Minimal data rates and invariance entropy

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***Abstract*— For compact locally controlled invariant subsets of the state space, minimal data rates for achieving invariance are characterized by the invariance entropy. In particular, for linear control systems with bounded control range, locally invariant sets are constructed and the associated minimal data rates are computed.**

I. INTRODUCTION

The purpose of this paper is to characterize the minimal data rates which are necessary to render a subset Q of the state space of a control system invariant. More precisely, let

$$\dot{x}(t) = f(x(t), u(t)),\ u \in \mathcal{U},$$

be a finite dimensional control system in $\mathbb{R}^n$ with admissible (open-loop) controls in a set $\mathcal{U}$ of functions defined on $\mathbb{R}$ with values in $\mathbb{R}^m$. We fix a subset Q of the state space $\mathbb{R}^n$ where the system should remain under the action of a controller. Often controls are generated by sending the information about the state at a time T to a controller which computes a time-dependent control function which is used on a certain time interval $[T, T+S]$ until the next information about the state arrives. Such devices are, in particular, used in nonlinear model-predictive control; see, e.g., Grüne [6]. If the digital communication to the controller is restricted, symbolic controllers may be used, which only need a limited amount of bits transferred to them per time unit in order to generate the desired control action. In any case, the result of all possible controller actions will be an, in general infinite, set $\mathcal{R}$ of time dependent control functions. Here we neglect the (for practical purposes dominant) question how to encode the information on the states which is sent over the communication channel. Instead we analyze the information that is needed in order to determine control functions in such a set $\mathcal{R}$.

Then the main topic of this paper is the problem when the minimal data rate (i.e. the average number of bits per time) equals the invariance entropy introduced in [2]. In the present paper, we require a property, which is stronger than the standard assumption of controlled invariance. For initial values in Q, it must be possible to steer the system into the interior of Q, without leaving a neighborhood of Q. This property, which we call local controlled invariance, is close to a hypothesis in [11].

Similar problems have received some attention in the last years; see e.g. [5], [7], [11]; the fundamental paper [11] by Nair, Evans, Mareels, and Moran establishes a relation to the notion of topological entropy from the theory of dynamical systems. In [2] we could relate minimal data rates to the so-called strict invariance entropy which is based on quantization of the state space. However, we can show finiteness of strict invariance entropy only under restrictive assumptions. The present paper instead relates minimal data rates to invariance entropy which is finite under weak hypotheses. The dissertation [8] and the papers [9], [10], [1] contain a number of further results for invariance entropy. Hopefully, the analysis put forward here will also be useful for understanding minimal data rates in networks of control systems.

In section II, a precise problem formulation is given and the notion of local controlled invariance is introduced. Section III presents the main result on the relation between minimal data rates and invariance entropy.

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II. MINIMAL DATA RATES AND LOCAL CONTROLLED INVARIANCE

In this section, the control problem is stated formally, and the notion of minimal data rates for almost invariance is introduced.

Consider a finite-dimensional control system in $\mathbb{R}^n$ described by

$$\dot{x}(t) = f(x(t), u(t)),\ u \in \mathcal{U}.$$

Here $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and Lipschitz-continuous with respect to the first argument. The control functions u are in a set $\mathcal{U}$ of admissible control functions defined on $[0, \infty)$ with values in $\mathbb{R}^m$. We assume that unique global solutions $\varphi(t, x_0, u), t \geq 0$, exist for controls $u \in \mathcal{U}$ and initial conditions $x(0) = x_0 \in \mathbb{R}^n$. The control task will be to keep the system in or near a given compact subset $Q \subset \mathbb{R}^n$ which we keep fixed throughout this paper.

A standard model supposes that the information on the state is encoded and transmitted via a digital communication channel, then decoded and used by the controller for the determination of a control. (For simplicity, we suppose that the information packets are received with no delay and no error.) Instead of analyzing in more detail the encoding and decoding of information about the state which is used by the controller, we concentrate on the information about the (open-loop) controls which are employed in order to keep the system near the set Q. A motivation is that the result of successful data transmissions is an, in general infinite, set of time dependent control functions keeping the system in or near Q. Encoding of the state must result in uniquely determined control functions, hence minimal data rates can be related to the encoding of sets of control functions.

