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INVARIANCE ENTROPY FOR CONTROL SYSTEMS*

FRITZ COLONIUS† AND CHRISTOPH KAWAN†

Abstract. For continuous time control systems, this paper introduces invariance entropy as a measure for the amount of information necessary to achieve invariance of weakly invariant compact subsets of the state space. Upper and lower bounds are derived; in particular, finiteness is proven. For linear control systems with compact control range, the invariance entropy is given by the sum of the real parts of the unstable eigenvalues of the uncontrolled system. A characterization via covers and corresponding feedbacks is provided.

Key words. nonlinear control, invariance, topological entropy, invariant covering

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1. Introduction.

This paper is concerned with the amount of information necessary to keep a continuous time control system in a given subset Q of the state space. We introduce "invariance entropy" that measures how often open loop control functions must be adjusted in order to avoid exit from a subset Q of the state space. Due to the analysis of the open loop problem this information measure does not depend on a specific class of feedback strategies and hence is intrinsic.

The increasing relevance of control systems with restricted digital communication channels has spurred interest in the information necessary for accomplishing control tasks. Early contributions are due to Delchamps [7], who considered quantized feedbacks for stabilization. Wong and Brockett [16, 15] study the influence of restricted communication channels. For the present paper, the work by Nair et al. [11] and Nair and Evans [9, 10] is fundamental. They develop a method to describe data rates necessary to render subsets Q of the state space invariant. Their approach is based on a notion describing for discrete time systems how many feedbacks defined on open covers of Q are necessary in order to make Q invariant (or asymptotically stable) up to time N; then they let N tend to infinity and take the infimum over all covers and obtain what they call feedback entropy. In particular, they show that this number is equal to the minimum data rate for a symbolic controller rendering Q invariant.

The present paper introduces various versions of open loop entropies and discusses their relations. Since topological entropy is a property of dynamical systems (see, e.g., Adler, Konheim, and McAndrew [2], Robinson [13], or Katok and Hasselblatt [8]), it would appear that a view of control systems as dynamical systems might be helpful. In fact, including the time shift along control functions, one obtains a dynamical system, the control flow (cf. Colonius and Kliemann [4]). This point of view is helpful in order to adapt several constructions traditionally used for topological entropy to control systems.

A preliminary definition of our information measure (see section 3 for precise definitions of invariance entropy) is the following: For systems with compact control

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http://www.siam.org/journals/sicon/48-3/71390.html
†Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany (fritz.colonius@math.uni-augsburg.de, christoph.kawan@math.uni-augsburg.de).

range let Q be a compact subset of the state space. Then, for $T > 0$, we let $r_{\text{inv}}(T; Q)$ be the minimal number of controls $u \in \mathcal{U}$ such that for every initial value $x \in Q$ there is u with corresponding trajectory $\varphi(t, x, u) \in Q$ for all $[0, T]$. Then we consider the exponential growth rate of these numbers as T tends to infinity,

$$\limsup_{T\to\infty} \frac{1}{T} \ln r_{\text{inv}}(T; Q).$$

A characteristic feature of this information measure is that no information on the present state of the system is involved. Our main results provide upper and lower bounds for the invariance entropy; in particular, it is shown that the invariance entropy is finite. For linear control systems (with compact control range) the invariance entropy is given by the sum of the real parts of the unstable eigenvalues. We remark that Nair et al. [11] also have a similar result for feedback entropy of linear control systems; however, they show this only for vanishing control range, not for arbitrary compact control range. Finally, we can also give a characterization of invariance entropy in terms of covers and a feedback construction akin to the contribution in [11].

The structure of the paper is as follows. Section 2 recalls some basic properties of control systems (mainly for notational purposes) and also recalls Bowen's definition of topological entropy. Section 3 introduces several variants of invariance entropy and their properties. Section 4 provides lower and upper bounds for the invariance entropy which can be computed directly from the right-hand side of the system. In particular, it is shown that the invariance entropy is finite. One of these bounds, together with a classical result by Bowen on entropy of linear maps, is used in section 5 to compute the invariance entropy of linear control systems. Finally, section 6 gives a characterization in terms of feedbacks defined on covers.

Notation. We write $\text{cl}(Y)$ for the closure of a subset Y of a topological space X and $\text{int}(Y)$ for the interior. The spectrum of a matrix $A \in \mathbb{R}^{d\times d}$ is denoted by $\sigma(A)$. $\#S$ denotes the cardinality of a set S.

2. Preliminaries.

In this preliminary section we recall some basic facts on nonlinear control systems, mainly to introduce some notation, and we also recall some properties of topological entropy for dynamical systems.

2.1. Control systems.

Let $d, m \in \mathbb{N}$, M be an open subset of $\mathbb{R}^d$, and $U \subset \mathbb{R}^m$ be compact. Let $f : M \times \mathbb{R}^m \to \mathbb{R}^d$ be a continuous mapping such that the partial derivative with respect to the first argument exists and depends continuously on both arguments. Define the set of *admissible control functions* by

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \mid u \text{ measurable and } u(t) \in U \text{ a.e.}\}.$$

The *shift flow* on $\mathcal{U}$ is given by

$$\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \Theta(t, u) := \Theta_t u \quad \text{with } (\Theta_t u)(s) := u(t + s) \quad \text{for all } t, s \in \mathbb{R}.$$

The family

$$\dot{x}(t) = f(x(t), u(t)), \quad u \in \mathcal{U}, \tag{2.1}$$

of ordinary differential equations is called a *control system.* For given initial value $x \in M$ and control function $u \in \mathcal{U}$ the solution of the initial value problem $x(0) = x$ will be denoted by $\varphi(t, x, u)$. Note that $\varphi(\cdot, x, u)$ is only a solution in the Carathéodory sense. That is, $\varphi(\cdot, x, u)$ is an absolutely continuous curve which satisfies the corresponding integral equation. Throughout we assume that solutions are defined globally. This assumption is justified by the fact that we consider only trajectories which do not leave a compact subset of the state space M (cf. Sontag [14, Prop. C.3.6]). Thus we obtain a cocycle $\varphi : \mathbb{R} \times M \times \mathcal{U} \to M$, i.e.,

$$\varphi(t+s,x,u) = \varphi(s,\varphi(t,x,u),\Theta_t u) \text{ for all } t,s \in \mathbb{R},\ x \in M,\ u \in \mathcal{U}. \tag{2.2}$$

The *positive and negative orbits* from $x \in M$ at time $t \geq 0$ are

$$\mathcal{O}_t^+(x) = \{\varphi(t,x,u) \mid u \in \mathcal{U}\}, \quad \mathcal{O}_t^-(x) = \{\varphi(-t,x,u) \mid u \in \mathcal{U}\}.$$

For $T > 0$ we denote

$$\mathcal{O}_{\leq T}^{\pm}(x) = \bigcup_{t\in[0,T]} \mathcal{O}_t^{\pm}(x), \quad \mathcal{O}^{\pm}(x) = \bigcup_{t\geq 0} \mathcal{O}_t^{\pm}(x).$$

A subset Q of the state space M is called *weakly invariant* (or *controlled invariant*) if for all $x \in Q$ there is some $u \in \mathcal{U}$ with $\varphi(t,x,u) \in Q$ for all $t \geq 0$ and Q is called *strongly invariant* if $\mathcal{O}^+(x) \subset Q$ for all $x \in Q$.

2.2. Topological entropy. We recall the definition of topological entropy for a uniformly continuous map $f : X \to X$ on a metric space (X,d): For all $n \in \mathbb{N}$ a metric on X is given by

$$d_{n,f}(x,y) := \max_{0\leq j\leq n-1} d(f^j(x), f^j(y)).$$

For $n \in \mathbb{N}$ and $\varepsilon > 0$ a set $E \subset X$ is called (n,ε)-separated (with respect to f) if $d_{n,f}(x,y) \geq \varepsilon$ for all $x,y \in E$ with $x \neq y$. A set $F \subset X$ (n,ε)-spans another set $K \subset X$ (with respect to f) if for all $x \in K$ there is some $y \in F$ with $d_{n,f}(x,y) < \varepsilon$.

For a compact set $K \subset X$ let $r(n,\varepsilon,K,f)$ be the minimal cardinality of a set F which (n,ε)-spans K, and let $s(n,\varepsilon,K,f)$ be the maximal cardinality of an (n,ε)-separated set $E \subset K$. Define

$$h_{\text{span}}(\varepsilon,K,f) := \limsup_{n\to\infty} \frac{1}{n} \ln r(n,\varepsilon,K,f), \quad h_{\text{sep}}(\varepsilon,K,f) := \limsup_{n\to\infty} \frac{1}{n} \ln s(n,\varepsilon,K,f).$$

With these definitions the following statements hold true. (For a proof, see, for instance, Bowen [3, Lem. 1, p. 402].)

(i) $r(n,\varepsilon,K,f) \leq s(n,\varepsilon,K,f) \leq r(n,\frac{\varepsilon}{2},K,f) < \infty$.

(ii) If $\varepsilon_1 < \varepsilon_2$, then $h_{\text{span}}(\varepsilon_1,K,f) \geq h_{\text{span}}(\varepsilon_2,K,f)$ and $h_{\text{sep}}(\varepsilon_1,K,f) \geq h_{\text{sep}}(\varepsilon_2,K,f)$.

Hence the following definitions make sense:

$$h_{\text{top}}(K,f) := \lim_{\varepsilon\searrow 0} h_{\text{span}}(\varepsilon,K,f) = \lim_{\varepsilon\searrow 0} h_{\text{sep}}(\varepsilon,K,f),$$

$$h_{\text{top}}(f) := \sup_{K\subset X \text{ compact}} h_{\text{top}}(K,f).$$

$h_{\text{top}}(f)$ is called the *topological entropy* of f. In general $h_{\text{top}}(f)$ depends on the given metric. But if X is compact, it is a topological invariant.

