COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **10**, Number **3**, May **2011**

doi:10.3934/cpaa.2011.10.847

pp. **847–857**

TOPOLOGICAL CONJUGACY FOR AFFINE-LINEAR FLOWS AND CONTROL SYSTEMS

Fritz Colonius

Institut für Mathematik, Universität Augsburg
86135 Augsburg, Germany

Alexandre J. Santana

Departamento de Matemática, Universidade Estadual de Maringá
Maringá-PR, 87020-900, Brazil

Abstract. Hyperbolic affine-linear flows on vector bundles possess unique bounded solutions on the real line. Hence they are topologically skew conjugate to their linear parts. This is used to show a classification of inhomogeneous bilinear control systems.

1. Introduction.

The main subject of this paper are topological conjugacies of affine-linear control systems in $\mathbb{R}^d$. In particular, we will generalize the classical result (see e.g. Robinson [8]) that, in case of hyperbolicity, two linear autonomous differential equations are topologically conjugate if and only if the dimensions of the stable subspaces coincide. Here we consider control systems of the form

$$\dot{x} = A_0x + a_0 + \sum_{i=1}^{m} u_i(t)[A_ix + a_i],\ u = (u_1, ..., u_m) \in \mathcal{U}, \tag{1}$$

where A_i are in the set $\mathfrak{gl}(d,\mathbb{R})$ of real $d \times d$-matrices, $a_i \in \mathbb{R}^d$, and $\mathcal{U} := \{u \in L_\infty(\mathbb{R},\mathbb{R}^m),\ u(t) \in U$ for all $t \in \mathbb{R}\}$ is the set of admissible control functions with values in a set $U \in \mathbb{R}^m$. The solutions are in the Carathéodory sense; i.e. they satisfy the corresponding integral equation. We denote the solutions with initial condition $x(0) = x_0 \in \mathbb{R}^d$ by $\psi(t, x_0, u), t \in \mathbb{R}$.

Frequently, systems of this form are called inhomogeneous bilinear control systems in contrast to the homogeneous case where $a_i = 0$ for all i (D. Elliott [7] proposes to call systems of the form (1) 'biaffine'). Control system (1) defines a dynamical system (or flow) on $\mathcal{U} \times \mathbb{R}^d$ by

$$\Psi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d,\ \Psi_t(u, x) = (\theta_tu, \psi(t, x, u)), \tag{2}$$

where $(\theta_tu)(s) := u(t + s), s \in \mathbb{R}$, is the shift on $\mathcal{U}$. In fact, Ψ satisfies the flow properties $\Psi_0 = \mathrm{id}$ and $\Psi_{t+s} = \Psi_t\circ\Psi_s$ for $t, s \in \mathbb{R}$. If the control range U is compact and convex, the set $\mathcal{U}$ of admissible control functions is a compact metrizable space endowed with the weak* topology of L_∞ and Ψ is a continuous skew product flow (cf. Colonius and Kliemann [5] for details on this construction). We require that topological conjugacies of such flows respect the skew product structure; i.e., also for the topological conjugacies on the vector bundles $\mathcal{U} \times \mathbb{R}^d$ the first component

2000 *Mathematics Subject Classification.* Primary: 37B55, 37N35; Secondary: 93B10.

Key words and phrases. Topological conjugacy, affine-linear flows, bilinear control systems.

The second author is supported CAPES grant 4741/07-2.

should be independent of the second component; cf. also the monograph Cong [6] which includes an exposition of equivalences and normal forms for nonautonomous linear differential equations (emphasizing results based on the ergodic theory). For linear skew product flows (which cover the homogeneous case with $a_i = 0$ for all i in (1)), such topological skew conjugacies have been characterized by Ayala, Colonius, and Kliemann in [3]. Here we generalize their results to the affine case. Note that linear control systems $\dot{x} = Ax + Bu$ do not define linear flows, but affine flows. The work by Baratchart, Chyba, and Pomet [4] uses a similar notion of conjugacy for control systems (cf. Remark 5). This paper also contains a discussion of various conjugation notions for control systems. Here we only note that our notion of conjugacy is different from the notion of state equivalence, where the state transformations are not allowed to depend on the control functions (cf. Agrachev and Sachkov [1, Section 9.2].)

In Section 2, we discuss in the more general context of affine-linear flows on vector bundles the existence of unique solutions e_0 which are bounded on the real line; this holds provided that the flows are hyperbolic. Here we rely on methods from Aulbach and Wanner [2], where general nonautonomous differential equations of Carathéodory type are treated. This is used to prove that hyperbolic affine-linear flows are topologically skew conjugate to their linear part provided that an additional continuity property holds. In Section 3 we show that control systems of the form (1) define affine-linear skew product flows and that they are topologically skew conjugate to their homogeneous parts. In particular, using the classification for homogeneous bilinear control systems from [3], we obtain a classification for these inhomogeneous bilinear control systems.