We use the following mathematical formalization. Let $\varepsilon > 0$ and consider a set $\mathcal{R}(\varepsilon) \subset \mathcal{U}$ of controls which are sufficient for the control task to keep the system within the ε-neighborhood of the set Q for all $t \geq 0$, i.e., for all $x \in Q$ there is $u \in \mathcal{R}(\varepsilon)$ with $\mathrm{dist}(\varphi(t,x,u),Q) < \varepsilon$ for all $t \geq 0$. We want to determine which information has to be transmitted through a digital communication channel in order to determine a control function in such a set $\mathcal{R}(\varepsilon)$. Let

$$\mathcal{R}(T,\varepsilon) := \{u_{|[0,T]} \mid u \in \mathcal{R}(\varepsilon)\}$$

be the corresponding set of controls which are used up to time T. It appears reasonable to assume that $\mathcal{R}(T,\varepsilon)$ only contains finitely many elements, if only a finite amount of information is received up to time T. In this case, the finitely many elements of $\mathcal{R}(T,\varepsilon)$ can be encoded by symbols given by finite sequences of 0's and 1's in sets

$$\Sigma_k := \{(s_0 s_1 s_2 ... s_{k-1}) \text{ with } s_i \in \{0,1\}\}, k \in \mathbb{N}.$$

Let $B(\mathcal{R}(T,\varepsilon))$ be the minimal number k such that the controls used up to time T can be encoded by Σ_k. Equivalently, there is an injective map from $\mathcal{R}(T,\varepsilon)$ to Σ_k; or, the number of control functions on $[0,T]$ is bounded above by 2^k. The number of bits determining an element of Σ_k are $\log_2(2^k) = k$. If up to time T we use symbols in $\Sigma_{B(\mathcal{R}(\varepsilon))}$, then the data rate on the time interval $[0,T]$ is defined as $\frac{1}{T}B(\mathcal{R}(T,\varepsilon))$ and the required asymptotic data rate for $\mathcal{R}(\varepsilon)$ is $\limsup_{T\to\infty} \frac{1}{T}B(\mathcal{R}(T,\varepsilon))$.

Definition 1: With the notions introduced above, the asymptotic minimal data rate for controlled almost invariance of $Q \subset \mathbb{R}^n$ is

$$R_{\mathrm{inv}}(Q) := \lim_{\varepsilon\to 0} \inf_{\mathcal{R}(\varepsilon)} \limsup_{T\to\infty} \frac{1}{T} B(\mathcal{R}(T,\varepsilon)),$$

where the infimum is taken over all sets $\mathcal{R}(\varepsilon) \subset \mathcal{U}$ of controls such that for every element x in Q there is $u \in \mathcal{R}(\varepsilon)$ with $\mathrm{dist}(\varphi(t,x,u),Q) < \varepsilon$ for all $t \geq 0$. For brevity, we call $R_{\mathrm{inv}}(Q)$ just the invariance data rate for Q.

The discussion of invariance entropy in [2] presupposes that the set Q is compact and controlled invariant, i.e., for all $x \in Q$ there is an admissible control $u \in \mathcal{U}$ such that the corresponding trajectory $\varphi(t,x,u)$ remains in Q for $t \geq 0$. In the present context, we need the following variant of this condition.

Definition 2: A compact set $Q \subset \mathbb{R}^n$ is **locally controlled invariant** if for all $\varepsilon > 0$ and every $x \in Q$ there are $T > 0$ and $v \in \mathcal{U}$ with

$$\mathrm{dist}(\varphi(t,x,v),Q) < \varepsilon \text{ for } t \in [0,T] \quad (1)$$

and

$$\varphi(T,x,v) \in \mathrm{int}Q. \quad (2)$$

Hence for initial values in Q, the system can be steered into the interior of Q, without leaving a neighborhood of Q.

Remark 3: For control-affine systems with convex and compact control range, local controlled invariance implies controlled invariance, if the times T in (1), (2) are bounded away from 0. This follows, since here a sequence of control functions u_n keeping the system within distance $\frac{1}{n}$ of Q has a cluster point and the trajectories depend continuously on the controls (see, e.g., [3].)

The following proposition draws an immediate consequence of local controlled invariance.

