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Spectral Theory for Perturbed Systems

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This paper presents an overview of topological, smooth, and control techniques for dynamical systems and their interrelations for the study of perturbed systems. We concentrate on spectral analysis via linearization of systems. Emphasis is placed on parameter dependent perturbed systems and on a comparison of the Markovian and the dynamical structure of systems with Markov diffusion perturbation process. A number of applications is provided.

1 Introduction

The theory of perturbed systems may be viewed as part of dynamical systems theory: Differential equations with deterministic (time varying) perturbations can be understood as skew product flows (see Sell [47]), systems with stochastic perturbations as flows over a probability space (see L. Arnold [2]), and (open loop) control systems as flows over the space of admissible control functions (see Colonius/Kliemann [15]). The common feature of these approaches is that perturbed systems are viewed as specific skew product flows, in which the structure of the base flow determines the nature of the perturbation under consideration and the kind of techniques that are appropriate for the analysis of the systems. Furthermore, there are direct connections between these different classes of perturbed systems such as the support theorem due to Stroock/Varadhan [48] that links Markov diffusion systems and control systems. Oseledets' multiplicative ergodic theory is one of the main tools for the study of stochastically perturbed systems. The reason for this is simple: For a linear flow on a vector bundle with flow invariant measure on the base space, Oseledets' theorem states that almost all points are Lyapunov regular, describing the exponential convergence and divergence of the trajectories. This allows for a detailed analysis of the perturbed flow, including stability, invariant manifolds, entropy, pressure, etc. As a consequence, a dynamic concept of stochastic bifurcation can be formulated and applied to a great number of systems. On the other hand, several spectral concepts for linear flows on vector bundles have been proposed using ideas of topological dynamics, while Oseledets' theory is a 'measurable' theory. Most notable among the topological concepts is the 'dichotomy spectrum', based on exponential dichotomies Sacker/Sell [44], which yields continuous bundle decompositions and corresponding invariant manifolds.

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The purpose of this paper is to study the different spectral concepts for perturbed flows and to discuss their connections, including the associated bundle decompositions. We will proceed from general linear flows on vector bundles to perturbed system with a common fixed point. For the Markov diffusion perturbation model we compare some results obtained via this flow point of view to those obtained via stochastic analysis and the theory of Markov semigroups. We hope that this approach allows the reader to see clearly the differences and similarities between the different concepts.

Section 2 recalls general properties of stochastic and deterministic perturbed systems. Section 3 discusses spectral concepts for the general case of linear flows on vector bundles, and Section 4 specializes the results to specific families of perturbed systems. In Section 5 parameter dependent systems are considered, including a brief discussion of bifurcation results. Applications to robust stability of perturbed systems and to feedback stabilizability of control systems are briefly mentioned in Section 6.

2 Stochastic Systems, Control Flows, and Diffusion Processes - Basic Concepts

Perturbed systems, as we understand them in this paper, consist of two components, namely the perturbation model and the system model. A natural framework for these systems are skew product flows, which we consider the starting point of our theory. In this section we recall several classes of perturbed systems and describe their relation to skew product flows. An important aspect of our set-up is that all spaces and the dynamical systems on them have topological properties which aid in the qualitative analysis in the subsequent sections.

On the most abstract level, a perturbation model is given by a continuous flow on a topological space $\mathcal{U}$

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \tag{1}$$

i.e., it holds that $\theta_t \circ \theta_s = \theta_{t+s}$ and $\theta_0 = \mathrm{id}$. (We will often write θ_t for the map $\theta(t, \cdot)$.) Note that the flow (1) is defined on the two sided time interval $\mathbb{R}$, and hence $\theta_t^{-1} = \theta_{-t}$ for all $t \in \mathbb{R}$. The model of a system perturbed by θ is a continuous skew product flow on the topological product space $\mathcal{U} \times M$

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \tag{2}$$

whose first component, the perturbation (1), affects the system component φ, but not vice versa. In particular, the φ–component itself is not a flow. The skew product flow Φ is a prototype of a deterministically perturbed system in continuous time. In a stochastic perturbation model one has, in addition to (2), a probability measure P on the Borel σ–algebra of $\mathcal{U}$, which is invariant under the flow θ, i.e., $\theta_t P = P$ for all $t \in \mathbb{R}$. This set–up differs from the one treated by Arnold [2] in the way that we require $\mathcal{U}$ to be a topological space and θ to be continuous, while Arnold's perturbation model is just measurable.