2. **Bounded solutions for affine-linear flows.**

The purpose of this section is to show that for hyperbolic affine-linear flows there exist unique bounded solutions depending continuously on the base points. Then skew conjugacy is characterized.

We start by defining affine-linear flows on vector bundles. Recall that a linear skew product flow $\Phi = (\theta, \varphi)$ on a vector bundle $B \times \mathbb{R}^d$ with compact metric base space B is a continuous map

$$\Phi : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d$$

with the flow properties $\Phi_0 = \mathrm{id}$ and $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for $t, s \in \mathbb{R}$, and

$$\Phi_t(b, x) = (\theta_t b, \varphi(t, b, x)) \text{ for } (t, b, x) \in \mathbb{R} \times B \times \mathbb{R}^d,$$

where θ is a continuous flow on the base space B and $\varphi(t, b, x)$ is linear in x; i.e. $\varphi(t, b, \alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \varphi(t, b, x_1) + \alpha_2 \varphi(t, b, x_2)$ for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^d$.

Definition 2.1. Let $B \times \mathbb{R}^d$ be a vector bundle with compact metric base space B. A continuous map $\Psi = (\theta, \psi) : \mathbb{R} \times B \times \mathbb{R}^d \to B \times \mathbb{R}^d$ is called an affine-linear Carathéodory skew product flow on $B \times \mathbb{R}^d$ if there are a linear skew product flow $\Phi = (\theta, \varphi)$ and a function $f : B \to L_\infty(\mathbb{R}, \mathbb{R}^d)$ such that f satisfies

$$f(b, t + s) = f(\theta_s(b), t) \text{ for all } b \in B \text{ and almost all } t, s \in \mathbb{R}, \tag{3}$$

and for all $(t, b, x) \in \mathbb{R} \times B \times \mathbb{R}^d$

$$\Psi_t(b, x) = \Phi_t(b, x) + \int_0^t \Phi_{t-s}(\theta_s b, f(b, s)) \mathrm{d}s. \tag{4}$$

Here we write $f(b,s) := f(b)(s), s \in \mathbb{R}$. For brevity, we call the flows defined above just affine-linear flows. The base flows of Ψ and Φ coincide and the integral in (4) is a Lebesgue integral in the $\mathbb{R}^d$-component only. Hence, in terms of ψ, this equation means

$$\psi(t,b,x) = \varphi(t,b,x) + \int_0^t \varphi(t-s, \theta_s b, f(b,s))\mathrm{d}s, \tag{5}$$

and the flow property of Ψ is expressed by the cocycle property

$$\psi(t+s,b,x) = \psi(t, \theta_s b, \psi(s,b,x)).$$

The following proposition shows that Definition 2.1, in fact, defines a flow.

Proposition 1. *Let Ψ be an affine-linear flow as defined above. Then Ψ satisfies the flow properties*

$$\Psi_0 = \mathrm{id} \text{ and } \Psi_{t+s} = \Psi_t \circ \Psi_s \text{ for } t,s \in \mathbb{R}.$$

Proof. The property $\Psi_0 = \mathrm{id}$ is obvious and one computes

$$\begin{aligned}
&\Psi_{t+s}(b,x)\\
&= \Phi_{t+s}(b,x) + \int_0^{t+s} \Phi_{t+s-\tau}(\theta_\tau b, f(b,\tau))\mathrm{d}\tau\\
&= \Phi_t \circ \Phi_s(b,x) + \int_0^s \Phi_{t+s-\tau}(\theta_\tau b, f(b,\tau))\mathrm{d}\tau + \int_s^{t+s} \Phi_{t+s-\tau}(\theta_\tau b, f(b,\tau))\mathrm{d}\tau\\
&= \Phi_t \circ \Phi_s(b,x) + \int_0^s \Phi_t \circ \Phi_{s-\tau}(\theta_\tau b, f(b,\tau))\mathrm{d}\tau + \int_0^t \Phi_{t-\tau}(\theta_{\tau+s} b, f(b,\tau+s))\mathrm{d}\tau\\
&= \Phi_t \circ \Phi_s(b,x) + \int_0^s \Phi_t \circ \Phi_{s-\tau}(\theta_\tau b, f(b,\tau))\mathrm{d}\tau + \int_0^t \Phi_{t-\tau}(\theta_{\tau+s} b, f(\theta_s b,\tau))\mathrm{d}\tau\\
&= \Phi_t(\Psi_s(b,x)) + \int_0^t \Phi_{t-\tau}(\theta_{\tau+s} b, f(\theta_s b,\tau))\mathrm{d}\tau\\
&= \Psi_t(\Psi_s(b,x)).
\end{aligned}$$

□

Remark 1. Continuity of the map Ψ defined in (4) is equivalent to the property that the inhomogeneous term

$$a(t,b) := \int_0^t \Phi_{t-s}(\theta_s b, f(b,s))\mathrm{d}s,\ t \in \mathbb{R}, b \in B, \tag{6}$$

is continuous. Sufficiency follows from continuity of Φ, necessity follows by setting $x = 0$ in (4).