Now consider a continuous semiflow $\Phi : \mathbb{R}_0^+ \times X \to X$ on the metric space (X,d). For brevity we denote the time-t-map $\Phi(t,\cdot) : X \to X$ by Φ_t. We assume that Φ is uniformly continuous in the following sense (cf. section 5 of [3]):

$$\forall t_0,\varepsilon > 0 :\ \exists \delta > 0 :\ \forall t \in [0,t_0],\ x,y \in X :\ d(x,y) < \delta \Rightarrow d(\Phi_t(x),\Phi_t(y)) < \varepsilon. \tag{2.3}$$

We want to define the topological entropy of Φ in a way analogous to how we did for maps. To this end, we introduce for every real number $T > 0$ the metric

$$d_{T,\Phi}(x, y) := \max_{t\in[0,T]} d(\Phi_t(x), \Phi_t(y)).$$

As for maps we can define (T, ε)-separated and (T, ε)-spanning sets. For instance, we call a set $E \subset X$ (T, ε)-separated if for all distinct $x, y \in E$ one has $d_{T,\Phi}(x, y) \geq \varepsilon$. If $K \subset X$ is a compact set, the quantities $r(T, \varepsilon, K, \Phi)$, $s(T, \varepsilon, K, \Phi)$, $h_{\mathrm{span}}(\varepsilon, K, \Phi)$, $h_{\mathrm{sep}}(\varepsilon, K, \Phi)$, $h_{\mathrm{top}}(K, \Phi)$, and $h_{\mathrm{top}}(\Phi)$ are defined just as for maps.

The following lemma, which will be needed in section 5, relates the topological entropy of a semiflow to the topological entropy of a map.

LEMMA 2.1. *The topological entropy of the semiflow Φ equals the topological entropy of its time-one-map $h_{\mathrm{top}}(\Phi) = h_{\mathrm{top}}(\Phi_1)$.*

Proof. Fix a compact set $K \subset X$ and real numbers $T, \varepsilon > 0$. Let $F \subset X$ be a set which (T, ε)-spans K with respect to the semiflow Φ and define $n \in \mathbb{N}$ to be the greatest natural number such that $n - 1 \leq T$. Then for every $x \in K$ there is some $y \in F$ with $\max_{t\in[0,T]} d(\Phi_t(x), \Phi_t(y)) < \varepsilon$. Since $\Phi_j = (\Phi_1)^j$ for all $j \in \mathbb{N}_0$ this implies

$$d_{n,\Phi_1}(x, y) = \max_{0\leq j\leq n-1} d((\Phi_1)^j(x), (\Phi_1)^j(y)) \leq \max_{t\in[0,T]} d(\Phi_t(x), \Phi_t(y)) < \varepsilon.$$

Thus F (n, ε)-spans the set K with respect to the map Φ_1, which implies that $r(n, \varepsilon, K, \Phi_1) \leq r(T, \varepsilon, K, \Phi)$. It follows that

$$\begin{aligned} h_{\mathrm{span}}(\varepsilon, K, \Phi_1) &= \limsup_{n\to\infty} \frac{1}{n} \ln r(n, \varepsilon, K, \Phi_1) \leq \limsup_{n\to\infty} \frac{1}{n} \ln r(n, \varepsilon, K, \Phi) \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \ln r(T, \varepsilon, K, \Phi) = h_{\mathrm{span}}(\varepsilon, K, \Phi). \end{aligned}$$

Consequently, $h_{\mathrm{span}}(\Phi_1) \leq h_{\mathrm{span}}(\Phi)$. In order to show the converse inequality, let $T, \varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ according to (2.3) with $t_0 = 1$. Let $n \in \mathbb{N}$ be the smallest natural number such that $T \leq n-1$ and let $F \subset X$ be a set which (n, δ)-spans K with respect to Φ_1. Then for every $x \in K$ there is some $y \in F$ such that $d_{n,\Phi_1}(x, y) < \delta$. For every $t \in [0, T]$ there are unique $j \in \{0, 1, \ldots, n - 1\}$ and $s \in [0, 1)$ such that $t = j + s$, which implies

$$\begin{aligned} d(\Phi_t(x), \Phi_t(y)) &= d(\Phi_s(\Phi_j(x)), \Phi_s(\Phi_j(x))) \\ &= d(\Phi_s((\Phi_1)^j(x)), \Phi_s((\Phi_1)^j(x))) < \varepsilon. \end{aligned}$$

Consequently, F is also (T, ε)-spanning the set K with respect to the semiflow Φ. Now for given $T > 0$ let $n = n(T)$ denote the smallest integer with $T \leq n - 1$. Then it follows that

$$\begin{aligned} h_{\mathrm{span}}(\varepsilon, K, \Phi) &= \limsup_{T\to\infty} \frac{1}{T} \ln r(T, \varepsilon, K, \Phi) \leq \limsup_{T\to\infty} \frac{1}{T} \ln r(n, \delta, K, \Phi_1) \\ &\leq \limsup_{n\to\infty} \frac{1}{n-2} \ln r(n, \delta, K, \Phi_1) \\ &= \limsup_{n\to\infty} \frac{1}{n} \ln r(n, \delta, K, \Phi_1) = h_{\mathrm{span}}(\delta, K, \Phi_1). \end{aligned}$$

Thus $h_{\mathrm{top}}(K, \Phi) \leq h_{\mathrm{top}}(K, \Phi_1)$ and $h_{\mathrm{top}}(\Phi) \leq h_{\mathrm{top}}(\Phi_1)$. ◻

3. Definition and elementary properties. This section presents the definition of several versions of invariance entropy. Basic properties of these notions are derived.

Consider the control system (2.1). Let $K, Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant. For given $T, \varepsilon > 0$ we call $\mathcal{S} \subset \mathcal{U}$ a (T,ε)-*spanning* set for (K,Q) if for every $x \in K$ there exists $u \in \mathcal{S}$ with

$$\varphi(t,x,u) \in N_\varepsilon(Q) = \{p \in M \mid \exists\, q \in Q:\ d(p,q) < \varepsilon\} \quad \text{for all } t \in [0,T];$$

here d denotes the Euclidean distance (note that this notion is different from the one used for topological entropy). By $r_{\mathrm{inv}}(T,\varepsilon,K,Q)$ we denote the minimal cardinality of a (T,ε)-spanning set. A set $\mathcal{S}^* \subset \mathcal{U}$ is called T-*spanning* for (K,Q) if for every $x \in K$ there exists $u \in \mathcal{S}^*$ with

$$\varphi(t,x,u) \in Q \quad \text{for all } t \in [0,T].$$

The minimal cardinality of a T-spanning set is denoted by $r^*_{\mathrm{inv}}(T,K,Q)$. If there is no finite T-spanning set, we define $r^*_{\mathrm{inv}}(T,K,Q) := \infty$. Let $0 < T_1 < T_2$. Since every (T_2,ε)-spanning (T_2-spanning) set is obviously also (T_1,ε)-spanning (T_1-spanning), it follows that

$$r_{\mathrm{inv}}(T_1,\varepsilon,K,Q) \le r_{\mathrm{inv}}(T_2,\varepsilon,K,Q) \quad \text{and} \quad r^*_{\mathrm{inv}}(T_1,K,Q) \le r^*_{\mathrm{inv}}(T_2,K,Q).$$

Since every (T,ε_1)-spanning set is also (T,ε_2)-spanning if $\varepsilon_1 < \varepsilon_2$, we obtain

$$r_{\mathrm{inv}}(T,\varepsilon_1,K,Q) \ge r_{\mathrm{inv}}(T,\varepsilon_2,K,Q) \quad \text{for} \quad \varepsilon_1 < \varepsilon_2. \tag{3.1}$$

We define the *invariance entropy* $h_{\mathrm{inv}}(K,Q)$ and the *strict invariance entropy* $h^*_{\mathrm{inv}}(K,Q)$ by

$$h_{\mathrm{inv}}(\varepsilon,K,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r_{\mathrm{inv}}(T,\varepsilon,K,Q),$$

$$h_{\mathrm{inv}}(K,Q) := \lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon,K,Q),$$

$$h^*_{\mathrm{inv}}(K,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T,K,Q).$$

From (3.1) it follows that the limit $\lim_{\varepsilon\searrow 0} h_{\mathrm{inv}}(\varepsilon,K,Q)$ is well defined. If $K = Q$, we often suppress the argument K. Thus we write, e.g., $r_{\mathrm{inv}}(T,\varepsilon,Q)$ instead of $r_{\mathrm{inv}}(T,\varepsilon,Q,Q)$.

Remark 3.1. In general, it is not true that for the strict invariance entropy the numbers $r^*_{\mathrm{inv}}(T,K,Q)$ are finite. Hence we introduce the weaker version $h_{\mathrm{inv}}(K,Q)$. In section 4 we will show that $h_{\mathrm{inv}}(K,Q)$ as defined above is finite. Compare also Example 5.1.

The following proposition summarizes some basic properties of these quantities.

PROPOSITION 3.1. *Let $K, Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant for system* (2.1).

(i) $r_{\mathrm{inv}}(T,\varepsilon,K,Q) < \infty$ *for all* $T, \varepsilon > 0$.

(ii) $r^*_{\mathrm{inv}}(T,Q)$ *is either finite for all* $T > 0$ *or for none.*

(iii) *The function* $T \mapsto \ln r^*_{\mathrm{inv}}(T,Q)$ *is subadditive and consequently*

$$h^*_{\mathrm{inv}}(Q) = \lim_{T\to\infty} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T,Q) = \inf_{T>0} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T,Q).$$

(iv) $h_{\mathrm{inv}}(K,Q) \leq h^*_{\mathrm{inv}}(K,Q)$.

Proof.

(i) Since Q is weakly invariant, for every $x \in K$ there exists some $u_x \in \mathcal{U}$ with $\varphi(\mathbb{R}^+_0, x, u_x) \subset Q$. Since $N_\varepsilon(Q)$ is open in M and solutions depend continuously on the initial value, for every $x \in K$ there exists a neighborhood W_x with $\varphi(t, W_x, u_x) \subset N_\varepsilon(Q)$ for all $t \in [0,T]$. The family $\{W_x\}_{x\in K}$ is an open cover of K. By compactness one can choose a finite subcover $\{W_{x_j}\}_{j=1,\dots,n}$, $x_1,\dots,x_n \in K$. It follows that $\mathcal{S} := \{u_{x_1},\dots,u_{x_n}\}$ is a (T,ε)-spanning set for (K,Q). Hence $r_{\mathrm{inv}}(T,\varepsilon,K,Q) \leq n < \infty$.