The specific perturbations treated in this paper are L^∞–functions with compact range. In the deterministic case for the perturbation (1), we consider the following set–up taken from [18].

Let $U \subset \mathbb{R}^m$ be compact and convex, with $0 \in \mathrm{int}U$, the interior of U. Denote by $\mathcal{U} = \{u : \mathbb{R} \to U, \text{ measurable}\}$ the perturbation space, equipped with the weak* topology of

$L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}^m))^*$. This space is compact and metrizable. The flow θ is given by the time shift

$$\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \theta_t(u(\cdot)) = u(t + \cdot), \tag{3}$$

resulting in a continuous dynamical system. Some standard interpretations of the model (3) are time varying perturbations with a given range, as they are used in robustness theory, or open loop control functions, as they are common in control theory.

In a stochastic perturbation model we are also given a θ–invariant probability measure P on $\mathcal{U}$. One way to arrive at such a measure is given by the Kolmogorov construction for stationary processes: Let $\eta : \mathbb{R} \times \Omega \to U$ be a stationary stochastic process on a probability space $(\Omega, \mathfrak{F}', P')$, with continuous trajectories. Let $C(\mathbb{R}, U)$ be the space of continuous functions on $\mathbb{R}$ with values in U, and $\overline{\mathfrak{F}}$ the σ–algebra on $C(\mathbb{R}, U)$, generated by the cylinder sets. Then the process η induces a probability measure $\overline{P}$ on $(C(\mathbb{R}, U), \overline{\mathfrak{F}})$, which is invariant under the shift in $C(\mathbb{R}, U)$. We imbed $C(\mathbb{R}, U)$ into $\mathcal{U}$, extend $\overline{\mathfrak{F}}$ to the Borel σ–algebra $\mathfrak{F}$ of $\mathcal{U}$, and extend $\overline{P}$ to a measure P on $\mathfrak{F}$, which is invariant under the shift θ in (3), compare Gichman/Skorochod [25] for details on Kolmogorov's construction. Note that the extension of the trajectory space to $\mathcal{U}$ allows us to use topological properties of the flow θ in (3).

The specific systems treated in this paper are smooth systems with affine perturbations: Let M be a paracompact C^∞–manifold of dimension $d < \infty$, and let $X_0, X_1, \ldots, X_m$ be C^∞–vector fields on M. The system dynamics are given by the ordinary differential equation

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \ on\ M. \tag{4}$$

where $u(\cdot) \in \mathcal{U}$.

Since we restrict ourselves to global flows, we assume that (4) has a unique solution $\varphi(t, x, u)$ for all $(u, x) \in \mathcal{U} \times M$ with $\varphi(0, x, u) = x$, which is defined for all $t \in \mathbb{R}$. Sufficient conditions for this are, e.g., globally Lipschitz continuous vector fields or compactness of M, since we assume U to be compact. Equation (4), together with the perturbation (3), defines the system flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \tag{5}$$

which is a continuous skew product flow.

The rest of this introductory part is devoted to Markov diffusion systems and to approaches for the analysis of their qualitative behavior. We start from a stochastic perturbation given by a stochastic differential equation on a C^∞–manifold N (of finite dimension)

$$d\eta = Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_j(\eta) \circ dW_j, \tag{6}$$

where $Y_0, Y_1, ..., Y_\ell$ are C^∞–vector fields on N and '$\circ$' denotes the symmetric (Stratonovich) stochastic differential. (We refer the reader to Arnold [1] and Ikeda/Watanabe [31] for basic facts on stochastic differential equations.) We assume that Equation (6) admits at least one stationary Markov solution, see e.g. Khasminskii [35]. We force this solution to be the unique stationary Markov one by imposing a Lie algebra rank condition of the form

$$\dim \mathcal{LA}\{Y_1, \ldots, Y_\ell\}(q) = \dim N \text{ for all } q \in N. \tag{7}$$