Remark 2. In Definition 2.1, the range of the map f is taken as $L_\infty(\mathbb{R}, \mathbb{R}^d)$. This is due to our intention to treat control system (1) with bounded control range U. Certainly, the consideration of affine-linear flows with other ranges of the affine term f makes sense; e.g. one could require that the values of f are locally integrable functions. Furthermore, consideration of general vector bundles, which only locally are products, is certainly worthwhile.

Next we clarify the relation between the homogeneous equation described by the linear flow Φ and the inhomogeneous equation.

Lemma 2.2. *Let Ψ be an affine-linear flow on the vector bundle $B\times\mathbb{R}^d$ and consider initial values $(b,x_1),(b,x_2)\in B\times\mathbb{R}^d$. Then the difference of the corresponding solutions is a solution of the homogeneous system with initial value $(b,x_1-x_2)\in B\times\mathbb{R}^d$; i.e. it satisfies*

$$\Psi_t(b,x_1)-\Psi_t(b,x_2)=\Phi_t(b,x_1-x_2) \text{ for all } t\in\mathbb{R}.$$

Proof. This is immediate from the definition (4) and linearity of $\Phi_t(b,x)$ in the second argument. □

The next lemma, which is a modification of Aulbach and Wanner [2, Lemma 3.2], shows existence and continuous dependence of bounded solutions provided that exponential stability holds. We start with the following notational remarks and assumptions.

For an affine-linear flow (4) suppose that the bundle $B\times\mathbb{R}^d$ admits a decomposition into the Whitney sum of two vector subbundles

$$B\times\mathbb{R}^d=\mathcal{V}^1\oplus\mathcal{V}^2, \tag{7}$$

where $\mathcal{V}^1$ and $\mathcal{V}^2$ are invariant under the linear flow Φ. Thus for every $b\in B$ we have a decomposition

$$\mathbb{R}^d=\mathcal{V}^1_b\oplus\mathcal{V}^2_b$$

into linear subspaces $\mathcal{V}^1_b$ and $\mathcal{V}^2_b$ which have dimensions independent of the base point $b\in B$. We identify $\{b\}\times\mathcal{V}^i_b$ with $\mathcal{V}^i_b$ and note that the linear flow Φ leaves the subbundles invariant in the following sense:

$$\Phi_t(b,x)\in\mathcal{V}^i_{\theta_t b} \text{ for } x\in\mathcal{V}^i_b \text{ and } t\in\mathbb{R}.$$

We denote the restricted linear flows on $\mathcal{V}^i$ by Φ^i_t and write $\left\|\Phi^1_t(b,\cdot)\right\|$ for the norm of the linear map $x\mapsto\varphi^1_t(b,x):\mathcal{V}^1_b\to\mathcal{V}^i_{\theta_t b}$. Decompose the affine term accordingly,

$$f(b,s)=f^1(b,t)+f^2(b,t) \tag{8}$$

with $f^i(b,t)\in\mathcal{V}^i_b, t\in\mathbb{R}, b\in B$. If f^i satisfies property (3), then $\mathcal{V}^i$ is also invariant under the affine-linear flow Ψ^i defined by

$$\Psi^i_t(b,x)=\Phi^i_t(b,x)+\int_0^t\Phi^i_{t-s}(b,f^i(b,s))\mathrm{d}s. \tag{9}$$

Lemma 2.3. *Consider the affine-linear flow (4) and assume that the following conditions are satisfied:*

(i) the linear part Φ of Ψ admits a decomposition (7) into invariant subbundles $\mathcal{V}^1$ and $\mathcal{V}^2$, where $\mathcal{V}^1$ is stable: there are constants $\alpha>0$ and $K\geq 1$ such for the restricted flow Φ^1 on $\mathcal{V}^1$

$$\left\|\Phi^1_t(b,\cdot)\right\|\leq Ke^{-\alpha t} \text{ for all } t\geq 0 \text{ and all } b\in B; \tag{10}$$

(ii) the affine term f^1 defined by (8) satisfies property (3), and there is $M>0$ with

$$\left\|f^1(b)\right\|_\infty\leq M \text{ for all } b\in B. \tag{11}$$

Then for every $b\in B$ there is a unique bounded solution $e^1(b,t),t\in\mathbb{R}$, for the flow Ψ^1.

If the map

$$b\mapsto\int_{-\infty}^0\varphi^1(-s,\theta_s b,f^1(b,s))\mathrm{d}s:B\to\mathbb{R}^d \tag{12}$$

is continuous, then the map $e^1:B\times\mathbb{R}\to\mathbb{R}^d$ is continuous.

Proof. The proof consists of two steps.