(ii) Assume that $r^*_{\mathrm{inv}}(T_0,Q) < \infty$ for some $T_0 > 0$. Then

$$r^*_{\mathrm{inv}}(T,Q) \leq r^*_{\mathrm{inv}}(T_0,Q) < \infty \quad \text{for all } T \in (0,T_0).$$

For $T > T_0$ choose $k \in \mathbb{N}$ with $kT_0 \geq T$ and let $\mathcal{S}^* = \{u_1,\dots,u_n\}$ be a minimal T_0-spanning set, i.e., $n = r^*_{\mathrm{inv}}(T_0,Q)$. For every k-tuple $(i_0,i_1,\dots,i_{k-1})$ with $i_j \in \{1,\dots,n\}$ for $j = 0,1,\dots,k-1$ we define a control function $u_{i_0,i_1,\dots,i_{k-1}} \in \mathcal{U}$ by

$$u_{i_0,i_1,\dots,i_{k-1}}(t) := u_{i_j}(t - jT_0) \ \text{ for all } t \in [jT_0,(j+1)T_0),\ j = 0,1,\dots,k-1.$$

The function $u_{i_0,i_1,\dots,i_{k-1}}$ may be extended arbitrarily to $\mathbb{R}\backslash[0,kT_0)$. By this construction we obtain n^k control functions. Consider the set

$$\mathcal{S}^*_k := \left\{u_{i_0,i_1,\dots,i_{k-1}} \mid (i_0,i_1,\dots,i_{k-1}) \in \{1,\dots,n\}^k\right\}.$$

Now let $x_0 \in Q$. Since $\mathcal{S}^*$ is T_0-spanning there exists $u_{i_0} \in \mathcal{S}^*$ with $\varphi([0,T_0],x_0,u_{i_0}) \subset Q$. Let $x_1 := \varphi(T_0,x_0,u_{i_0})$. Then again, there exists $u_{i_1} \in \mathcal{S}^*$ with $\varphi([0,T_0],x_1,u_{i_1}) \subset Q$. Next we define $x_2 := \varphi(T_0,x_1,u_{i_1})$ and repeat this process until, after k steps, we have obtained control functions $u_{i_0},u_{i_1},\dots,u_{i_{k-1}}$. From the cocycle property (2.2) and the definition of $u_{i_0,i_1,\dots,i_{k-1}}$ it follows that

$$\varphi([0,kT_0],x_0,u_{i_0,i_1,\dots,i_{k-1}}) \subset Q.$$

This implies that $\mathcal{S}^*_k$ is a (kT_0)-spanning set and thus

$$r^*_{\mathrm{inv}}(T,Q) \leq r^*_{\mathrm{inv}}(kT_0,Q) \leq \#\mathcal{S}^*_k = n^k < \infty,$$

which proves the assertion.

(iii) If $r^*_{\mathrm{inv}}(T,Q) = \infty$ for all $T > 0$, the assertion is trivial. So by (ii) we may assume that $r^*_{\mathrm{inv}}(T,Q) < \infty$ for all $T > 0$. With arguments similar to those in the proof of (ii) one can show that

$$r^*_{\mathrm{inv}}(T+S,Q) \leq r^*_{\mathrm{inv}}(T,Q) \cdot r^*_{\mathrm{inv}}(S,Q) \ \text{ for all } T,S > 0.$$

This implies subadditivity of the monotone increasing function $T \mapsto \ln r^*_{\mathrm{inv}}(T,Q)$, $(0,\infty) \to \mathbb{R}^+_0$. Hence the limit exists and equals the infimum $\inf_{T>0} \frac{1}{T} \ln r^*_{\mathrm{inv}}(T,Q)$ (see [6, Lem. 1.21, p. 14] for a proof of the latter).

(iv) Every T-spanning set is obviously also (T,ε)-spanning for all $\varepsilon > 0$, and thus $r_{\mathrm{inv}}(T,\varepsilon,K,Q) \leq r^*_{\mathrm{inv}}(T,K,Q)$ for all $T,\varepsilon > 0$. This implies $h_{\mathrm{inv}}(\varepsilon,K,Q) \leq h^*_{\mathrm{inv}}(K,Q)$ for all $\varepsilon > 0$ and hence $h_{\mathrm{inv}}(K,Q) \leq h^*_{\mathrm{inv}}(K,Q)$. ∎

Remark 3.2. From Proposition 3.1(ii) and (iii) it follows that $h^*_{\mathrm{inv}}(Q) < \infty$ if and only if $r^*_{\mathrm{inv}}(T,Q) < \infty$ for one $T > 0$ if and only if $r^*_{\mathrm{inv}}(T,Q) < \infty$ for all $T > 0$.

In order to compute upper bounds for $h_{\mathrm{inv}}(K,Q)$ it will be useful to define another quantity which will be called the *strong invariance entropy* for (K,Q). To this end, we introduce the *lift* of the weakly invariant set Q, defined by

$$\mathcal{Q} := \{(x,u) \in Q \times \mathcal{U} \mid \varphi(t,x,u) \in Q \text{ for all } t \geq 0\}.$$

For given $T, \varepsilon > 0$ a set $\mathcal{S}^+ \subset \mathcal{Q}$ is called *strongly* (T,ε)*-spanning* for (K,Q) if for every $x \in K$ there exists $(y,u) \in \mathcal{S}^+$ with

$$d_{T,u}(x,y) := \max_{t\in[0,T]} d(\varphi(t,x,u),\varphi(t,y,u)) < \varepsilon.$$

By $r^+_{\text{inv}}(T,\varepsilon,K,Q)$ we denote the minimal cardinality of a strongly (T,ε)-spanning set. As for $r_{\text{inv}}(T,\varepsilon,K,Q)$ it follows by continuous dependence on initial conditions that $r^+_{\text{inv}}(T,\varepsilon,K,Q)$ is finite. We define

$$h^+_{\text{inv}}(\varepsilon,K,Q) := \limsup_{T\to\infty} \frac{1}{T} \ln r^+_{\text{inv}}(T,\varepsilon,K,Q),$$

$$h^+_{\text{inv}}(K,Q) := \lim_{\varepsilon\searrow 0} h^+_{\text{inv}}(\varepsilon,K,Q).$$

Obviously, $r^+_{\text{inv}}(T,\varepsilon,K,Q)$, considered as a function of T and ε, has the same monotonicity properties as $r_{\text{inv}}(T,\varepsilon,K,Q)$. Again, for $K = Q$ we drop the corresponding argument.

PROPOSITION 3.2. *Let $K,Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant for system* (2.1). *Then $h_{\text{inv}}(K,Q) \leq h^+_{\text{inv}}(K,Q)$.*

Proof. Let $\mathcal{S}^+ = \{(y_1,u_1),\dots,(y_n,u_n)\}$ be a minimal strongly (T,ε)-spanning set for (K,Q) and define $\mathcal{S} := \{u_1,\dots,u_n\}$. We want to show that $\mathcal{S}$ is (T,ε)-spanning. To this end, pick $x \in K$ arbitrarily. Then there exists $i \in \{1,\dots,n\}$ with $d(\varphi(t,x,u_i),\varphi(t,y_i,u_i)) < \varepsilon$ for all $t \in [0,T]$. Since $\mathcal{S}^+ \subset \mathcal{Q}$ we have $\varphi(t,y_i,u_i) \in Q$ for all $t \geq 0$, which implies $\varphi(t,x,u_i) \in N_\varepsilon(Q)$ for $t \in [0,T]$. Hence

$$r_{\text{inv}}(T,\varepsilon,K,Q) \leq \#\mathcal{S} \leq \#\mathcal{S}^+ = r^+_{\text{inv}}(T,\varepsilon,K,Q).$$

Consequently, also $h_{\text{inv}}(\varepsilon,K,Q) \leq h^+_{\text{inv}}(\varepsilon,K,Q)$ and $h_{\text{inv}}(K,Q) \leq h^+_{\text{inv}}(K,Q)$. ∎

In order to prove the next proposition we need the following technical lemma.

LEMMA 3.3. *For any functions $f_1,\dots,f_N : \mathbb{R}^+_0 \to (0,\infty)$ $(N \in \mathbb{N})$ it holds that*

$$\limsup_{T\to\infty} \frac{1}{T} \ln \left(\sum_{i=1}^N f_i(T)\right) \leq \max_{i=1,\dots,N} \limsup_{T\to\infty} \frac{1}{T} \ln f_i(T).$$

Proof. For brevity we write

$$\lambda(f) := \limsup_{T\to\infty} \frac{1}{T} \ln f(T)$$

for any function $f : \mathbb{R}^+_0 \to (0,\infty)$. We define $g : \mathbb{R}^+_0 \to (0,\infty)$ by

$$g(T) := \max_{i=1,\dots,N} f_i(T) \quad \text{for all } T \in \mathbb{R}^+_0.$$

Then

$$\lambda\left(\sum_{i=1}^N f_i\right) \leq \lambda(Ng) = \limsup_{T\to\infty} \frac{1}{T}\left[\ln(N) + \ln(g(T))\right] = \lambda(g).$$

Thus, it suffices to show that $\lambda(g) \leq \max_{i=1,\dots,N} \lambda(f_i)$. Let $(T_k)_{k\in\mathbb{N}}$, $T_k \in \mathbb{R}^+_0$, be a sequence with $T_k \to \infty$ and

$$\lambda(g) = \lim_{k\to\infty} \frac{1}{T_k} \ln \max_{i=1,\dots,N} f_i(T_k).$$

Obviously, there exists an $i_0 \in \{1,\dots,N\}$ such that $f_{i_0}(T_k) = \max_{i=1,\dots,N} f_i(T_k)$ for infinitely many $k \in \mathbb{N}$. Let $(T_{n_k})_{k\in\mathbb{N}}$ be a corresponding subsequence. Then we obtain

$$\lambda(g) = \lim_{k\to\infty} \frac{1}{T_{n_k}} \ln f_{i_0}(T_{n_k}) \leq \lambda(f_{i_0}) \leq \max_{i=1,\dots,N} \lambda(f_i). \qquad \square$$

The following proposition summarizes some more properties of both invariance entropy and strict invariance entropy.

PROPOSITION 3.4. *Let $K, Q \subset M$ be nonvoid compact sets with $K \subset Q$ and Q weakly invariant for system* (2.1).

(i) *If there exist finitely many controls $u_1,\dots,u_n \in \mathcal{U}$ such that for every point $x \in K$ there exists $i \in \{1,\dots,n\}$ with $\varphi(\mathbb{R}_0^+, x, u_i) \subset Q$, then*

$$h_{\text{inv}}(K,Q) = h^*_{\text{inv}}(K,Q) = 0.$$

In particular this holds if K is finite or if Q is strongly invariant.