In (7) we have used the following notation: Let $\mathfrak{X}(N)$ be the set of vector fields on N, and let $\mathfrak{Y} \subset X(N)$ be a subset. $\mathcal{LA}(\mathfrak{Y})$ denotes the Lie algebra generated by $\mathfrak{Y}$ in $\mathfrak{X}(N)$, which induces a distribution Δ (in the differential geometric sense) in the tangent bundle TN. For $q \in N$, the vector space $\mathcal{LA}\{\mathfrak{Y}\}(q) \subset T_qN$ is the distribution Δ evaluated at q. Condition (7) guarantees (see Kunita [37]) that equation (6) has a unique stationary Markov solution η_t^*, which we extend to all $t \in \mathbb{R}$, compare Arnold [2]. We consider this process η_t^* as a background noise, which is mapped via a surjective function

$$f : N \to U \tag{8}$$

onto the perturbation space $U \subset \mathbb{R}^m$, compare [17, Lemma 3.17] and [14]. Then $\xi_t = f(\eta_t^*)$ is a stationary stochastic process on U. Combining this perturbation model with the system (4) we arrive at the Markov diffusion process

$$\begin{array}{rcl} d\eta & = & Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_i(\eta) \circ dW_j, \quad \eta_0 = \eta_0^*, \\ \dot{x} & = & X_0(x) + \sum_{i=1}^{m} f_i(\eta_t) X_i(x) \end{array} \tag{9}$$

on the state space $N \times M$.

The behavior of the system (9) can now be studied using a variety of approaches:

- Stochastic analysis, compare, e.g., the standard references Ikeda/Watanabe [31] or Ethier/ Kurtz [24],
- Stochastic flows, compare Arnold [2],
- Imbedding of the stationary process η_t^* into the flow (5) as described above,
- Connections with control theory via the support theorem of Stroock and Varadhan [48].

In this paper we will use a combination of the last two approaches. To this end we briefly describe a version of the support theorem that is suitable for our purposes, compare Kunita [36], [38], Ichihara/Kunita [30] or Arnold [2].

Let L be a finite dimensional C^∞–manifold and consider the stochastic differential equation

$$dz = Z_0(z)dt + \sum_{k=1}^{r} Z_k(z) \circ dW_k, \tag{10}$$

with C^∞ vector fields $Z_0, \ldots, Z_r$. Denote by $C_p(\mathbb{R}^+, L)$ the space of continuous functions $w : [0, \infty) \to L$ with $w(0) = p \in L$, equipped with the topology of uniform convergence on compact time intervals. For the initial value $p \in L$, the stochastic differential equation induces a probability measure P_p on $C_p(\mathbb{R}^+, L)$ which, intuitively, assigns to each Borel set B in $C_p(\mathbb{R}^+, L)$ the probability that the functions in B appear as trajectories of the solution of (10). Stroock and Varadhan [48] associate with (10) formally a control system of the form

$$\dot{z} = Z_0(z) + \sum_{k=1}^{r} w_k(t) Z_k \tag{11}$$

with control functions $w \in \mathcal{W} = \{w : [0,\infty) \to \mathbb{R}^r,\ piecewise\ constant\}$. We denote by $\psi(\cdot,p,w)$ the solutions of (11) with initial value $\psi(0,p,w) = p$, and by $\Psi_p = \{\psi(\cdot,p,w),\ w \in \mathcal{W}\} \subset C_p(\mathbb{R}^+, L)$ the set of all such solutions. The support theorem now states

$$\operatorname{supp} P_p = \operatorname{cl} \Psi_p, \tag{12}$$

where 'supp' denotes the support of a measure (i.e. the smallest closed subset of full measure), and the closure 'cl' is taken in $C_p(\mathbb{R}^+, L)$. In the form (12) the support theorem is not yet suitable for the study of (9) with $L = N \times M$, because it refers only to fixed initial conditions, the control functions in *W* are taken to be piecewise constant, and we would have to choose controls with values in $\mathbb{R}^\ell$ to first analyze the η–, and then the x–component of (9). However, Kunita [37] shows that under the Lie algebra rank condition $\dim \mathcal{LA}\{Z_1,\dots,Z_r\}(p) = \dim L$ for all $p \in L$ we have

$$\operatorname{cl} \Psi_p = C_p(\mathbb{R}^+, L). \tag{13}$$

This, together with an appropriate concept of controllability regions, will allow us to reduce the control analysis to the system (4) with control functions in $\mathcal{U}$, and hence to the study of the skew product flow (5).