(I) The linear flow Φ has only the trivial bounded solution on $\mathbb{R}$. In fact, let $\varphi(t,b,x)$ be a bounded solution on $\mathbb{R}$. Then for $t \geq 0$

$$\begin{aligned}\|x\| &= \|\varphi(0,b,x)\| = \|\varphi(t,\theta_{-t}b,\varphi(-t,b,x))\| \\ &\leq \left\|\Phi_t^1(\theta_{-t}b,\cdot)\right\| \|\varphi(-t,b,x)\| \\ &\leq Ke^{-\alpha t}\sup_{s\in\mathbb{R}} \|\varphi(s,b,x)\|.\end{aligned}$$

The right hand side converges to 0 for $t \to \infty$ and hence $x = 0$ follows.

(II) Now suppose that $f^1(b,t)$ is given by (8). Uniqueness of the bounded solution follows immediately from step (I), since by Lemma 2.2 the difference of two bounded solutions is a bounded solution of the homogeneous equation. In order to show existence, define

$$e^1 : B \times \mathbb{R} \to \mathbb{R}^d \text{ by } e^1(b,t) := \int_{-\infty}^{t} \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s. \tag{13}$$

Note that the integral indeed exists, since for all $b \in B$ and $s \leq t$

$$\left\|\varphi^1(t-s,\theta_s b,f^1(b,s))\right\| \leq \left\|\Phi_{t-s}^1(\theta_s b,\cdot)\right\| \sup\{\left\|f^1(b,s)\right\|\} \leq KMe^{\alpha(t-s)}.$$

In order to show the continuity property, fix $t_0 \in \mathbb{R}$ and $b_0 \in B$. Then, denoting the characteristic function of the interval $(-\infty,t]$ by $\chi_{(-\infty,t]}$, one finds

$$\begin{aligned}&\left|e^1(b,t) - e^1(b_0,t_0)\right| \\ &= \left|\int_{-\infty}^{t} \varphi^1(t-s,\theta_s b,f^1(b,s))\mathrm{d}s - \int_{-\infty}^{t_0} \varphi^1(t_0-s,\theta_s b_0,f^1(b_0,s))\mathrm{d}s\right| \\ &\leq \left|\int_{\mathbb{R}} \chi_{(-\infty,t]}(s)\varphi^1(t-s,\theta_s b,f^1(b,s))\mathrm{d}s - \int_{\mathbb{R}} \varphi^1(t_0-s,\theta_s b,f^1(b,s))\mathrm{d}s\right| \\ &+ \left|\int_{0}^{t_0} \varphi^1(t_0-s,\theta_s b,f^1(b,s))\mathrm{d}s - \int_{0}^{t_0} \varphi^1(t_0-s,\theta_s b_0,f^1(b_0,s))\mathrm{d}s\right| \\ &+ \left|\int_{-\infty}^{0} \varphi^1(t_0-s,\theta_s b,f^1(b,s))\mathrm{d}s - \int_{-\infty}^{0} \varphi^1(t_0-s,\theta_s b_0,f^1(b_0,s))\mathrm{d}s\right|\end{aligned}$$

For $(t,b) \to (t_0,b_0)$, the first summand converges to 0 by Lebesgue's theorem on dominated convergence, since

$$\chi_{(-\infty,t]}(s)\varphi_{t-s}^1(\theta_s b,f^1(b,s)) \to \chi_{(-\infty,t_0]}(s)\varphi_{t_0-s}^1(\theta_s b_0,f^1(b_0,s)) \text{ for all } t,s,$$

and the integrands are bounded. The second summand converges to 0 by the same arguments. The third summand equals

$$\begin{aligned}&\left|\int_{-\infty}^{0} \varphi^1(t_0,b,\varphi^1(-s,\theta_s b,f^1(b,s)))\mathrm{d}s - \int_{-\infty}^{0} \varphi^1(t_0,b_0,\varphi^1(-s,\theta_s b_0,f^1(b_0,s)))\mathrm{d}s\right| \\ &= \left|\varphi^1(t_0,b,\textstyle\int_{-\infty}^{0}\varphi^1(-s,\theta_s b,f^1(b,s))\mathrm{d}s) - \varphi^1(t_0,b_0,\int_{-\infty}^{0}\varphi^1(-s,\theta_s b_0,f^1(b_0,s))\mathrm{d}s)\right|.\end{aligned}$$

Then the continuity assumption for the map (12) together with continuity of φ^1 shows that this converges to 0.

Now let us show that the function $e^1(b,t)$ defined in (13) is a solution for the flow Ψ^1. In fact, formula (4) is satisfied for f^1, since

$$\begin{aligned} e^1(b,t) &= \int_{-\infty}^{t} \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s \\ &= \int_0^t \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s + \int_{-\infty}^{0} \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s \\ &= \int_0^t \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s + \varphi^1(t,b,\textstyle\int_{-\infty}^{0}\varphi^1(-s,\theta_s b, f^1(b,s))\mathrm{d}s) \\ &= \int_0^t \varphi^1(t-s,\theta_s b, f^1(b,s))\mathrm{d}s + \varphi^1(t,b,x) \end{aligned}$$

for

$$x := e(b,0) = \int_{-\infty}^{0} \varphi^1(-s,\theta_s b, f^1(b,s))\mathrm{d}s.$$

□

Next we generalize this result to hyperbolic systems.