(ii) *For all $\varepsilon > 0$ and $\tau > 0$*

$$h_{\text{inv}}(\varepsilon,K,Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \ln r_{\text{inv}}(n\tau,\varepsilon,K,Q). \tag{3.2}$$

(iii) *Let $K_i \subset K$, $i = 1,\dots,N$, be closed subsets of K with $K = \bigcup_{i=1}^N K_i$. Then*

$$h_{\text{inv}}(K,Q) = \max_{i=1,\dots,N} h_{\text{inv}}(K_i,Q).$$

(iv) *Consider for every $s > 0$ the control system*

$$\dot{x}(t) = s \cdot f(x(t),u(t)), \quad u \in \mathcal{U}. \tag{3.3}$$

Then Q is weakly invariant for each of these systems. Let $h_{\text{inv},s}(K,Q)$ denote the corresponding invariance entropy. Then it holds that

$$h_{\text{inv},s}(K,Q) = s \cdot h_{\text{inv}}(K,Q) \quad \text{for all } s > 0.$$

Assertions (ii), (iii), *and* (iv) *remain valid for the strict invariance entropy.*

Proof.

(i) From the assumptions it immediately follows that for all $T, \varepsilon > 0$ one has $r_{\text{inv}}(T,\varepsilon,K,Q) \leq r^*_{\text{inv}}(T,K,Q) \leq n$. This implies $\frac{1}{T}\ln r^*_{\text{inv}}(T,K,Q) \leq \frac{\ln(n)}{T} \to 0$ for $T \to \infty$ and thus $h_{\text{inv}}(K,Q) \leq h^*_{\text{inv}}(K,Q) = 0$.

(ii) Obviously, the left-hand side of (3.2) is not less than the right-hand side. In order to show the reverse inequality, let $(T_k)_{k\in\mathbb{N}}$ be a sequence converging to ∞. Then for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $n_k\tau \leq T_k \leq (n_k+1)\tau$, and $n_k \to \infty$ for $k \to \infty$. If $T_1 \leq T_2$, then $r_{\text{inv}}(T_1,\varepsilon,K,Q) \leq r_{\text{inv}}(T_2,\varepsilon,K,Q)$, which implies

$$r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q)$$

and consequently

$$\frac{1}{T_k}\ln r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq \frac{1}{n_k\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q).$$

This yields

$$\limsup_{k\to\infty}\frac{1}{T_k}\ln r_{\text{inv}}(T_k,\varepsilon,K,Q) \leq \limsup_{k\to\infty}\frac{1}{n_k\tau}\ln r_{\text{inv}}((n_k+1)\tau,\varepsilon,K,Q).$$

Since

$$\frac{1}{n_k\tau}\ln r_{\mathrm{inv}}((n_k+1)\tau,\varepsilon,K,Q)=\frac{n_k+1}{n_k}\frac{1}{(n_k+1)\tau}\ln r_{\mathrm{inv}}((n_k+1)\tau,\varepsilon,K,Q)$$

and $\frac{n_k+1}{n_k}\to 1$ for $k\to\infty$, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\frac{1}{T_k}\ln r_{\mathrm{inv}}(T_k,\varepsilon,K,Q)&\le\limsup_{k\to\infty}\frac{1}{n_k\tau}\ln r_{\mathrm{inv}}(n_k\tau,\varepsilon,K,Q)\\&\le\limsup_{k\to\infty}\frac{1}{k\tau}\ln r_{\mathrm{inv}}(k\tau,\varepsilon,K,Q).\end{aligned}$$

This proves the claim.

(iii) If $\mathcal{S}$ is a minimal (T,ε)-spanning set for (K,Q), then $\mathcal{S}$ is also (T,ε)-spanning for (K_i,Q). Thus we obtain $r_{\mathrm{inv}}(T,\varepsilon,K_i,Q)\le r_{\mathrm{inv}}(T,\varepsilon,K,Q)$, implying

$$\max_{i=1,\dots,N}h_{\mathrm{inv}}(K_i,Q)\le h_{\mathrm{inv}}(K,Q).$$

On the other hand, if $\mathcal{S}_i$ is a minimal (T,ε)-spanning set for (K_i,Q), $i=1,\dots,N$, then $\mathcal{S}:=\bigcup_{i=1}^N\mathcal{S}_i$ is (T,ε)-spanning for (K,Q). This yields

$$r_{\mathrm{inv}}(T,\varepsilon,K,Q)\le\#\mathcal{S}\le\sum_{i=1}^N\#\mathcal{S}_i=\sum_{i=1}^N r_{\mathrm{inv}}(T,\varepsilon,K_i,Q).$$

By Lemma 3.3 we obtain

$$h_{\mathrm{inv}}(\varepsilon,K,Q)\le\limsup_{T\to\infty}\frac{1}{T}\ln\sum_{i=1}^N r_{\mathrm{inv}}(T,\varepsilon,K_i,Q)\le\max_{i=1,\dots,N}h_{\mathrm{inv}}(\varepsilon,K_i,Q),$$

which yields the result.

(iv) Let φ_s denote the cocycle of system (3.3). Let $(x,u)\in M\times\mathcal{U}$ and define $\tilde{u}(t):\equiv u(ts)$. Then obviously $\tilde{u}\in\mathcal{U}$ and for all $t\in\mathbb{R}$ it holds that

$$\varphi_s(\tfrac{t}{s},x,\tilde{u})=x+\int_0^{t/s}sf(\varphi_s(\tau,x,\tilde{u}),\tilde{u}(\tau))d\tau=x+\int_0^t f(\varphi_s(\tfrac{\tau}{s},x,\tilde{u}),u(\tau))d\tau.$$

This proves that $\varphi_s(\frac{t}{s},x,\tilde{u})=\varphi(t,x,u)$ for all $t\in\mathbb{R}$. From that we can conclude that there exists a one-to-one correspondence between the (T,ε)-spanning sets of system (2.1) and the $(\frac{T}{s},\varepsilon)$-spanning sets of system (3.3), which preserves the cardinality. This implies $r_{\mathrm{inv}}(T,\varepsilon,K,Q)=r_{\mathrm{inv},s}(\frac{T}{s},\varepsilon,K,Q)$ and thus

$$\begin{aligned}h_{\mathrm{inv}}(\varepsilon,K,Q)&=\limsup_{T\to\infty}\frac{1}{T}\ln r_{\mathrm{inv}}(T,\varepsilon,K,Q)\\&=s\cdot\limsup_{T\to\infty}\frac{1}{sT}r_{\mathrm{inv}}(\tfrac{T}{s},\varepsilon,K,Q)=s\cdot h_{\mathrm{inv},s}(\varepsilon,K,Q).\end{aligned}$$

This proves the assertion.

Finally, analogous arguments show that assertions (ii)–(iv) are also valid for the strict invariance entropy. ∎

Remark 3.3. Proposition 3.4(ii) shows that for all time steps $\tau > 0$ one obtains the same result. Hence from the invariance entropy one cannot deduce any information on maximum allowable time steps (cf. also Nesic and Teel [12]).

The next theorem shows that the invariance entropy cannot increase under semiconjugation.

THEOREM 3.5. *Consider two control systems $\dot{x} = f(x,u)$ and $\dot{y} = g(y,v)$ on M and N with corresponding solutions $\varphi(t,x,u)$ and $\psi(t,y,v)$ and control spaces $\mathcal{U}$ and $\mathcal{V}$ corresponding to control ranges U and V. Let $\pi : M \to N$ be a continuous map and $h : \mathcal{U} \to \mathcal{V}$ any map with the semiconjugation property*

$$\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u)) \quad \text{for all } x \in M,\ u \in \mathcal{U},\ t \geq 0. \tag{3.4}$$

Then

$$h_{\text{inv}}(\pi(K),\pi(Q)) \leq h_{\text{inv}}(K,Q)$$

if $K \subset Q \subset M$ are compact and Q is weakly invariant. The analogous statement holds for the strict invariance entropy.

Equation (3.4) *holds, in particular, if $\pi : M \to N$ is a continuously differentiable map and $H : U \to V$ a continuous map such that*

$$D\pi_x f(x,u) = g(\pi(x),H(u)) \quad \text{for all } (x,u) \in M \times U. \tag{3.5}$$

Proof. By the assumptions it is clear that $\pi(K) \subset \pi(Q) \subset N$ are nonvoid compact sets. Equation (3.4) implies weak invariance of $\pi(Q)$ with respect to the system on N: If $y \in \pi(Q)$, then there exists $x \in Q$ with $\pi(x) = y$. Let $u \in \mathcal{U}$ be a control function with $\varphi(t,x,u) \in Q$ for all $t \geq 0$. It follows that

$$\psi(t,y,h(u)) = \psi(t,\pi(x),h(u)) \overset{(3.4)}{=} \pi(\varphi(t,x,u)) \in \pi(Q) \quad \text{for all } t \geq 0.$$

Now let $T, \varepsilon > 0$. Since π is uniformly continuous on the compact set Q there exists $\delta > 0$ with $\pi(N_\delta(Q)) \subset N_\varepsilon(\pi(Q))$. Let $\mathcal{S} \subset \mathcal{U}$ be a minimal (T,δ)-spanning set for (K,Q) and define $\tilde{\mathcal{S}} := h(\mathcal{S})$. For any $y \in \pi(K)$ there exists $x \in K$ with $\pi(x) = y$. Let $u \in \mathcal{S}$ such that $\varphi([0,T],x,u) \subset N_\delta(Q)$. Then $h(u) \in \tilde{\mathcal{S}}$ and $\psi([0,T],\pi(x),h(u)) \subset \pi(N_\delta(Q)) \subset N_\varepsilon(\pi(Q))$. This shows that $\tilde{\mathcal{S}}$ is (T,ε)-spanning for $(\pi(K),\pi(Q))$. Consequently,

$$h_{\text{inv}}(\varepsilon,\pi(K),\pi(Q)) \leq h_{\text{inv}}(\delta,K,Q) \leq h_{\text{inv}}(K,Q).$$

For $\varepsilon \searrow 0$ we obtain $h_{\text{inv}}(\pi(K),\pi(Q)) \leq h_{\text{inv}}(K,Q)$. It is even easier to see that the same inequality holds for the strict invariance entropy.