3 Spectra for Linear Flows on Vector Bundles

Let us fix some notation for linear flows on vector bundles. For the precise definitions and standard results see, e.g. [33], or [18, Appendix B.1] which contains all the specifics needed here. A vector bundle over a compact metric space B with finite dimensional (Hilbert space) fibers is denoted by $\pi : \mathcal{V} \to B$, its projective bundle is $\pi : \mathbb{P}\mathcal{V} \to B$, and the zero section is Z. A linear flow Ψ on $\pi : \mathcal{V} \to B$ is a flow on $\mathcal{V}$ preserving fibers such that the induced maps on the fibers are linear, i.e. for all $\alpha \in \mathbb{R}$ and $v_1, v_2 \in \mathcal{V}$ with $\pi(v_1) = \pi(v_2)$, and all $t \in \mathbb{R}$ one has $\pi(\Psi(t,v_1)) = \pi(\Psi(t,v_2))$ and $\Psi(t,\alpha(v_1+v_2)) = \alpha\Psi(t,v_1)+\alpha\Psi(t,v_2)$. The cocycle associated with Ψ is the family of linear maps $\{\Psi_{b,t} = \Psi(t,\cdot)_{|\mathcal{V}_b} : \mathcal{V}_b \to \mathcal{V}_{\pi\Psi(t,b)}, t \in \mathbb{R}, b \in B\}$, where $\mathcal{V}_b$ is the fiber over $b \in B$ and $\pi\Psi$ is the induced flow on the base space B. Note that Ψ induces a flow $\mathbb{P}\Psi$ on the projective bundle $\mathbb{P}\mathcal{V}$. The Lyapunov exponents of the linear flow Ψ are defined for $v \in \mathcal{V}$ as

$$\lambda(v) = \limsup_{t\to\infty} \frac{1}{t} \log |\Psi_t v|, \quad v \in \mathcal{V}\backslash Z, \tag{14}$$

and the Lyapunov spectrum of Φ is the collection of all Lyapunov exponents

$$\Sigma_{\mathrm{Ly}} = \{\lambda(v), v \in \mathcal{V}\backslash Z\}. \tag{15}$$

Unfortunately, the Lyapunov spectrum of a flow does not possess reasonable regularity or continuity properties (compare Lyapunov [42], Hahn [29], or [18, Section 2.8]). Different spectral concepts that have been proposed in the literature try to remedy these shortcomings.

We begin with the Morse spectrum [16]. For $\varepsilon, T > 0$ an (ε, T)–chain ζ in $\mathbb{P}\mathcal{V}$ is given by $n \in \mathbb{N}$, $p_0, \ldots, p_n \in \mathcal{V}$, and $t_0, \ldots, t_{n-1} \geq T$ with $\mathrm{d}(\mathbb{P}\Psi_{t_i} p_i, p_{i+1}) < \varepsilon$ for all i. A chain recurrent component $\mathcal{M}$ is a maximal subset of $\mathbb{P}\mathcal{V}$ such that any two points p, q in $\mathcal{M}$ are for every $\varepsilon, T > 0$ connected by an (ε, T)-chain. Define the chain exponent of an (ε, T)-chain ζ as

$$\lambda(\zeta) = (\sum_{i=0}^{n-1} t_i)^{-1} \sum_{i=0}^{n-1} (\log(|v_i|)) \tag{16}$$

with arbitrary $v_i \in \mathbb{P}^{-1}(p_i)$. For a chain recurrent component $\mathcal{M}$ of $\mathbb{P}\Psi$ we set

$$\begin{aligned} \Sigma_{Mo}(\mathcal{M}) \quad = \quad & \{\lambda \in \mathbb{R}, \text{ there are } \varepsilon^k \to 0, T^k \to \infty \text{ and } (\varepsilon^k, T^k)\text{-chains} \\ & \zeta^k \text{ in } \mathcal{M} \text{ with } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty\}. \end{aligned} \tag{17}$$