Corollary 1. *Consider the affine-linear flow Ψ in (4) and assume that the following conditions are satisfied:*

(i) the linear part Φ of Ψ is hyperbolic. Thus it admits a decomposition (7) into invariant subbundles $\mathcal{V}^1$ and $\mathcal{V}^2$, where $\mathcal{V}^1$ is stable and $\mathcal{V}^2$ is unstable such that the restrictions Φ^1 and Φ^2 of Φ to $\mathcal{V}^1$ and $\mathcal{V}^2$, respectively, satisfy for constants $\alpha > 0$ and $K_1, K_2 \geq 1$ and for all $b \in B$

$$\left\|\Phi_t^1(b,\cdot)\right\| \leq K_1 e^{-\alpha t} \text{ for } t \geq 0 \text{ and } \left\|\Phi_t^2(b,\cdot)\right\| \leq K_2 e^{\alpha t} \text{ for } t \leq 0;$$

(ii) the affine terms f^1 and f^2 defined by (8) satisfy property (3), and there is $M > 0$ with

$$\|f(b)\|_\infty \leq M \text{ for all } b \in B.$$

Then for every $b \in B$ there is a unique bounded solution $e(b,t), t \in \mathbb{R}$, for the flow Ψ.

If the maps $B \to \mathbb{R}^d$

$$b \mapsto \int_{-\infty}^{0} \varphi^1(-s,\theta_s b, f(b,s))\mathrm{d}s \text{ and } b \mapsto \int_{-\infty}^{0} \varphi^2(s,\theta_{-s} b, f(b,-s))\mathrm{d}s \tag{14}$$

are continuous, then the map $e : \mathbb{R} \times B \to \mathbb{R}^d$ is continuous.

Proof. Since $\left\|f^i(b,t)\right\| \leq \|f(b,t)\|$, property (11) holds for f^1 and f^2. Hence Lemma 11 implies the existence of unique bounded solutions $e^1(t,b)$ $t \in \mathbb{R}, b \in B$. Applying this lemma to the time inverse of Ψ^2, one also finds unique bounded solutions $e^2(t,b)$. This yields bounded solutions

$$e(b,t) := e^1(b,t) + e^2(b,t)$$

for Ψ. Conversely, a bounded solution for Ψ projects down to bounded solutions of Ψ^1 and Ψ^2. This shows uniqueness of $e(b,t)$. Step (II) of the proof for Lemma 11 applied to the flow Ψ^1 and the time inverse of Ψ^2 implies continuity of the map $(t,b) \mapsto e(b,t)$. □

Corollary 1 will allow us to derive results on topological conjugacy. More specifically, we use the following notion of topological conjugacy which respects the skew product structure (cf. [3, Definition 2.3]).

Definition 2.4. Let $\Psi^1 = (\theta^1, \psi^1)$ and $\Psi^2 = (\theta^2, \psi^2)$ be affine linear flows on vector bundles $B^1 \times \mathbb{R}^d$ and $B^2 \times \mathbb{R}^d$, respectively. We say that Ψ^1 and Ψ^2 are topologically skew conjugate if there exists a skew homeomorphism

$$H = (h_B, h) : B^1 \times \mathbb{R}^d \to B^2 \times \mathbb{R}^d$$

such that $H(\Psi^1_t(b,x)) = \Psi^2_t(H(b,x))$; i.e., $h_B : B^1 \to B^2, h : B^1 \times \mathbb{R}^d \to \mathbb{R}^d$ are maps with

$$h_B(\theta^1_t b)) = \theta^2_t(h_B(b)) \text{ for all } t \in \mathbb{R} \text{ and } b \in B^1, \tag{15}$$

$$h(\theta^1_t b, \psi^1(t,b,x)) = \psi^2(t, h_B(b), h(b,x)) \text{ for all } t \in \mathbb{R}, b \in B^1, \text{ and } x \in \mathbb{R}^d. \tag{16}$$

Thus topological skew conjugacy requires that the base flows are topologically conjugate via the homeomorphism h_B and (16) holds. The next theorem shows that hyperbolic affine-linear flows are skew conjugate to their linear part.

Theorem 2.5. *Consider the affine-linear flow Ψ in (4) and assume that the following conditions are satisfied:*

(i) the linear part Φ of Ψ is hyperbolic with stable subbundle $\mathcal{V}^1$ and unstable subbundle $\mathcal{V}^2$;

(ii) the affine terms f^1 and f^2 defined by (8) satisfy property (3), and there is $M > 0$ with

$$\|f(b)\|_\infty \le M \text{ for all } b \in B.$$

(iii) The maps : $B \to \mathbb{R}^d$

$$b \mapsto \int_{-\infty}^0 \varphi^1(-s, \theta_s b, f(b,s))\mathrm{d}s \text{ and } b \mapsto \int_{-\infty}^0 \varphi^2(s, \theta_{-s} b, f(b,-s))\mathrm{d}s$$

are continuous.