In order to see the second assertion, recall that the solution $\varphi(\cdot,x,u) : \mathbb{R} \to M$ is the unique absolutely continuous curve with $\varphi(0,x,u) = x$ and

$$\frac{d}{dt}\varphi(t,x,u) = f(\varphi(t,x,u),u(t)) \quad \text{for all } t \in \mathbb{R} \text{ where } \frac{d}{dt}\varphi(t,x,u) \text{ exists.}$$

By the chain rule we obtain for all $t \in \mathbb{R}$ where $\frac{d}{dt}\varphi(t,x,u)$ exists

$$\begin{aligned}\frac{d}{dt}\pi(\varphi(t,x,u)) &= D\pi_{\varphi(t,x,u)}\frac{d}{dt}\varphi(t,x,u) = D\pi_{\varphi(t,x,u)}f(\varphi(t,x,u),u(t)) \\ &\overset{(3.5)}{=} g(\pi(\varphi(t,x,u)),H(u(t))).\end{aligned}$$

It follows that $\pi(\varphi(\cdot,x,u)) : \mathbb{R} \to N$ is an absolutely continuous curve on N with $\pi(\varphi(0,x,u)) = \pi(x)$ which satisfies the differential equation $\dot{y} = g(y, H(u))$ almost everywhere. Let $h : \mathcal{U} \to \mathcal{V}$ be defined by $h(u)(t) := H(u(t))$ for all $u \in \mathcal{U}$ and $t \in \mathbb{R}$. Since H is a continuous map from U to V, $t \mapsto H(u(t))$ is measurable for every $u \in \mathcal{U}$ and $H(u(t)) \in V$ for almost all $t \in \mathbb{R}$, which shows that h is well defined. By uniqueness of solutions it follows that $\pi(\varphi(t,x,u)) = \psi(t,\pi(x),h(u))$. ∎

4. General bounds. For simplicity we assume throughout this section that $M = \mathbb{R}^d$. We will provide rough bounds for $h_{\text{inv}}(K,Q)$—one lower and one upper bound—which can be computed directly from the right-hand side of the system. Since the upper bound is always finite, finiteness of $h_{\text{inv}}(K,Q)$ also follows.

In the following proof we denote by $\text{div}_{\text{x}} f(x,u)$ the divergence of the function f with respect to the first variable, i.e.,

$$\text{div}_{\text{x}} f(x,u) = \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i}(x,u) = \text{tr}\frac{\partial f}{\partial x}(x,u),$$

where $f_1,\dots,f_d : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ are the coordinate functions of f.

THEOREM 4.1. *Consider control system* (2.1) *with* $M = \mathbb{R}^d$. *Let* $K, Q \subset \mathbb{R}^d$ *be nonvoid compact sets with* $K \subset Q$ *and* Q *being weakly invariant. Then, if the Lebesgue measure* $\lambda^d(K)$ *of* K *is positive, the following estimate holds:*

$$h_{\text{inv}}(K,Q) \geq \max\left\{0, \min_{(x,u)\in Q\times U} \text{div}_{\text{x}} f(x,u)\right\}. \tag{4.1}$$

Proof. For arbitrary $T, \varepsilon > 0$ let $\mathcal{S} = \{u_1,\dots,u_n\}$ be a minimal (T,ε)-spanning set for (K,Q). Define the following sets:

$$K_j := \{x \in K \mid \varphi([0,T],x,u_j) \subset N_\varepsilon(Q)\}, \quad j = 1,\dots,n.$$

By openness of $N_\varepsilon(Q)$ and continuous dependence on initial conditions, K_j is open in K and hence a Borel set. Since $\varphi(t,K_j,u_j) \subset N_\varepsilon(Q)$ for all $t \in [0,T]$ and $j = 1,\dots,n$ we obtain in particular

$$\lambda^d(\varphi(T,K_j,u_j)) \leq \lambda^d(N_\varepsilon(Q)) \quad \text{for} \quad j = 1,\dots,n.$$

Moreover, by the transformation theorem and Liouville's trace formula we get for all $j \in \{1,\dots,n\}$

$$\begin{aligned}\lambda^d(\varphi(T,K_j,u_j)) &= \int_{K_j} \left|\det \frac{\partial \varphi}{\partial x}(T,x,u_j)\right| dx \\ &\geq \lambda^d(K_j) \cdot \inf_{\substack{(x,u)\in K\times\mathcal{U},\\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \left|\det \frac{\partial \varphi}{\partial x}(T,x,u)\right| \\ &= \lambda^d(K_j) \cdot \inf_{\substack{(x,u)\in K\times\mathcal{U},\\ \varphi([0,T],x,u)\subset N_\varepsilon(Q)}} \exp\left(\int_0^T \text{div}_{\text{x}} f(\varphi(s,x,u),u(s))ds\right).\end{aligned}$$

In the rest of this proof, $\inf_{(x,u)}$ denotes the infimum over all $(x,u) \in K \times \mathcal{U}$ with $\varphi([0,T],x,u) \subset N_\varepsilon(Q)$. One finds

$$\begin{aligned} &\inf_{(x,u)} \exp\left(\int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds\right) \\ &\geq \exp\left(T \cdot \min_{(x,u)\in \mathrm{cl}(N_\varepsilon(Q))\times U} \mathrm{div}_\mathrm{x} f(x,u)\right) \\ &= \min_{(x,u)\in \mathrm{cl}(N_\varepsilon(Q))\times U} \exp\left(T \cdot \mathrm{div}_\mathrm{x} f(x,u)\right); \end{aligned} \tag{4.2}$$

hence the infimum is positive. We may assume that $\lambda^d(K_1) = \max_{j=1,\dots,n} \lambda^d(K_j)$. This implies

$$\begin{aligned} \lambda^d(K) &\leq \sum_{j=1}^n \lambda^d(K_j) \leq n \cdot \max_{j=1,\dots,n} \lambda^d(K_j) = n \cdot \lambda^d(K_1) \\ &\leq n \cdot \frac{\lambda^d(\varphi(T,K_1,u_1))}{\inf_{(x,u)} \exp\left(\int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds\right)} \\ &\leq n \cdot \frac{\lambda^d(N_\varepsilon(Q))}{\inf_{(x,u)} \exp\left(\int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds\right)}. \end{aligned}$$

Consequently, for $n = r_{\mathrm{inv}}(T,\varepsilon,K,Q)$ we obtain the estimate

$$n \geq \frac{\lambda^d(K)}{\lambda^d(N_\varepsilon(Q))} \inf_{(x,u)} \exp\left(\int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds\right).$$

Taking the logarithm on both sides, dividing by T, and letting T go to infinity yields the inequality

$$h_{\mathrm{inv}}(\varepsilon,K,Q) \geq \limsup_{T\to\infty} \frac{1}{T} \inf_{(x,u)} \int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds.$$

Again using estimate (4.2) we also find

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{T} \inf_{(x,u)} \int_0^T \mathrm{div}_\mathrm{x} f(\varphi(s,x,u),u(s))ds \\ &\geq \limsup_{T\to\infty} \min_{(x,u)\in \mathrm{cl}(N_\varepsilon(Q))\times U} \mathrm{div}_\mathrm{x} f(x,u) = \min_{(x,u)\in \mathrm{cl}(N_\varepsilon(Q))\times U} \mathrm{div}_\mathrm{x} f(x,u). \end{aligned}$$

Letting ε tend to zero we obtain (4.1). ∎

The next theorem, whose proof is a modification of [8, Thm. 3.3.9, p. 124], provides an upper bound for the strong invariance entropy and hence for the invariance entropy. For the proof recall the definition of fractal dimension: Let $Z \subset X$ be a totally bounded subset of a metric space (X,d) and let $b(\varepsilon,Z)$ be the minimal cardinality of a cover of Z by ε-balls. Then the fractal dimension of Z is defined by

$$\dim_F(Z) := \limsup_{\varepsilon\searrow 0} \frac{\ln b(\varepsilon,Z)}{\ln(1/\varepsilon)} \in \mathbb{R} \cup \{\infty\}.$$

The fractal dimension depends on the metric and is not a topological invariant. But for a relatively compact open subset of a differentiable manifold it equals the topological dimension.

THEOREM 4.2. *Consider control system* (2.1) *with* $M = \mathbb{R}^d$. *Let* $K, Q \subset \mathbb{R}^d$ *be nonvoid compact sets with* $K \subset Q$ *and* Q *being weakly invariant. Then, with* $L := \max_{(x,u)\in Q\times U} \|\frac{\partial f}{\partial x}(x,u)\|$, *the following estimate holds:*

$$h^+_{\mathrm{inv}}(K,Q) \leq L \dim_F(K) \leq Ld. \tag{4.3}$$

Proof. Let $T, \varepsilon > 0$ be given. Then one can choose a C^1-function $\theta : \mathbb{R}^d \to [0,1]$ with compact support such that

$$\theta(x) = 1 \quad \text{for all } x \in N_\varepsilon(Q)$$

holds (see [1, Prop. 5.5.8, p. 380]). We define $\tilde{f}(x,u) := \theta(x)f(x,u)$, $\tilde{f} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$. Then $\tilde{f}$ is continuous and continuously differentiable with respect to the first argument. Consider the control system

$$\dot{x}(t) = \tilde{f}(x(t), u(t)), \quad u \in \mathcal{U}. \tag{4.4}$$

The right-hand side of this system is globally bounded, and thus solutions exist globally (see, e.g., [14, Prop. C.3.7]). We denote the cocycle associated with (4.4) by ψ. Note that

$$\psi([0,T],x,u) \subset N_\varepsilon(Q) \Rightarrow \psi(t,x,u) = \varphi(t,x,u) \quad \text{for all } t \in [0,T].$$

Define

$$L_\varepsilon := \max_{(x,u)\in \mathrm{supp}(\theta)\times U} \left\| \frac{\partial \tilde{f}}{\partial x}(x,u) \right\| = \max_{(x,u)\in \mathbb{R}^d\times U} \left\| \frac{\partial \tilde{f}}{\partial x}(x,u) \right\|.$$

Then L_ε is a global Lipschitz constant for $\tilde{f}$ on $\mathbb{R}^d \times U$ with respect to the first variable, that is,

$$\|\tilde{f}(x_1,u) - \tilde{f}(x_2,u)\| \leq L_\varepsilon \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in \mathbb{R}^d, \ u \in U.$$

Note that Q is also weakly invariant with respect to system (4.4) and the lift $\mathcal{Q}$ is the same for systems (2.1) and (4.4). Also the strongly (T,ε)-spanning sets of system (4.4) coincide with those of system (2.1).