Proposition 4: Suppose that $Q \subset \mathbb{R}^n$ is compact and locally controlled invariant. Then for all $\varepsilon > 0$ there are $\delta > 0$ and finitely many times $T_1, ..., T_s > 0$ and controls $v_1, ..., v_s \in \mathcal{U}, s = s(\varepsilon,\delta)$, such that for every x with $\mathrm{dist}(x,Q) \leq \delta$ there are T_j, v_j with

$$\mathrm{dist}(\varphi(t,x,v_j),Q) < \varepsilon \text{ for } t \in [0,T_j] \text{ and } \varphi(T_j,x,v_j) \in Q. \quad (3)$$

Proof: Let $\varepsilon > 0$. By assumption, for every $x \in Q$ there are $T_x > 0$ and $v_x \in \mathcal{U}$ with $\mathrm{dist}(\varphi(t,x,v_x),Q) < \varepsilon$ for all $t \in [0,T_x]$ and $\varphi(T_x,x,v) \in \mathrm{int}Q$. By continuous dependence on the initial value for all y in an open neighborhood one has $\mathrm{dist}(\varphi(t,y,v_x),Q) < \varepsilon$ for all $t \in [0,T_x]$ and $\varphi(T_x,y,v_x) \in \mathrm{int}Q$. Then the compact set Q can be covered by finitely many of these neighborhoods. Hence for all x in a closed δ-neighborhood of Q the assertion holds. ■

Remark 5: It is the property obtained in Proposition 4 as a consequence of local controlled invariance which will be used in the rest of this paper.

III. Minimal data rates and invariance entropy

This section shows that the invariance entropy coincides with the invariance data rate of a compact set Q, if Q is locally controlled invariant.

Recall the following definition of invariance entropy from [2]; see also [8] for far reaching further results on this concept.

Definition 6: Let $Q \subset \mathbb{R}^n$ be compact. For given $T, \varepsilon > 0$ we call $\mathcal{S} \subset \mathcal{U}$ a (T,ε)-spanning set for Q, if for every $x \in Q$ there exists $u \in \mathcal{S}$ with

$$\mathrm{dist}(\varphi(t,x,u),Q) < \varepsilon \text{ for all } t \in [0,T].$$

Let $r_{\mathrm{inv}}(T,\varepsilon,Q)$ denote the minimal number of elements of a (T,ε)-spanning set. Then the invariance entropy $h_{\mathrm{inv}}(Q)$ is defined by

$$h_{\mathrm{inv}}(\varepsilon,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon),$$
$$h_{\mathrm{inv}}(Q) := \lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon).$$

Our main result is that the minimal data rate $R_{\mathrm{inv}}(Q)$ for almost invariance of a compact locally controlled invariant set Q coincides with its invariance entropy.

Theorem 7: Let Q be a compact set in $\mathbb{R}^n$ with nonvoid interior and assume that Q is locally controlled invariant. Then the invariance data rate $R_{\mathrm{inv}}(Q)$ of Q and the invariance entropy of Q are related by

$$R_{\mathrm{inv}}(Q) = \log_2 e \cdot h_{\mathrm{inv}}(Q).$$

Proof: First we show the upper bound on $R_{\mathrm{inv}}(Q)$. Let $\varepsilon, T > 0$ be given. By Proposition 4 there are $\delta > 0$ and finitely many times $T_1, ..., T_s > 0$ and controls $v_1, ..., v_s \in \mathcal{U}, s = s(\varepsilon, \delta)$, such that for every y with $\operatorname{dist}(y, Q) \leq \delta$ there are T_j, v_j with

$$\operatorname{dist}(\varphi(t, y, v_j), Q) < \varepsilon \text{ for } t \in [0, T_j] \text{ and } \varphi(T_j, y, v_j) \in Q.$$

Consider a (T, δ)-spanning set $\mathcal{S} = \{u_1, ..., u_k\} \subset \mathcal{U}$ for Q with minimal number $k = r_{\text{inv}}(T, \delta)$ of elements. Thus for every $x \in Q$ there exists $u_i \in \mathcal{S}$ with

$$\operatorname{dist}(\varphi(t, x, u_i), Q) < \delta \text{ for all } t \in [0, T].$$

In particular, one has $\operatorname{dist}(\varphi(T, x, u_i), Q) < \delta$. Next we define ks controls w_{ij} on intervals $[0, T + T_j]$ by

$$w_{ij}(t) = \begin{cases} u_i(t) & \text{for} \quad t \in [0, T] \\ v_j(t + T) & \text{for} \quad t \in (T, T + T_j]. \end{cases}$$