Then Ψ and its linear part Φ are topologically skew conjugate. Moreover, consider two affine-linear flows Ψ and $\hat{\Psi}$ satisfying assumptions (i), (ii), and (iii). Then they are topologically skew conjugate, if and only if the base flows are topologically conjugate and the dimensions of the stable subbundles coincide.

Proof. The second assertion follows from the first one and [3, Corollary 3.4], which states that two hyperbolic linear flows are topologically skew conjugate iff their base flows are topologically conjugate and the dimensions of their stable subbundles coincide. Hence it remains to show that Ψ and its linear part Φ are topologically skew conjugate. Their base flows coincide, hence we can take h_B as the identity on B. Define h as the translation with respect to the unique bounded solutions e, which exist by Corollary 1:

$$h(b,x) = x - e(b,0),\ (b,x) \in B \times \mathbb{R}^d,$$

The map $H : B \times \mathbb{R}^d \to B \times \mathbb{R}^d$ defined by

$$H(b,x) = (b, h(b,x)),\ (b,x) \in B \times \mathbb{R}^d,$$

is continuous and bijective with continuous inverse

$$H^{-1}(b,x) = (b, x + e(b,0)),\ (b,x) \in B \times \mathbb{R}^d.$$

Hence it is a skew homeomorphism. By Lemma 2.2, the difference of the solutions for the initial values x and $e(b,0)$ is a solution of the homogeneous system with initial value $(b, x - e(b,0)) \in B \times \mathbb{R}^d$,

$$\Psi_t(b,x) - \Psi_t(b, e(b,0)) = \Phi_t(b, x - e(b,0)) \text{ for all } t \in \mathbb{R}.$$

Since $e(\theta_t b, 0) = e(b,t)$, we find

$$\begin{aligned} h(\theta_t^1 b, \psi(t,b,x)) &= \psi(t,b,x) - e(\theta_t b, 0) \\ &= \psi(t,b,x) - e(b,t) \\ &= \psi(t,b,x) - \psi(t,b,e(b,0)) \\ &= \varphi(t,b,x-e(b,0)) \\ &= \varphi(t,b,h(b,x)). \end{aligned}$$

Hence equality (16) holds and $H = (\mathrm{id}_B, h)$ is the desired conjugacy. □

3. **Conjugacy in affine control system.** In this section we show that control systems of the form (1) define affine-linear flows. In case of hyperbolicity, they satisfy the assumptions of Theorem 2.5 and hence we obtain a classification of these control systems with respect to topological skew conjugacy.

Consider an affine control system

$$\dot{x} = A_0 x + a_0 + \sum_{i=1}^{m} u_i(t)[A_i x + a_i],\ u = (u_1, ..., u_m) \in \mathcal{U}, \tag{17}$$

given by $\{(A_0, a_0), \cdots, (A_{m-1}, a_{m-1})\}$, where $A_i \in \mathfrak{gl}(d,\mathbb{R})$, $a_i \in \mathbb{R}^d$ for $i = 0, \ldots, m$, and the control range $U \subset \mathbb{R}^m$ is assumed to be compact and convex.

Remark 3. A special case of (1) is

$$\dot{x} = A_0 x + a_0 + \sum_{i=1}^{m_1} u_i(t) A_i x + B u_0(t), \tag{18}$$

where B is a $d \times m_2$ matrix with columns $b_i \in \mathbb{R}^d$. This follows by defining $m = m_1 + m_2, A_i = 0$ for $i = m_1 + 1, ..., m_1 + m_2$, and $a_i = 0$ for $i = 1, ..., m_1, a_i = b_i$ for $i = m_1 + 1, ..., m_1 + m_2$. Note also that

$$\dot{x} = A_0 x + u(t) A_1 x + u(t) a_1$$

is not of the form (18).

System (17) can be written as

$$\dot{x} = A(u(t))x + a(u(t)),$$

where

$$A(u) := A_0 + \sum_{i=1}^{m} A_i u_i \text{ and } a(u) := \sum_{i=1}^{m} u_i a_i + a_0,\ u \in \mathbb{R}^m.$$

For $u \in \mathcal{U}$, the fundamental solution $X_u(t,s)$ of the homogeneous equation is given by

$$\frac{d}{dt} X_u(t,s) = A(u(t)) X_u(t,s),\ X_u(s,s) = I,\ t,s \in \mathbb{R}.$$

Then the solutions $x(t) = \psi(t,s,x_0,u)$ of (17) with initial condition $x(s) = x_0$ are given by

$$x(t) = X_u(t,s)x_0 + \int_s^t X_u(t,\tau) a(u(\tau)) \mathrm{d}\tau, t \in \mathbb{R}.$$

If $s = 0$ we omit this argument, i.e., $\psi(t,x_0,u) := \psi(t,0,x_0,u)$.