Now let $\mathcal{S}^+ = \{(y_1,u_1),\ldots,(y_n,u_n)\} \subset \mathcal{Q}$ be a minimal strongly (T,ε)-spanning set for (K,Q). Define the sets

$$N_i := \left\{ x \in \mathbb{R}^d \mid d_{T,u_i}(x,y_i) < \varepsilon \right\}, \quad i = 1,\ldots,n,$$

where

$$d_{T,u_i}(x,y_i) = \max_{t\in[0,T]} \|\psi(t,x,u_i) - \psi(t,y_i,u_i)\|.$$

It does not matter whether we consider trajectories of system (2.1) or of system (4.4), since the trajectory $\varphi(t,y_i,u_i)$ is contained in Q for $t \geq 0$ and $\varphi(t,x,u_i)$ is ε-close to it up to time T and thus contained in $N_\varepsilon(Q)$. By the definition of strongly spanning sets K is contained in $\bigcup_{i=1}^n N_i$. Let $x \in \mathbb{R}^d$ be a point with $\|x - y_i\| < e^{-L_\varepsilon T}\varepsilon$ for some $i \in \{1, \ldots, n\}$. It follows that

$$\|\psi(t, x, u_i) - \psi(t, y_i, u_i)\| \leq \|x - y_i\| + L_\varepsilon \int_0^t \|\psi(\tau, x, u_i) - \psi(\tau, y_i, u_i)\| d\tau \tag{4.5}$$

for all $t \geq 0$. By Gronwall's lemma this implies

$$\|\psi(t, x, u_i) - \psi(t, y_i, u_i)\| \leq \|x - y_i\| e^{L_\varepsilon T} < \varepsilon \quad \text{for all } t \in [0, T]$$

and hence also

$$\|\varphi(t, x, u_i) - \varphi(t, y_i, u_i)\| \leq \|x - y_i\| e^{L_\varepsilon T} < \varepsilon \quad \text{for all } t \in [0, T].$$

It follows that $x \in N_i$, and thus N_i contains the ball $B_{e^{-L_\varepsilon T}\varepsilon}(y_i)$. Now assume to the contrary that there exists a cover $\mathcal{V}$ of K consisting of $(e^{-L_\varepsilon T}\varepsilon)$-balls such that $N := \#\mathcal{V} < \#\mathcal{S}^+ = n$. Let these balls be centered at points $x_1, \ldots, x_N \in Q$, and assign to the point x_i a control function v_i with $(x_i, v_i) \in \mathcal{Q}$. Then the ball $B_{e^{-L_\varepsilon T}\varepsilon}(x_i)$ is contained in the set

$$V_i := \left\{ x \in \mathbb{R}^d \mid d_{T, v_i}(x, y_i) < \varepsilon \right\}, \quad i = 1, \ldots, N.$$

Thus, the set $\{(x_1, v_1), \ldots, (x_N, v_N)\}$ is also strongly (T, ε)-spanning, which contradicts the minimality of $\mathcal{S}^+$. It follows that

$$r_{\text{inv}}^+(T, \varepsilon, K, Q) \leq b(e^{-L_\varepsilon T}\varepsilon, K).$$

We have $\ln(1/(e^{-L_\varepsilon T}\varepsilon)) = \ln(e^{L_\varepsilon T}\varepsilon^{-1}) = L_\varepsilon T - \ln(\varepsilon)$, and thus

$$T = \frac{\ln(e^{L_\varepsilon T}\varepsilon^{-1}) + \ln(\varepsilon)}{L_\varepsilon} = \frac{\ln(e^{L_\varepsilon T}\varepsilon^{-1})}{L_\varepsilon} \left(1 + \frac{\ln(\varepsilon)}{\ln(e^{L_\varepsilon T}\varepsilon^{-1})} \right).$$

It follows that

$$\begin{aligned}
h_{\text{inv}}^+(\varepsilon, K, Q) &= \limsup_{T \to \infty} \frac{1}{T} \ln r_{\text{inv}}^+(T, \varepsilon, K, Q) \\
&\leq \limsup_{T \to \infty} \frac{1}{T} \ln b(e^{-L_\varepsilon T}\varepsilon, K) \\
&= L_\varepsilon \limsup_{T \to \infty} \frac{\ln b(e^{-L_\varepsilon T}\varepsilon, K)}{L_\varepsilon T} \\
&= L_\varepsilon \limsup_{T \to \infty} \frac{\ln b(e^{-L_\varepsilon T}\varepsilon, K)}{L_\varepsilon \frac{\ln(e^{L_\varepsilon T}\varepsilon^{-1})}{L_\varepsilon}(1 + \frac{\ln(\varepsilon)}{\ln(e^{L_\varepsilon T}\varepsilon^{-1})})} \\
&= L_\varepsilon \limsup_{T \to \infty} \frac{\ln b(e^{-L_\varepsilon T}\varepsilon, K)}{\ln(e^{L_\varepsilon T}\varepsilon^{-1})(1 + \frac{\ln(\varepsilon)}{\ln(e^{L_\varepsilon T}\varepsilon^{-1})})} \\
&= L_\varepsilon \limsup_{T \to \infty} \frac{\ln b(e^{-L_\varepsilon T}\varepsilon, K)}{\ln(e^{L_\varepsilon T}\varepsilon^{-1})} = L_\varepsilon \dim_F(K).
\end{aligned}$$

The Lipschitz constant L_ε was used only in (4.5) in order to obtain the estimate

$$\|\tilde{f}(\psi(\tau,x,u_i),v_i(\tau)) - \tilde{f}(\psi(\tau,y_i,v_i),u_i(\tau))\| \leq L_\varepsilon \|\psi(\tau,x,u_i) - \psi(\tau,y_i,u_i)\|$$

for $\tau \in [0,t]$. Since here $\psi(\tau,y_i,u_i) \in Q$ and, as we have seen later, $\|\psi(\tau,x,u_i) - \psi(\tau,y_i,u_i)\| < \varepsilon$ for all $\tau \in [0,t]$, the estimate we derived for $h^+_{\text{inv}}(\varepsilon,K,Q)$ holds also with

$$L_\varepsilon = \max_{(x,u)\in \text{cl}(N_\varepsilon(Q))\times U} \left\| \frac{\partial \tilde{f}}{\partial x}(x,u) \right\| = \max_{(x,u)\in \text{cl}(N_\varepsilon(Q))\times U} \left\| \frac{\partial f}{\partial x}(x,u) \right\|.$$

As ε tends to zero, the so-defined L_ε tends to $L := \max_{(x,u)\in Q\times U} \|\frac{\partial f}{\partial x}(x,u)\|$ and consequently

$$h^+_{\text{inv}}(K,Q) = \lim_{\varepsilon \searrow 0} h^+_{\text{inv}}(\varepsilon,K,Q) \leq \left(\lim_{\varepsilon \searrow 0} L_\varepsilon \right) \dim_F(K) = L \dim_F(K),$$

which proves the assertion. □

Example 4.1. With the preceding theorems we are now able to compute the invariance entropy for one-dimensional linear control systems of the form

$$\dot{x}(t) = ax(t) + u(t) =: f(x(t),u(t)), \quad u \in \mathcal{U},$$

with $a \geq 0$. In general, for one-dimensional control systems Theorems 4.1 and 4.2 yield

$$h_{\text{inv}}(K,Q) \in \left[\min_{(x,u)\in Q\times U} \frac{\partial f}{\partial x}(x,u), \max_{(x,u)\in Q\times U} \left| \frac{\partial f}{\partial x}(x,u) \right| \right]$$

if K has positive Lebesgue measure. Since $\frac{\partial f}{\partial x}(x,u) = a \geq 0$ for the linear control system, we obtain $h_{\text{inv}}(K,Q) = a$. In the next section we will compute the invariance entropy for linear control systems in arbitrary dimensions.

5. Linear control systems. In this section we compute the invariance entropy for control systems on $\mathbb{R}^d$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U}, \tag{5.1}$$

with matrices $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ and compact control range U. The solutions of (5.1) are given by the variations of constants formula:

$$\varphi(t,x,u) = e^{At}x + \int_0^t e^{A(t-s)}Bu(s)ds.$$

THEOREM 5.1. *Let $K,Q \subset \mathbb{R}^d$ be nonvoid compact sets with $K \subset Q$ and Q being weakly invariant. Let $\lambda_1,\dots,\lambda_d$ be the eigenvalues of A. Then the following estimate holds:*

$$h^+_{\text{inv}}(K,Q) \leq \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i).$$

If, in addition, K has positive Lebesgue measure, we have

$$h_{\text{inv}}(K,Q) = h^+_{\text{inv}}(K,Q) = \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i).$$

Proof. The proof is divided into three steps.

Step 1. We show that $h_{\text{inv}}^+(K,Q)$ is bounded from above by the sum of the positive eigenvalue real parts of A. To this end, consider the linear semiflow $\Phi(t,x) = e^{At}x$, $\Phi : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}^d$. With respect to the Euclidean norm, this semiflow is uniformly continuous in the sense of (2.3) since for all $t_0 > 0$, $t \in [0,t_0]$, and $x,y \in \mathbb{R}^d$ one has

$$\|e^{At}x - e^{At}y\| = \|e^{At}(x-y)\| \leq \|e^{At}\|\|x-y\| \leq \left(\max_{t\in[0,t_0]} \|e^{At}\|\right) \|x-y\|.$$

Hence by Lemma 2.1 the topological entropy $h_{\text{top}}(\Phi)$ equals the topological entropy of the time-one-map $\Phi_1(x) = e^{A}x$. By [3, Thm. 15] the topological entropy of the linear map Φ_1 is given by

$$h_{\text{top}}(\Phi_1) = \sum_{i:\ |\mu_i|>1} \ln|\mu_i|,$$

where $\mu_1,\dots,\mu_d$ are the eigenvalues of e^A. Since $|\mu_i| = \left|e^{\lambda_i}\right| = e^{\text{Re}(\lambda_i)}$ we obtain

$$h_{\text{top}}(\Phi) = h_{\text{top}}(\Phi_1) = \sum_{i:\ |e^{\lambda_i}|>1} \text{Re}(\lambda_i) = \sum_{i:\ \text{Re}(\lambda_i)>0} \text{Re}(\lambda_i).$$

Hence, it suffices to show that $h_{\text{inv}}^+(K,Q) \leq h_{\text{top}}(\Phi)$. To this end, for given $T,\varepsilon > 0$ let $E \subset Q$ be a maximal (T,ε)-separated set with respect to the semiflow Φ, say $E = \{y_1,\dots,y_n\}$. Then E is also (T,ε)-spanning the set Q which means that for all $x \in Q$ there is $j \in \{1,\dots,n\}$ with

$$\max_{t\in[0,T]} \|e^{At}x - e^{At}y_j\| < \varepsilon.$$

Since Q is weakly invariant, we can assign to each y_j ($j \in \{1,\dots,n\}$) a control function $u_j \in \mathcal{U}$ such that $\varphi(\mathbb{R}_0^+, y_j, u_j) \subset Q$. Let $\mathcal{S}^+ := \{(y_1,u_1),\dots,(y_n,u_n)\} \subset \mathcal{Q}$. Since $\varphi(t,x,u) - \varphi(t,y,u) = e^{At}x - e^{At}y$ for all $t \geq 0$, $x,y \in \mathbb{R}^d$, and $u \in \mathcal{U}$, we obtain that $\mathcal{S}^+$ is strongly (T,ε)-spanning for (Q,Q) and hence also for (K,Q). This implies

$$r_{\text{inv}}^+(T,\varepsilon,K,Q) \leq s(T,\varepsilon,Q,\Phi) \quad \text{for all } T,\varepsilon > 0$$

and consequently $h_{\text{inv}}^+(K,Q) \leq h_{\text{sep}}(Q,\Phi) = h_{\text{top}}(Q,\Phi) \leq h_{\text{top}}(\Phi)$.