Definition 3.1 Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$. For a chain recurrent component $L \subset B$ of the base flow $\pi\Psi$ on B the Morse spectrum is defined as

$$\Sigma_{Mo}(L) = \cup\{\Sigma_{\mathrm{Mo}}(\mathcal{M}), \begin{array}{c} \mathcal{M} \text{ is a chain recurrent component} \\ \text{of } \mathbb{P}\Psi \text{ and } \mathbb{P}\pi\mathcal{M} \subset L \end{array} \},$$

and the Morse spectrum of Ψ is

$$\Sigma_{\mathrm{Mo}} = \cup\{\Sigma_{\mathrm{Mo}}(\mathcal{M}), \mathcal{M} \text{ is a chain recurrent component of } \mathbb{P}\Psi\}.$$

Upon noticing that the chain recurrent components $\mathcal{M}_1, ..., \mathcal{M}_l$ of $\mathbb{P}\Psi$ form a Morse decomposition with order $\precsim$ (see, e.g., [18], Appendix B.2), we can formulate the main result on the Morse spectrum (compare [18, Theorem 5.1.6]).

Theorem 3.2 *Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$.*

(i) $\Sigma_{\mathrm{Ly}} \subset \Sigma_{\mathrm{Mo}} = \bigcup\{\Sigma_{\mathrm{Mo}}(L)$, *$L$ is chain recurrent component of $\pi\Psi\}$.*

(ii) $\Sigma_{\mathrm{Mo}}(L) = \bigcup_{i=1}^{\ell} \Sigma_{\mathrm{Mo}}(\mathcal{M}_i)$, *where the union is taken over the finitely many chain recurrent components of $\mathbb{P}\Psi_{|\mathbb{P}\pi^{-1}L}$ and $\ell \leq \dim \mathcal{V}$.*

(iii) $\Sigma_{\mathrm{Mo}}(\mathcal{M}_i) = [\kappa^*(\mathcal{M}_i), \kappa(\mathcal{M}_i)]$ *are (finite) intervals for each $i = 1, ..., \ell$ and $\mathcal{M}_i \preceq \mathcal{M}_j$ iff $\kappa^*(\mathcal{M}_i) < \kappa^*(\mathcal{M}_j)$ and $\kappa(\mathcal{M}_i) < \kappa(\mathcal{M}_j)$.*

(iv) *The chain recurrent components are linearly ordered, $M_1 \preceq ... \preceq \mathcal{M}_\ell$.*

The next result shows that the decomposition of the vector bundle $\mathcal{V}$ associated to the (intervals of the) Morse spectrum is indeed topologically nice. This theorem is a generalization of [46] to base flows with several chain recurrent components, compare [18, Theorem 5.2.6].

Theorem 3.3 *Let $\Psi : \mathcal{V} \to \mathcal{V}$ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$.*

(i) *For a chain recurrent component $L \subset B$ of the base flow $\pi\Psi$ the chain recurrent set of $\mathbb{P}\Psi_{|P\pi^{-1}L}$ has finitely many components $\mathcal{M}_i$, $i = 1, ..., \ell$ with $1 \leq \ell = \ell(L) \leq d = \dim \mathcal{V}$. Each $\mathcal{M}_i$ defines a continuous, constant dimensional subbundle of $\pi^{-1}L$ via $\mathcal{V}_i = \{v \in \pi^{-1}L$, $v \notin Z$ implies $\mathbb{P}v \in \mathcal{M}_i\}$. These subbundles form a Whitney sum $\pi^{-1}L = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_\ell$ of $\pi^{-1}L$.*

(ii) *Conversely, every chain recurrent component $\mathcal{M}$ of $\mathbb{P}\Psi$ is of the form above. In particular, $\mathbb{P}\Psi(\mathcal{M})$ is a chain recurrent component in B.*

We next discuss a spectral concept, introduced by Salamon and Zehnder [45], which we call the topological spectrum of a linear flow. For $\Psi : \mathcal{V} \to \mathcal{V}$ and $\lambda \in \mathbb{R}$ define

$$\Psi_t^\lambda = \exp(-\lambda t)\Psi_t,\ t \in \mathbb{R}. \tag{18}$$

The idea of the topological spectrum is based on the concept of isolated invariant sets: A compact subset $K \subset S$ is isolated invariant for a flow (S, Λ) if it is invariant ($\Lambda(t,x) \in K$ for all $t \in \mathbb{R}$, $x \in K$) and if there exists a neighborhood N of K such that $\Lambda(t,x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$.