Proposition 2. *Under the assumptions above, control system (17) defines an affine-linear flow* Ψ *given by (2) on the vector bundle* $\mathcal{U} \times \mathbb{R}^d$*, where* $\mathcal{U}$ *is endowed with a metric compatible with the weak* topology on* $L_\infty(\mathbb{R}, \mathbb{R}^m)$ *which can be identified with the dual space of* $L_1(\mathbb{R}, \mathbb{R}^m)$*.*

Proof. Since this system is control-affine, [5, Lemma 4.3.2] implies that the corresponding flow Ψ on $\mathcal{U} \times \mathbb{R}^d$ is continuous, uniformly on bounded t-intervals. Define

$$f : \mathcal{U} \to L_\infty(\mathbb{R}, \mathbb{R}^d),\ f(u,t) := a(u(t)) = \sum_{i=1}^m u_i(t)a_i + a_0, u \in \mathcal{U}, t \in \mathbb{R}.$$

Then $f(u, t+s) = a(u(t+s)) = f(\theta_s(u), t)$. We can write the variations-of-constants formula as

$$\psi(t, x_0, u) = X_u(t,0)x_0 + \int_0^t X_u(t,s)f(u,s)\mathrm{d}s.$$

Observe that the columns of $X_u(t,s)$ are the solutions of the bilinear control system

$$\dot{x} = A(u(t))x \text{ with } x(s) = e_i,$$

where e_i is the ith standard basis vector in $\mathbb{R}^d$. Thus they depend continuously on (t,u). Furthermore, $X_u(t,s) = X_{u(s+\cdot)}(t-s,0)$. We see that Ψ and the flow Φ associated with the homogeneous system $\dot{x} = A(u(t))x, u \in \mathcal{U}$, satisfy

$$\Psi_t(u,x) = \Phi_t(u,x) + \int_0^t \Phi_{t-s}(\theta_s u, f(u,s))\mathrm{d}s \text{ for } (t,u,x) \in \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d.$$

□

The following theorem is the main result of this paper. It presents a classification of affine (or inhomogeneous bilinear) control systems with respect to topological skew conjugacy.

Theorem 3.1. *Consider affine control system (17) with compact and convex control range* $U \subset \mathbb{R}^m$*, and its homogeneous part*

$$\dot{x} = A_0 x(t) + \sum_{i=1}^m u_i(t) A_i x(t) = A(u(t))x(t) \tag{19}$$

with associated flows Ψ *and* Φ *on* $\mathcal{U} \times \mathbb{R}^d$*, respectively. Assume that the linear flow* Φ *is hyperbolic. Then* Ψ *and* Φ *are topologically skew conjugate. Moreover, consider two affine control systems of the form (17). Then the associated affine-linear flows are topologically skew conjugate if and only if the shift flows on the sets of control functions are topologically conjugate and the dimensions of the stable subbundles coincide.*

Proof. We only have to verify the assumptions of Theorem 2.5. The hyperbolicity condition (i) holds by assumption. Concerning condition (ii), the affine term is here given by

$$f(u,t) := a(u(t)) = \sum_{i=1}^m u_i(t)a_i + a_0,$$

and, as noted in the proof of Proposition 2, property (3) holds:

$$f(u, t+s) = a(u(t+s)) = \sum_{i=1}^m u_i(t+s)a_i + a_0 = f(\theta_s(u), t).$$

The projections to the stable and unstable subbundles are linear. Since $f(u,t)$ is a linear combination of the a_i, its projections are determined by the projections of the a_i. Then property (3) also holds for the projections of f to the stable and the unstable subbundles. Uniform boundedness follows by compactness of the control range U.

It remains to show the continuity properties in (iii). The integrand in

$$\int_{-\infty}^{0} \varphi^1(-s,\theta_s u, f(u,s))\mathrm{d}s = \int_0^{\infty} \varphi^1(s,\theta_{-s}u, \textstyle\sum_{i=1}^m u_i(s)a_i + a_0)\mathrm{d}s$$

is bounded by

$$\begin{aligned}\left|\varphi(s,\theta_{-s}u, \sum_{i=1}^m u_i(s)a_i + a_0)\right| &= \left|\Phi_s^1(\theta_{-s}u,\cdot)\left[\textstyle\sum_{i=1}^m u_i(\tau)a_i + a_0\right]\right| \\ &\le \left\|\Phi_s^1(\theta_{-s}u,\cdot)\right\| \left|\textstyle\sum_{i=1}^m u_i(\tau)a_i + a_0\right| \\ &\le K_1 e^{-\alpha t}\sup_{u\in U}\left|\textstyle\sum_{i=1}^m u_i a_i + a_0\right|.\end{aligned}$$