Step 2. Under the assumption that $\lambda^d(K) > 0$ and $\text{Re}(\lambda_i) > 0$ for all $i \in \{1,\dots,d\}$ we prove that

$$h_{\text{inv}}(K,Q) \geq \sum_{i=1}^{d} \text{Re}(\lambda_i).$$

This is a consequence of Theorem 4.1: Let $f(x,u) = Ax + Bu$. Then it follows that

$$\text{div}_{\text{x}} f(x,u) = \text{tr}\frac{\partial f}{\partial x}(x,u) = \text{tr}(A) = \sum_{i=1}^{d} \lambda_i = \sum_{i=1}^{d} \text{Re}(\lambda_i).$$

The last equality holds since nonreal eigenvalues of a real matrix appear as pairs of complex conjugate numbers, and thus the imaginary parts in the sum cancel. By (4.1) the assertion follows.

Step 3. We prove the inequality $h_{\mathrm{inv}}(K,Q) \geq \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i)$ under the assumption $\lambda^d(K) > 0$ for arbitrary matrices A.

If all eigenvalue real parts of A are nonpositive, the assertion is true, since $h_{\mathrm{inv}}(K,Q) \geq 0$ holds anyway. Hence we may assume that there exists at least one eigenvalue with positive real part. We write $\mathbb{E}^s$, $\mathbb{E}^u$, and $\mathbb{E}^c$ for the corresponding stable, unstable, and center subspaces with respect to the flow $(t,x) \mapsto e^{At}x$. This furnishes the decomposition $\mathbb{R}^d = \mathbb{E}^u \oplus (\mathbb{E}^s \oplus \mathbb{E}^c)$. Consider the projection

$$\pi : \mathbb{R}^d \to \mathbb{E}^u, \quad x \mapsto x^u.$$

The map π is obviously C^1 and we can project our control system to $\mathbb{E}^u$: Let $f(x,u) = Ax + Bu$ and $g(y,u) = A\,|_{\mathbb{E}^u}\, y + \pi Bu$, $g : \mathbb{E}^u \times U \to \mathbb{E}^u$. Then we have

$$D\pi_x f(x,u) = \pi(Ax + Bu) = \pi Ax + \pi Bu = A\pi x + \pi Bu = g(\pi x, u),$$

and thus we can apply Theorem 3.5, which yields

$$h_{\mathrm{inv}}(K,Q) \geq h_{\mathrm{inv}}(\pi K, \pi Q).$$

Since the projected system on $\mathbb{E}^u$ again is a linear control system and all eigenvalue real parts of $A\,|_{\mathbb{E}^u} : \mathbb{E}^u \to \mathbb{E}^u$ are positive, we obtain by Step 2 that

$$h_{\mathrm{inv}}(K,Q) \geq \sum_{\lambda \in \sigma(A|\mathbb{E}^u)} \mathrm{Re}(\lambda) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i)$$

if $\pi K \subset \mathbb{E}^u$ has positive Lebesgue measure. In order to show the latter, let $s = \dim \mathbb{E}^u$ and let λ^s denote the s-dimensional Lebesgue measure on $\mathbb{E}^u$. Assume to the contrary that $\lambda^s(\pi K) = 0$, and consider the linear transformation

$$\alpha : \mathbb{R}^d \to \mathrm{im}(\pi) \oplus \ker(\pi), \quad x \mapsto (\pi x, x - \pi x).$$

On $\mathrm{im}(\pi) \oplus \ker(\pi)$ let $\langle \cdot, \cdot \rangle_e$ be the inner product given by

$$\langle (u_1, v_1), (u_2, v_2) \rangle_e = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^d$. The inner product $\langle \cdot, \cdot \rangle_e$ induces a norm $\| \cdot \|_e$ and a Lebesgue measure λ_e^d on $\mathrm{im}(\pi) \oplus \ker(\pi)$. Using the transformation theorem and the theorem of Fubini we obtain

$$\begin{aligned}
\lambda^d(K) \leq \lambda^d(\pi^{-1}\pi K) &= \int_{\mathbb{R}^d} \chi_{\pi^{-1}\pi K}(x)dx \\
&= \int_{\mathrm{im}(\pi)\oplus\ker(\pi)} \chi_{\pi^{-1}\pi K}(\alpha^{-1}(u,v)) |\det \alpha^{-1}|\ d(u,v) \\
&= |\det \alpha^{-1}| \int_{\mathrm{im}(\pi)} \int_{\ker(\pi)} \chi_{\pi^{-1}\pi K}(u+v) du\ dv \\
&= |\det \alpha^{-1}| \int_{\mathrm{im}(\pi)} \int_{\ker(\pi)} \chi_{\alpha(\pi^{-1}\pi K)}(u,v) du\ dv.
\end{aligned}$$

Since $\alpha(\pi^{-1}\pi K) = \pi K \times \ker(\pi)$ we obtain the contradiction

$$\begin{aligned}\lambda^d(K) &\leq |\det\alpha^{-1}| \int_{\mathrm{im}(\pi)} \int_{\ker(\pi)} \chi_{\pi K\times\ker(\pi)}(u,v)du\ dv \\ &= |\det\alpha^{-1}| \int_{\mathrm{im}(\pi)} \int_{\ker(\pi)} \chi_{\pi K}(u)\chi_{\ker(\pi)}(v)dv\ du \\ &= |\det\alpha^{-1}| \int_{\ker(\pi)} \chi_{\ker(\pi)}(v) \left(\int_{\mathrm{im}(\pi)} \chi_{\pi K}(u)du\right) dv \\ &= |\det\alpha^{-1}| \int_{\ker(\pi)} \chi_{\ker(\pi)}(v) \underbrace{\lambda^s(\pi K)}_{=0} dv = 0.\end{aligned}$$

This finishes the proof. □

Remarks 5.1.

(i) *In the case when* $\lambda^d(K) = 0$ *we cannot make a general statement about the exact value of* $h_{\mathrm{inv}}(K,Q)$. *If, e.g.,* K *is finite, then* $h_{\mathrm{inv}}(K,Q) = 0$. *But if the projection of* K *to* $\mathbb{E}^u(A)$ *has positive Lebesgue measure in* $\mathbb{E}^u(A)$, *then* $h_{\mathrm{inv}}(K,Q) = \sum_{i:\ \mathrm{Re}(\lambda_i)>0} \mathrm{Re}(\lambda_i)$ *anyway.*

(ii) *The existence of a nonvoid compact weakly invariant subset for the linear control system* (5.1) *can be guaranteed if the pair* (A,B) *is controllable, the matrix* A *is hyperbolic, and the control range* U *is compact and convex with nonvoid interior. Then there exists a unique control set* D *and its closure* $Q = \mathrm{cl}(D)$ *is compact (see Colonius and Spadini* [5, *Thm.* 4.1]*). It is easily seen to be weakly invariant. Moreover, it has nonvoid interior and hence positive Lebesgue measure.*

At the end of this section we want to show by an example that $h^*_{\mathrm{inv}}(Q) = \infty$ is possible even if $h_{\mathrm{inv}}(Q) = 0$.

Example 5.1. Consider the linear control system $\dot{x} = -x + u(t)$ on $\mathbb{R}$ with control range $U = [-1,1]$ $(d = m = 1)$. Let $Q \subset [-1,1]$ be an infinite compact set which is totally disconnected (e.g., a Cantor set or the closure of an infinite, countable, discrete, and bounded set). Then for every $x \in Q$ there exists a unique constant control function $u_x \in \mathcal{U}$ with $\varphi(t,x,u_x) = x$ for all $t \geq 0$, namely $u_x(t) \equiv x$. Thus, Q is weakly invariant. Since Q is totally disconnected, each point $x \in Q$ can be kept in Q for some positive time $T > 0$ only by making it a stationary point, i.e., by using the constant control function u_x. Consequently, since Q is infinite, one needs infinitely many control functions to obtain a T-spanning set for Q. By Theorem 5.1 one has $h_{\mathrm{inv}}(Q) = 0$ in this case.

Remark 5.1. In view of this example, it is tempting to conjecture that if $h^*_{\mathrm{inv}}(K,Q)$ happens to be finite, then it coincides with $h_{\mathrm{inv}}(K,Q)$. However, we cannot prove this conjecture.

6. Characterization via finite covers and relation to feedback entropy. In this last section we will give an alternative definition for the strict invariance entropy $h^*_{\mathrm{inv}}(Q)$ via finite covers of the set Q. Again, for simplicity we assume that $M = \mathbb{R}^d$. This definition will reveal a connection to the topological feedback entropy defined in [11] and will also provide a clearer view on what is measured by the quantity $h^*_{\mathrm{inv}}(Q)$. Again consider the general control system (2.1).