Definition 3.4 The topological spectrum of a linear flow Ψ is

$$\Sigma_{\mathrm{top}} = \{\lambda \in \mathbb{R},\ \text{the zero section } Z \text{ is not an isolated invariant set of } \Psi^\lambda\}.$$

The relation between the Morse and the topological spectrum is described in the next result, compare [18, Theorem 5.5.3].

Theorem 3.5 *For a linear flow Ψ on a vector bundle $\pi : \mathcal{V} \to B$ the inclusion $\Sigma_{\mathrm{Mo}} \subset \Sigma_{\mathrm{top}}$ holds. If the flow on the base space B is chain transitive. then we have equality of the spectra.*

Note that even in the case of a chain transitive base flow this result does not mean that Σ_{top} consists of the same intervals as the Morse spectrum, since the Morse intervals may overlap (compare, e.g., Example 7.3.19 in [18]) and the topological spectrum is defined globally for the flow Ψ. If the base flow is not chain transitive then strict inclusion may hold between the spectra, see [18, Example 5.5.4]. Combined with Theorem 3.2 (i), this also shows that the Lyapunov spectrum may be strictly included in the topological spectrum.

The topological spectrum is related to a bundle decomposition of $\mathcal{V}$ in the following way: If $\lambda \notin \Sigma_{\mathrm{top}}$, then the flow Ψ^λ from (18) has an unstable set $\mathcal{V}^{\lambda-} = \{v \in \mathcal{V},\ \emptyset \neq \omega^*(v) \subset Z\}$ and a stable set $\mathcal{V}^{\lambda+} = \{v \in \mathcal{V},\ \emptyset \neq \omega(v) \subset Z\}$ which intersect each fiber in a linear subspace and their projections $(\mathbb{P}\mathcal{V}^{\lambda+}, \mathbb{P}\mathcal{V}^{\lambda-})$ form an attractor–repeller pair of $\mathbb{P}\mathcal{V}$ in the sense of C. Conley. Theorem 3.3 (i) shows the following.

Corollary 3.6 *Assume that the base flow $\pi\Psi$ of a linear flow Ψ on a vector bundle is chain recurrent. Then there exists for any $\lambda \notin \Sigma_{\mathrm{top}}$ a number $k \in \{1, \ldots, \ell\}$ such that $\mathcal{V}^{\lambda-} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$ and $\mathcal{V}^{\lambda+} = \mathcal{V}_{k+1} \oplus \ldots \oplus \mathcal{V}_\ell$, where $\mathcal{V}_i,\ i = 1, ..., \ell$ are the subbundles of the Morse spectrum defined in Theorem 3.3 (i).*

The 'dichotomy spectrum' was introduced by Sacker/Sell [44], based on exponential dichotomies. Recall that a projection P on a vector bundle is a continuous map $P : \mathcal{V} \to \mathcal{V}$ with $P \circ P = P$ such that the restrictions to the fibers $P_b : \mathcal{V}_b \to \mathcal{V}_b$ are well defined maps for all $b \in B$. An exponential dichotomy is a projection $P \neq 0$, $P \neq id$ such that there are constants $K \geq 1$, $\alpha > 0$ with $|\Psi_t P \psi_{-s}| \leq K e^{-\alpha(t-s)}$ for $s \leq t$ and $|\Psi_t(\mathrm{id}-P)\psi_{-s}| \leq K e^{-\alpha(s-t)}$ for $s \geq t$.

Definition 3.7 The dichotomy spectrum of a linear flow Ψ is

$$\Sigma_{\mathrm{dich}} = \{\lambda \in \mathbb{R},\ \Psi^\lambda \text{ has no exponential dichotomy}\},$$

where Ψ^λ is defined as in (18).