Hence, invoking Lebesgue's theorem on dominated convergence, continuity follows if the integrand converges pointwise for $u \to u^0$ weak* in $\mathcal{U}$. One has

$$\begin{aligned}&\left|\varphi(s,\theta_{-s}u, f(u,s)) - \varphi(s,\theta_{-s}u^0, f(u^0,s))\right| \\ &\le \left|\varphi(s,\theta_{-s}u, \textstyle\sum_{i=1}^m u_i(s)a_i + a_0)\mathrm{d}s - \varphi(s,\theta_{-s}u^0, \textstyle\sum_{i=1}^m u_i(s)a_i + a_0)\mathrm{d}s\right| \\ &+ \left|\varphi(s,\theta_{-s}u^0, \textstyle\sum_{i=1}^m u_i(s)a_i + a_0) - \varphi(s,\theta_{-s}u^0, \textstyle\sum_{i=1}^m u_i^0(s)a_i + a_0)\right|.\end{aligned}$$

The variation-of constants formula shows that the first summand equals

$$\begin{aligned}&\int_0^s [X_{\theta_{-s}u}(t,\tau) - X_{\theta_{-s}u^0}(t,\tau)]a(u(\tau))\mathrm{d}\tau \\ &\le \int_0^s \left\|X_{\theta_{-s}u}(s,\tau) - X_{\theta_{-s}u^0}(s,\tau)\right\| \mathrm{d}\tau \,\sup_{u\in U}\left|\textstyle\sum_{i=1}^m u_i a_i + a_0\right|.\end{aligned}$$

The integrand is bounded and converges pointwise to 0. Hence the first summand converges to 0 for $u \to u_0$.

The second summand equals

$$\begin{aligned}&\int_0^s X_{\theta_{-s}u^0}(s,\tau)[a(u(\tau)) - a(u^0(\tau))]\mathrm{d}\tau \\ &= \int_{\mathbb{R}} \chi_{[0,s]}(\tau) X_{\theta_{-s}u^0}(s,\tau)[a(u(\tau)) - a(u^0(\tau))]\mathrm{d}\tau.\end{aligned}$$

Since a is affine linear, weak* convergence of u to u^0 implies that also $a(u(\cdot))$ converges weak* to $a(u^0(\cdot))$ in $L_\infty(\mathbb{R},\mathbb{R}^d)$, hence also the second summand converges to 0 and continuity of the first map in (iii) is established. Analogously, one shows continuity of the second map. □

Remark 4. Criteria for topological conjugacy of the shift flows on the base space of control functions are given in [3, Corollary 6.3].

Remark 5. Theorem 3.1, in particular, applies to linear control systems $\dot{x} = Ax + Bu$ with compact convex control range. For hyperbolic matrix A, it follows that these control systems are topologically conjugate to the uncontrolled system $\dot{x} = Ax$; the conjugacy map $h : \mathcal{U}\times\mathbb{R}^d \to \mathbb{R}^d$ is $h(u,x) = x - e(u,t)$, where $e(u,t) := -A^{-1}Bu(t)$. This is similar to the Hartman-Grobman theorem by Baratchart, Chyba, and Pomet [4, Theorem 3.7], who show that (locally around an equilibrium and for small control ranges) control systems are conjugate to the uncontrolled system obtained by linearization about the equilibrium.

REFERENCES

[1] A. A. Agrachev and Y. L. Sachkov, "Control Theory from a Geometric Viewpoint," Springer-Verlag, New York, 2004.

[2] B. Aulbach and T. Wanner, *Integral manifolds for Carathéodory type differential equations in Banach spaces*, in "Six Lectures on Dynamical Systems" (eds. B. Aulbach and F. Colonius), World Scientific, (1996), 45–119.

[3] V. Ayala, F. Colonius and W. Kliemann, *On topological equivalence of linear flows with applications to bilinear control systems*, J. Dynamical and Control Systems, **13** (2007), 337–362.

[4] L. Baratchart, M. Chyba and J.-P. Pomet, *A Grobman-Hartman theorem for control systems,* J. Dynamics and Differential Equations, **19** (2007), 95–107.

[5] F. Colonius and W. Kliemann, "The Dynamics of Control," Birkhäuser, Boston, 2000.

[6] Nguyen Dinh Cong, "Topological Dynamics of Random Dynamical Systems," Oxford Math. Monogr., Clarendon Press 1997.

[7] D. L. Elliott, "Bilinear Control Systems. Matrices in Action," Applied Mathematical Sciences, **169** Springer-Verlag, New York, 2009.

[8] C. Robinson, "Dynamical Systems. Stability, Symbolic Dynamics, and Chaos," CRC Press, 1999.

Received March 2009; revised September 2009.

E-mail address: `fritz.colonius@math.uni-augsburg.de`
E-mail address: `ajsantana@uem.br`