Since we may assume $\delta < \varepsilon$, we see that for every $x \in Q$ there are a control w_{ij} and a time T_j with

$$\begin{aligned} &\operatorname{dist}(\varphi(t, x, w_{ij}), Q) < \varepsilon \text{ for } t \in [0, T + T_j] \text{ and} \\ &\quad \varphi(T + T_j, x, w_{ij}) \in Q. \end{aligned}$$

Note that the associated data rate for the set of controls w_{ij} on the interval $[0, T + \min_j T_j]$ is, with $s = s(\varepsilon, \delta)$,

$$\begin{aligned} &\leq \frac{1}{T + \min_j T_j} [\log_2(ks) + 1] \\ &\leq \frac{1}{T} [\log_2 k + \log_2 s + 1]. \end{aligned}$$

Since $\varphi(T+T_j, x, w_{ij}) \in Q$ we can apply one of the controls u_i to $\varphi(T+T_j, x, w_{ij})$ in the next time interval $[T+T_j, T+T_j+T]$. We extend the controls w_{ij} to $[0, \infty)$ by taking all the possible combinations of the u_i and the v_j (naturally, these extensions are, in general, not periodic) and obtain a set $\mathcal{R}_0(\varepsilon) \subset \mathcal{U}$ of controls such that for every element x in Q there is $u \in \mathcal{R}_0(\varepsilon)$ with $\operatorname{dist}(\varphi(t, x, u), Q) < \varepsilon$ for all $t \geq 0$.

A rough estimate for the number of controls on an interval $[0, S]$ with $S = mT$ is

$$\leq (ks)^m = [r_{\text{inv}}(T, \delta) s(\varepsilon, \delta)]^m.$$

Hence the associated data rate satisfies

$$\begin{aligned} \frac{1}{S} B(\mathcal{R}_0(S, \varepsilon)) &\leq \frac{1}{mT} [\log_2 r_{\text{inv}}(T, \delta)^m + \log_2 s(\varepsilon, \delta)^m] \\ &= \frac{1}{T} \log_2 r_{\text{inv}}(T, \delta) + \frac{1}{T} \log_2 s(\varepsilon, \delta). \end{aligned}$$

Note that the number $s(\varepsilon, \delta)$ does not depend on T. Hence for every $l \in \mathbb{N}$ one finds $T_l > 0$ such that for all $T > T_l$ one has

$$\begin{aligned} &\frac{1}{T} \log_2 r_{\text{inv}}(T, \delta) + \frac{1}{T} \log_2 s(\varepsilon, \delta) \\ &< \frac{1}{T} \log_2 r_{\text{inv}}(T, \delta) + \frac{1}{l}. \end{aligned}$$

We find that

$$\begin{aligned} &\limsup_{S \to \infty} \frac{1}{S} B(\mathcal{R}_0(S, \varepsilon)) \\ &\leq \limsup_{T \to \infty} \frac{1}{T} \log_2 r_{\text{inv}}(T, \delta) \\ &= \log_2 e \cdot \limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{inv}}(T, \delta). \end{aligned}$$

Taking (i) the infimum over all sets $\mathcal{R}(\varepsilon)$ of controls such that for every element x in Q there is a control with $\operatorname{dist}(\varphi(t, x, u), Q) < \varepsilon$ for all $t \geq 0$; then (ii) the limit for $\delta \to 0$, and, finally, taking (iii) the limit for $\varepsilon \to 0$, we conclude.

$$\begin{aligned} R_{\text{inv}}(Q) &= \lim_{\varepsilon \to 0} \inf_{\mathcal{R}(\varepsilon)} \limsup_{T \to \infty} \frac{1}{T} B(\mathcal{R}(T, \varepsilon)) \\ &\leq \log_2 e \cdot \lim_{\delta \to 0} \limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{inv}}(T, \delta) \\ &= \log_2 e \cdot \lim_{\delta \to 0} h_{\text{inv}}(\delta, Q) \\ &= \log_2 e \cdot h_{\text{inv}}(Q). \end{aligned}$$