Theorem 3.8 *For a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ the dichotomy spectrum consists of disjoint intervals, whose endpoints are Lyapunov exponents of the flow. The inclusion $\Sigma_{\text{top}} \subset \Sigma_{\text{dich}}$ holds, and if the base flow is chain recurrent, then one has equality.*

For a proof of this theorem see Sell [47], Theorem IV.9. In general, the inclusion can be strict, see e.g. Robinson [43], p. 435. The bundle decomposition induced by the dichotomy spectrum is, for $\lambda \notin \Sigma_{\text{dich}}$, of the form $\mathcal{V} = \overline{\mathcal{V}}^{\lambda+} \oplus \overline{\mathcal{V}}^{\lambda-}$ with $\overline{\mathcal{V}}^{\lambda+} = P^\lambda \mathcal{V}$ and $\overline{\mathcal{V}}^{\lambda-} = (\text{id}-P^\lambda)\mathcal{V}$. Since these decompositions are exponentially separated subbundles (compare [18, Chapter 5] for the details) and by Corollary 3.6 we obtain for the different bundle decompositions the following result.

Corollary 3.9 *Let $L \subset B$ be a chain recurrent component of the base flow. Let $\{\mathcal{M}_i, i \in I\}$ be a maximal collection of Morse sets of the projective flow $\mathbb{P}\Psi$ such that the union of the Morse spectral intervals $J = \bigcup_{i\in I} \Sigma_{\text{Mo}}(\mathcal{M}_i)$ is an interval itself.*

(i) *J is an interval of the topological and of the dichotomy spectrum.*

(ii) *$\bigoplus_{i\in I} \mathcal{V}_i$ is the corresponding subbundle of $\mathcal{V}$ for both spectra, here the $\mathcal{V}_i$ are defined as in Theorem 3.3 (i).*

Examples show that the decomposition of $\mathcal{V}$ according to the Morse spectrum can be strictly finer than the one induced by either the topological or the dichotomy spectrum, compare, e.g., Example 5.5.12 in [18]. Note that the latter spectra are globally defined for a linear flow Ψ, as are their associated invariant subbundles, while the Morse spectrum and its invariant subbundles are defined 'locally' over each chain recurrent component of the base flow.

The spectral concepts presented so far are based on different concepts of topological dynamics, resulting in spectral intervals that contain the Lyapunov spectrum, and in continuous decompositions of the vector bundle $\mathcal{V}$ into associated invariant subbundles. An inner approximation of the Lyapunov spectrum through Lyapunov regular points is given by the Oseledets spectrum, see e.g. Arnold [2], or Katok/Hasselblatt [34].

Theorem 3.10 *Let Ψ be a linear flow on the vector bundle $\pi : \mathcal{V} \to B$ and let μ be a $\pi\Psi$–invariant probability measure of the base flow. Then there exists a measurable subset $\Gamma \subset B$ with $\mu\Gamma = 1$ such that for all $b \in \Gamma$ there are numbers $\lambda_i(b)$, $i = 1 \dots r(b) \leq \dim \mathcal{V}$ and a measurable decomposition $\mathcal{V}_b = L_b^1 \oplus \dots \oplus L_b^{r(b)}$ with $\lim_{t\to\pm\infty} \frac{1}{t} \log |\Psi(t, v)| = \lambda_i(b)$ iff $v \in L_b^i$. Furthermore, if μ is ergodic, then $r(b)$ and $\lambda_i(b)$ are constant μ–almost everywhere.*

Based on Theorem 3.10 one defines the Oseledets spectrum.

Definition 3.11 The Oseledets spectrum of a linear flow Ψ on a vector bundle $\pi : \mathcal{V} \to B$ with $\pi\Psi$–invariant measure μ on the base space is given by

$$\Sigma_{\text{Os}}(\mu) = \{\lambda \in \mathbb{R}, \lambda = \lambda_i(b) \; for \; some \; b \in \Gamma, \; i = 1 \dots r(b)\},$$

where we have used the notation from Theorem 3.10.

The relation between the Oseledets spectrum and the other spectral concepts is described in the next result.

