some compact invariant control set C with $C = \mathrm{cl\ int}C$, then $C\times E \subset \mathrm{supp}\mu$ and, in particular, $\mu(C\times E) > 0$.*

Proof. Suppose, contrary to the assertion, that there is $(y,j) \in (C\times E)\setminus \mathrm{supp}\mu$. Since $C = \mathrm{cl\ int}C$, we may assume that $y \in \mathrm{int}C$. Thus there is an open neighborhood $\mathcal{W}(y,j) \subset C\times E$ with

$$\mathcal{W}(y,j) \cap \mathrm{supp}\mu = \emptyset. \tag{4.2}$$

Pick $(x_0,i_0) \in \mathrm{supp}\mu \cap (C\times E)$. Then there exists $y_0 \in \mathcal{O}^+(x_0)$ with $(y_0,j) \in \mathcal{W}(y,j)$. By Lemma 4.3(i), we find $\delta > 0$ and an open neighborhood $\mathcal{W}(x_0)$ of x_0 such that for all $z \in \mathcal{W}(x_0)$ and $i \in E$

$$P_n\left((z,i),\mathcal{W}(y,j)\right) \geq \delta \text{ and, clearly, } \mu(\mathcal{W}(x)\times E) > 0. \tag{4.3}$$

This implies, with $\delta_1 := \delta\cdot\mu(\mathcal{W}(x_0)\times E) > 0$,

$$\int_{M\times E} P_n\left((z,i),\mathcal{W}(y,j)\right)d\mu \geq \int_{\mathcal{W}(x_0)\times E} P_n\left((z,i),\mathcal{W}(y,j)\right)d\mu \geq \delta_1,$$

and hence

$$
\begin{aligned}
1 &= \int_{M\times E} P_n\left((z,i), M\times E\right) d\mu \\
&= \int_{M\times E} P_n\left((z,i), (M\times E)\setminus(\mathcal{W}(y,j))\right) d\mu + \int_{M\times E} P_n\left((z,i), \mathcal{W}(y,j)\right) d\mu \\
&\geq \int_{M\times E} P_n\left((z,i), (M\times E)\setminus(\mathcal{W}(y,j))\right) d\mu + \delta_1.
\end{aligned}
$$

Using also (4.1) and (4.2), we obtain the contradiction

$$
\begin{aligned}
1 = \mu(\mathrm{supp}\mu) &= \int_{M\times E} P_n\left((z,i), \mathrm{supp}\mu\right) d\mu \\
&\leq \int_{M\times E} P_n\left((z,i), (M\times E)\setminus(\mathcal{W}(y,j))\right) d\mu \leq 1-\delta_1 < 1.
\end{aligned}
$$

□

Finally, we discuss the ergodic case. Recall that an invariant measure μ is ergodic (extremal) if it cannot be written as a proper convex combination of invariant measures.

THEOREM 4.7. *(i) Assume that system (2.1) with controls in* $L^\infty(\mathbb{R}_+, \mathrm{co}(S))$ *is locally accessible on a compact positively invariant set* M*. Then for every ergodic measure* μ *of the discrete-time process* $(\tilde{Z}_n)_n$ *there is a compact invariant control set* C *with* $\mathrm{supp}\mu = C\times E$.

(ii) Conversely, let C *be a compact invariant control set. Then there exists an ergodic measure with support equal to* $C\times E$ *and every invariant measure with support contained in* $C\times E$ *has support equal to* $C\times E$.

(iii) Assume that for some x *in a compact invariant control set* C *the Lie algebra* $\mathcal{LA}(F^0,\ldots,F^m)$ *has full rank at* x*. Then there is a unique invariant measure* μ *supported by* $C\times E$ *(hence* μ *is ergodic) and nonnegative constant* c *and* ρ *with* $\rho<1$ *such that for all* $(x,i)\in C\times E$ *and Borel sets* $A\subset C$

$$
|\mathbb{P}_{x,i}[\tilde{Z}_n\in A]-\mu(A)| \leq c\rho^n, n\in\mathbb{N}. \tag{4.4}
$$

Proof. (i) Theorem 4.5 shows that $\mathrm{supp}\mu\subset\bigcup_r C_r\times E$ and it remains to prove the converse inclusion. In view of Proposition 4.6, we have to show that the support of μ can intersect only one set of the form $C_r\times E$ for an invariant control set C_r. Let $(x,i)\in\mathrm{supp}\mu$ for some x in the interior of C_r and some $i\in E$. Then $\mu(C_r\times E)>0$ and for $A\subset C_r\times E$

$$
\mu(A) = \int_{C_r\times E} P((x,i),A)d\mu.
$$

Here it suffices to integrate over $C_r\times E$, since for the other points (x,i) in the support of μ one has that $x\in C_s, s\neq r$, implying that the probability to reach C_r vanishes. Hence the conditional probability measure induced by μ on $C_r\times E$ is invariant. If

$$
\mu\left(\bigcup\nolimits_{s\neq r} C_s\times E\right) > 0,
$$

then μ is not ergodic, since it can be written as a proper convex combination of the conditional probability measures induced by μ on $C_r\times E$ and on $\bigcup_{s\neq r} C_s\times E$, respectively

(ii) Existence follows from Feller continuity, compactness and positive invariance of $C \times E$. The second statement follows from Proposition 4.6.

(iii) Uniqueness and the exponential estimate follow from Benaïm, Le Borgne, Malrieu and Zitt [5, Theorem 4.5]. In fact, this theorem considers for a compact positively invariant set M the set Γ defined in (2.12) and assumes that there is $x \in \Gamma$ such that the Lie algebra $\mathcal{LA}(F^0, \ldots, F^m)$ has full rank at x. Then it concludes that (4.4) holds for all Borel sets $A \subset \Gamma$. Here we choose $M = C$. Then Proposition 2.14 implies $C = \Gamma$ and the assertion follows. ∎

Finally, we note the following consequence for the continuous-time process.

THEOREM 4.8. *The assertions of Theorem 4.7 also hold for the continuous-time process $(Z_t)_{t\geq 0}$ defined in (3.1). In particular, if for some x in a compact invariant control set C the Lie algebra $\mathcal{LA}(F^0, \ldots, F^m)$ has full rank at x, there is a unique invariant measure μ supported by $C \times E$ (hence μ is ergodic) and constants $c > 1$ and $\alpha > 0$ such that for all $(x, i) \in C \times E$ and Borel sets $A \subset C \times E$*

$$|\mathbb{P}_{x,i}[Z_t \in A] - \mu(A)| \leq ce^{-\alpha t}, t \geq 0. \tag{4.5}$$

Proof. The analogues of assertions (i) and (ii) in Theorem 4.7 hold, since by [5, Proposition 2.4] there is a homeomorphism between the invariant measures of the discrete-time process and the continuous-time process mapping ergodic measures onto ergodic measures; by [5, Lemma 2.6] the homeomorphism preserves the supports. The exponential convergence (4.5) follows by [5, Theorem 4.6]. ∎

REMARK 4.9. *If instead of rank condition for the Lie algebra $\mathcal{LA}(F^0, \ldots, F^m)$ the rank of the smallest Lie algebra containing all "control vector fields" $F^i - F^0, i \neq 0$, in (2.4) and all Lie brackets with $F^i, i = 0, ..., n$, is considered, one can even show convergence of the distributions, cp. [5, Theorem 4.4].*

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