For the converse inequality, we again fix $\varepsilon > 0$ and suppose that we have a set $\mathcal{R}(\varepsilon)$ of controls such that for every element x in Q there is $u \in \mathcal{R}(\varepsilon)$ with $\operatorname{dist}(\varphi(t, x, u), Q) < \varepsilon$ for all $t \geq 0$. This immediately implies that for $T > 0$ the set $\mathcal{R}(T, \varepsilon) := \{u_{|[0,T]} \mid u \in \mathcal{R}(\varepsilon)\}$ is (T, ε)-spanning for Q. If $\mathcal{R}(T, \varepsilon)$ can be encoded by Σ_k with minimal $k = B(\mathcal{R}(T, \varepsilon))$, then the number l of controls in $\mathcal{R}(T, \varepsilon)$ is equal to or less than 2^k. Hence one finds for the associated data rate

$$\begin{aligned} \frac{1}{T} B(\mathcal{R}(T, \varepsilon)) &\geq \frac{1}{T} \log_2 l \\ &\geq \frac{1}{T} \log_2 r_{\text{inv}}(T, \varepsilon) \\ &= \log_2 e \cdot \frac{1}{T} \ln r_{\text{inv}}(T, \varepsilon). \end{aligned}$$

Taking the limit superior for $T \to \infty$, the infimum over all sets $\mathcal{R}(\varepsilon)$ of controls such that for every element x in Q there is a control with $\operatorname{dist}(\varphi(t, x, u), Q) < \varepsilon$ for all $t \geq 0$ and, finally, taking the limit for $\varepsilon \to 0$, we conclude

$$R_{\text{inv}}(Q) \geq \log_2 e \cdot h_{\text{inv}}(Q).$$

■

Remark 8: Theorem 7 shows in particular, that the minimal data rate $R_{\text{inv}}(Q)$ remains invariant under topological conjugation of the state space; see [2, Theorem 3.5].

Next we discuss as an example minimal data rates for linear control systems. Recall (see e.g. [3]) that a subset $D \subset \mathbb{R}^n$ with nonvoid interior is a control set if it is a maximal set such that for all $x \in D$ one has $D \subset \operatorname{cl}\mathcal{O}^+(x)$, where $\mathcal{O}^+(x) := \{y \in \mathbb{R}^n$, there are $u \in \mathcal{U}$ and $t > 0$ with $\varphi(t, x, u) = y\}$ is the reachable set from $x \in \mathbb{R}^n$.

Consider linear control systems given by

$$\dot{x} = Ax + Bu \text{ with } u(t) \in U, \tag{4}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and the control range $U \subset \mathbb{R}^m$ is compact and convex and contains the origin in its interior. If the *reachability rank condition*

$$\text{rank}[B, AB, ..., A^{n-1}] = n \tag{5}$$

holds, then there exists a unique control set D with nonvoid interior, and D is bounded iff A is hyperbolic, i.e., A has no eigenvalue with vanishing real part (see [4]). The closure $\text{cl}D$ and the interior $\text{int}D$ of D are controlled invariant and $\text{cl}D$ equals the closure of $\text{int}D$. The next proposition shows, how to construct locally controlled invariant sets from control sets.

Proposition 9: Consider linear control systems (4) with control ranges $U^\rho := \rho \cdot U, \rho > 0$. Assume that A is hyperbolic and that the reachability rank condition (5) holds, and denote the corresponding control sets by D^ρ. Then for every $\rho \in (0,1)$ the closure of the control set D^ρ is locally controlled invariant for the system with control range U.

Proof: Since, for every $\rho > 0$ the closure $\text{cl}D^\rho$ is controlled invariant for the system with control range $U^\rho \subset U$, it is also controlled invariant for each system with control range $U^{\rho'}$ with $\rho' > \rho$.

The reachability rank condition implies that the linear and continuous map

$$u \mapsto \varphi(T, x, u) : L_\infty([0,T], \mathbb{R}^m) \to \mathbb{R}^n$$

is surjective. Let

$$\mathcal{U}^{\rho'} := \{u \in L_\infty([0,T], \mathbb{R}^m) \mid u(t) \in U^{\rho'} \text{ for } t \in [0,T]\}.$$