For a finite cover $\mathcal{A}$ of Q let $c(\mathcal{A}|Q)$ denote the minimal cardinality of a subcover. We say that a triple $(\mathcal{A}, v, \tau)$ is *invariantly covering* Q if τ is a positive real number,

$\mathcal{A}$ is a finite cover of Q, and $v : \mathcal{A} \to \mathcal{U}$ is a map assigning a control function $v_A \in \mathcal{U}$ to each $A \in \mathcal{A}$ with

$$\varphi(t, A, v_A) \subset Q \quad \text{for all } t \in [0, \tau].$$

If Q is invariantly covered by a triple $(\mathcal{A}, v, \tau)$, where $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ is ordered, then we set $v_a := v_{A_a}$ for $a = 1, \ldots, q$. For every $N \in \mathbb{N}$ and every N-tuple $(a_0, a_1, \ldots, a_{N-1}) \in \{1, \ldots, q\}^N$ we define the control function

$$v_{a_0,a_1,\ldots,a_{N-1}}(t) := v_{a_j}(t - j\tau) \quad \text{for all } t \in [j\tau, (j+1)\tau), \quad j = 0, 1, \ldots, N-1,$$

and the set

$$Q_{a_0,a_1,\ldots,a_{N-1}} := \left\{x \in Q \mid \varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in A_{a_j},\ j = 0, 1, \ldots, N-1\right\}. \tag{6.1}$$

For every $a \in \{1, \ldots, q\}$ we define the diffeomorphism

$$f_a : \mathbb{R}^d \to \mathbb{R}^d, \quad f_a(x) := \varphi(\tau, x, v_a).$$

This yields

$$Q_{a_0,a_1,\ldots,a_{N-1}} = A_{a_0} \cap \bigcap_{j=1}^{N-1} (f_{a_{j-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})^{-1}(A_{a_j}) \tag{6.2}$$

since

$$\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) = f_{a_{j-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0}$$

for all $j = 1, \ldots, N-1$ by the cocycle property (2.2).

Let $\mathcal{A}_N$ be the family of all the sets $Q_{a_0,a_1,\ldots,a_{N-1}}$:

$$\mathcal{A}_N := \left\{Q_{a_0,a_1,\ldots,a_{N-1}} \mid (a_0, a_1, \ldots, a_{N-1}) \in \{1, \ldots, q\}^N\right\}.$$

Then $\mathcal{A}_N$ is also a finite cover of Q (moreover, it is an open cover if $\mathcal{A}$ is an open cover, since in this case openness follows immediately from (6.2)): For every $x \in Q$ we find at least one N-tuple $(a_0, a_1, \ldots, a_{N-1})$ (which may be not unique) with $\varphi(j\tau, x, v_{a_0,a_1,\ldots,a_{N-1}}) \in A_{a_j}$ for $j = 0, 1, \ldots, N-1$, which follows by the invariant covering property of $(\mathcal{A}, v, \tau)$. Now we define

$$h^*_{\text{inv}}(\mathcal{A}, v, \tau) := \frac{1}{\tau} \lim_{N\to\infty} \frac{\ln c(\mathcal{A}_N|Q)}{N}. \tag{6.3}$$

It can be easily shown that $h^*_{\text{inv}}(\mathcal{A}, v, \tau)$ does not depend on the ordering of the set $\mathcal{A}$. The existence of the limit above follows from a subadditivity argument: Let $N, M \in \mathbb{N}$ and let $\tilde{\mathcal{A}}_N$ and $\tilde{\mathcal{A}}_M$ be minimal subcovers of $\mathcal{A}_N$ and $\mathcal{A}_M$, respectively. If $Q_1 = Q_{a_0,a_1,\ldots,a_{N-1}} \in \tilde{\mathcal{A}}_N$ and $Q_2 = Q_{b_0,b_1,\ldots,b_{M-1}} \in \tilde{\mathcal{A}}_M$, then

$$Q_1 \cap (f_{a_{N-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})^{-1}(Q_2) = Q_{a_0,a_1,\ldots,a_{N-1},b_0,b_1,\ldots,b_{M-1}}.$$

Consequently, we can define a map $\alpha : \tilde{\mathcal{A}}_N \times \tilde{\mathcal{A}}_M \to \mathcal{A}_{M+N}$ which maps the pair $(Q_{a_0,a_1,\ldots,a_{N-1}}, Q_{b_0,b_1,\ldots,b_{M-1}})$ to

$$Q_{a_0,a_1,\ldots,a_{N-1}} \cap (f_{a_{N-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})^{-1}(Q_{b_0,b_1,\ldots,b_{M-1}}).$$

The image of α is obviously a subcover of $\mathcal{A}_{M+N}$ and hence

$$c(\mathcal{A}_{M+N}|Q) \leq \#\alpha(\tilde{\mathcal{A}}_N \times \tilde{\mathcal{A}}_M) \leq \#(\tilde{\mathcal{A}}_N \times \tilde{\mathcal{A}}_M)$$
$$= \#\tilde{\mathcal{A}}_N \cdot \#\tilde{\mathcal{A}}_M = c(\mathcal{A}_N|Q) \cdot c(\mathcal{A}_M|Q).$$

This proves subadditivity of the sequence $(\ln c(\mathcal{A}_N|Q))_{N\in\mathbb{N}}$.

THEOREM 6.1. *For the control system* (2.1) *the strict invariance entropy and the entropy* (6.3) *defined via covers satisfy*

$$h^*_{\text{inv}}(Q) = \inf_{(\mathcal{A},v,\tau)} h^*_{\text{inv}}(\mathcal{A}, v, \tau), \tag{6.4}$$

where the infimum is taken over all triples which are invariantly covering Q. If there exists no such triple, then the infimum is defined as ∞.

Proof. If $h^*_{\text{inv}}(Q) = \infty$, then by Proposition 3.1(iii) $r^*_{\text{inv}}(T, Q) = \infty$ for all $T > 0$, which implies that there exists no invariantly covering triple $(\mathcal{A}, v, \tau)$. Hence, in this case the assertion holds.

Now assume that $h^*_{\text{inv}}(Q) < \infty$. Let $(\mathcal{A}, v, \tau)$, $\mathcal{A} = \{A_1, \ldots, A_q\}$, be a triple which is invariantly covering Q. We will show that

$$c(\mathcal{A}_N|Q) \geq r^*_{\text{inv}}(N\tau, Q) \quad \text{for all } N \in \mathbb{N}, \tag{6.5}$$

which implies

$$h^*_{\text{inv}}(\mathcal{A}, v, \tau) = \frac{1}{\tau} \lim_{N\to\infty} \frac{1}{N} \ln c(\mathcal{A}_N|Q) \geq \lim_{N\to\infty} \frac{1}{N\tau} \ln r^*_{\text{inv}}(N\tau, Q) = h^*_{\text{inv}}(Q).$$

The latter equality follows from Proposition 3.4(ii), which also holds for the strict invariance entropy. In order to show (6.5) let $\tilde{\mathcal{A}}_N$ be a minimal subcover of $\mathcal{A}_N$ and define

$$S_N := \left\{ v_{a_0,a_1,\ldots,a_{N-1}} \mid Q_{a_0,a_1,\ldots,a_{N-1}} \in \tilde{\mathcal{A}}_N \right\}.$$

Since $\tilde{\mathcal{A}}_N$ is covering Q, for every $x \in Q$ there is $(a_0, a_1, \ldots, a_{N-1})$ with $x \in Q_{a_0,a_1,\ldots,a_{N-1}} \in \tilde{\mathcal{A}}_N$. By (6.1) this implies, in particular, $\varphi([0, N\tau], x, v_{a_0,a_1,\ldots,a_{N-1}}) \subset Q$, which shows that S_N is $(N\tau)$-spanning and thus (6.5) holds.

It remains to show that there exists a sequence $(\mathcal{A}^k, v^k, \tau^k)$ of triples which are invariantly covering Q with $h^*_{\text{inv}}(\mathcal{A}^k, v^k, \tau^k) \to h^*_{\text{inv}}(Q)$ for $k \to \infty$. To this end, let $\tau^k := k$ and let $\mathcal{S}_k := \{v^k_1, \ldots, v^k_{n_k}\} \subset \mathcal{U}$ be a minimal k-spanning set for Q. Define

$$A_j := \left\{ x \in Q \mid \varphi([0, k], x, v^k_j) \subset Q \right\}, \quad j = 1, \ldots, n_k. \tag{6.6}$$

Then $\mathcal{A}^k := \{A_1, \ldots, A_{n_k}\}$ is a cover of Q. Let v^k be defined by $v^k(A_j) := v^k_j$ for $j = 1, \ldots, n_k$. Then it immediately follows that $(\mathcal{A}^k, v^k, \tau^k)$ is invariantly covering Q. We obtain

$$h^*_{\text{inv}}(\mathcal{A}^k, v^k, \tau^k) = \frac{1}{k} \lim_{N\to\infty} \frac{\ln c(\mathcal{A}^k_N|Q)}{N} = \frac{1}{k} \inf_{N\in\mathbb{N}} \frac{\ln c(\mathcal{A}^k_N|Q)}{N}$$
$$\leq \frac{1}{k} \ln \#\mathcal{A}^k = \frac{1}{k} \ln r^*_{\text{inv}}(k, Q).$$

Since $\frac{1}{k}\ln r^*_{\mathrm{inv}}(k,Q)$ converges to $h^*_{\mathrm{inv}}(Q)$ for $k\to\infty$ we find for every $\varepsilon>0$ some $k_0\in\mathbb{N}$ such that $\frac{1}{k}\ln r^*_{\mathrm{inv}}(k,Q)-h^*_{\mathrm{inv}}(Q)\le\varepsilon$ for all $k\ge k_0$. This implies

$$h^*_{\mathrm{inv}}(Q)\le h^*_{\mathrm{inv}}(\mathcal{A}^k,v^k,\tau^k)\le h^*_{\mathrm{inv}}(Q)+\varepsilon\quad\text{for all } k\ge k_0,$$

which proves the claim. □

Remark 6.1. The characterization of strict invariance entropy, given in the preceding theorem, is very similar to the definition of *strong topological feedback entropy* introduced in Nair et al. [11]. The differences are, first, that we consider continuous time systems, while topological feedback entropy is defined for time-discrete control systems of the form

$$x_{k+1}=F(x_k,u_k),\quad k\ge 0,$$

where the state space X is a topological space and the controls u_k are taken from an arbitrary set U. Furthermore, here a compact set $Q\subset X$ with nonvoid interior is considered such that there is another compact set $Q'\subset\mathrm{int}(Q)$ with the following property: For every $x_0\in Q$ there is a control $u_0\in U$ with $x_1=F(x_0,u_0)\in\mathrm{int}(Q')$. This invariance condition—called strong invariance in [11]—differs from the weak invariance that we impose on the set Q. For example, if Q is the closure of a variant control set with nonvoid interior, then there are always points on the boundary of Q which cannot be steered to the interior. The strong invariance condition in [11], which is tailored for stabilization problems, also makes it possible to consider only open covers of Q.

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