Then $\mathcal{U}^\rho \subset \text{int}\mathcal{U}^{\rho'}$ and the open mapping theorem implies that the corresponding reachable sets at time $T > 0$

$$\mathcal{O}_T^{\rho',+}(x) := \{y \in \mathbb{R}^n, \text{ there is } u \in \mathcal{U}^{\rho'} \text{ with } \varphi(T,x,u) = y\}$$

satisfy $\mathcal{O}_T^{\rho,+}(x) \subset \text{int}\mathcal{O}_T^{\rho',+}(x)$. Similarly, one sees that $\mathcal{O}_T^{\rho,-}(x) \subset \text{int}\mathcal{O}_T^{\rho',-}(x)$, where

$$\mathcal{O}_T^{\rho',-}(x) := \{y \in \mathbb{R}^n, \text{ there is } u \in \mathcal{U}^{\rho'} \text{ with } \varphi(T,y,u) = x\}.$$

If $x \in \text{int}D^\rho$ one finds a control $u \in \mathcal{U}^\rho$ with $\varphi(T,x,u) \in D^\rho$. Let $x \in \text{cl}D^\rho$. Then there are $x_n \in \text{int}D^\rho$ and $u_n \in \mathcal{U}^\rho$ with $x_n \to x$ and $\varphi(T,x_n,u_n) \in D^\rho$ and $\varphi(T,x_n,u_n) \to \varphi(T,x,u)$ and it follows that

$$\varphi(T,x,u) \in \text{cl}D^\rho \cap \mathcal{O}^{\rho,+}(x) \subset \text{cl int}D^\rho \cap \text{int}\mathcal{O}^{\rho'}(x).$$

Thus one finds a control $v \in \mathcal{U}^{\rho'} \subset \mathcal{U}^1 = \mathcal{U}$ with $\varphi(T,x,v) \in \text{int}D^\rho$. By choosing $\rho' - \rho > 0$ small enough, one sees that x can be steered into $\text{int}D^\rho$ without leaving an ε-neighborhood of $\text{cl}D^\rho$. ■

Remark 10: The construction of locally controlled invariant sets in Proposition 9 can be generalized to control-affine systems with compact and convex control range, provided that the so-called ρ-inner pair condition holds; see [3, Lemma 4.7.3 and Proposition 4.5.17].

The following example illustrates the preceding results.

Example.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t), \qquad u(t) \in U,$$

with $U = [-1,1]$. The solutions are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{5t}x_0 \\ e^{-t}y_0 \end{pmatrix} + \int_0^t u(s) \begin{pmatrix} e^{5(t-s)} \\ e^{s-t} \end{pmatrix} ds.$$

There is a unique control set D^ρ with nonvoid interior,

$$D^\rho = (-\frac{\rho}{5}, \frac{\rho}{5}\rho) \times [-\rho, \rho]$$

and

$$h_{\text{inv}}(\text{cl}D^\rho) = 1.$$

By Proposition 9 the sets $\text{cl}D^\rho$ are locally controlled invariant for the systems with control range $\rho \cdot U$ with $\rho \in (0,1)$. Hence, according to Theorem 7, the minimal data rates $R_{\text{inv}}(\text{cl}D^{\rho_1})$ for these systems are equal to $5\log_2 e$.

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References

[1] F. Colonius, U. Helmke, Entropy of controlled-invariant subspaces, 2009, submitted.

[2] F. Colonius, C. Kawan, Invariance entropy for control systems, SIAM J. Control Optimization 48(2009), 1701-1721.

[3] F. Colonius, W. Kliemann, The Dynamics of Control, Birkhäuser, Boston 2000.

[4] F. Colonius, M. Spadini, Uniqueness of Local Control Sets J. Dynamical and Control Systems, 9 (2003), 513 - 530

[5] C. de Persis, n-bit stabilization of n-dimensional nonlinear systems in feedforward form, IEEE Trans.Aut. Control, 50 (2005), pp.299–311.

[6] L. Grüne, Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems, SIAM J. Control Optim. 48(2009), 1206-1228.

[7] V. Gupta, A. Dana, J. Hespanha, R. Murray, and B. Hassibi, Data transmission over networks for estimation and control, IEEE Trans. Aut. Control, 54 (2009), pp.~1807–1819.

[8] C. Kawan, Invariance Entropy for Control Systems, Dissertation, Universität Augsburg 2009.

[9] C. Kawan, Upper and lower estimates for invariance entropy, Universität Augsburg, 2010, submitted.

[10] C. Kawan, Invariance entropy of control sets, Universität Augsburg 2010, submitted.

[11] G. Nair, R. J. Evans, I. Mareels, and W. Moran, Topological feedback entropy and nonlinear stabilization, IEEE Trans. Aut. Control, 49 (2004), 1585-1597.