Theorem 3.12 *Let Ψ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$.*

(i) *$\bigcup_\mu \Sigma_{\text{Os}}(\mu) \subset \Sigma_{\text{Ly}} \subset \Sigma_{\text{Mo}}$, where the union is taken over all $\pi\Psi$–invariant measures μ.*

(ii) *For every boundary point κ of the Morse spectrum (compare Theorem 3.2 (iii)) there exists an ergodic, $\pi\Psi$–invariant measure μ such that $\kappa \in \Sigma_{\mathrm{Os}}(\mu)$. In particular, κ is a regular Lyapunov exponent of the flow Ψ.*

Proof. Part (i) is obvious from Theorems 3.2 and 3.10. For part (ii) let $\mathcal{M}$ be a chain recurrent component of $\mathbb{P}\Psi$ with Morse interval $[\kappa^*(\mathcal{M}), \kappa(\mathcal{M})]$ according to Theorem 3.2 (iii). Then there exist by [18, Theorem 5.4.12] ergodic, $\mathbb{P}\Psi$–invariant measures μ^* and μ with support in $\mathcal{M}$ such that $\kappa^*(\mathcal{M}) = \lim_{t\to\pm\infty} \frac{1}{t} \log |\Psi(t, v))|$ for μ^*–almost all $\mathbb{P}v \in \mathbb{P}\mathcal{V}$, and $\kappa(\mathcal{M}) = \lim_{t\to\pm\infty} \frac{1}{t} \log |\Psi(t, v)|$ for μ–almost all $\mathbb{P}v \in \mathbb{P}\mathcal{V}$. Hence the marginals $\pi\mu^* = \mathbb{P}\pi(\mu^*)$ and $\pi\mu = \mathbb{P}\pi(\mu^*)$ on B have support in some chain recurrent component of $(B, \pi\Psi)$ and they are ergodic for this flow. Therefore $\kappa^*(\mathcal{M}) \in \Sigma_{\mathrm{Os}}(\pi\mu^*)$ and $\kappa(\mathcal{M}) \in \Sigma_{\mathrm{Os}}(\pi\mu)$. □

For general linear flows on vector bundles it is not clear, under which conditions the inclusions in Theorem 3.12(i) become equalities. For perturbation flows (5) we will show in the next section that equality holds almost always. The next result discusses the Oseledets decomposition of the bundle $\mathcal{V}$. It is based on an observation by Latushkin and Stepin [39].

Theorem 3.13 *Let Ψ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with chain recurrent base flow $\pi\Psi$. Let $\mathcal{V}^j$ be the subbundles of $\mathcal{V}$ corresponding to the finest Morse decomposition of the projective flow $\mathbb{P}\Psi$, compare Theorem 3.3. Let μ be an ergodic, $\pi\Psi$–invariant measure on B.*

(i) *For μ–almost all $b \in B$ and for every subspace L_b^i, $i \in \{1 \ldots r\}$ from Theorem 3.10 there exists a subspace $\mathcal{V}_b^j$ such that $L_b^i \subset \mathcal{V}_b^j$.*

(ii) *If every Morse spectral bundle $\mathcal{V}^j$ has nonvoid intersection with an Oseledets bundle L^i, then these bundles coincide for μ–almost all $b \in B$. In this case, the Oseledets bundles have a continuous extension to B. This holds, in particular, if the Morse bundles are one–dimensional.*

In general, the number of (measurable) Oseledets bundles may be strictly greater than the number of (continuous) bundles corresponding to the Morse spectrum, see [18, Example 5.5.18]. An ergodic, $\pi\Psi$–invariant measure μ 'picks' out of the spectral Morse intervals $r \leq \dim \mathcal{V}$ (deterministic) numbers λ_i which constitute, μ–almost surely, the Lyapunov spectrum of the flow Ψ. This fact allows, e.g., to base a stochastic bifurcation theory on 'the' largest Lyapunov exponent, while in topological spectral theory one has to deal with spectral intervals, compare the next section for further consequences.

To finish our discussion of spectral theory for linear flows on vector bundles, we remark that each spectral concept and its associated invariant subbundle decomposition comes with its invariant manifold theory. We refer the reader to Arnold [2] for the Oseledets case, [18, Section 5.6] for the Morse case, and to Aulbach/Wanner [8] for the case of exponential dichotomies.

4 Spectra of Perturbed Flows

So far we have considered general linear flows on vector bundles. We now show how to imbed the perturbed system (5) into this set–up, and draw some consequences for the Markov diffusion system (9). We start with the C^∞-system dynamics

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \text{ on } M$$

with $u \in \mathcal{U}$ as in (3). The linearization of (4) along trajectories yields a system on the tangent bundle TM described by

$$\frac{d}{dt} Tx(t) = TX_0(Tx(t)) + \sum_{i=1}^{m} u_i(t) TX_i(Tx(t)) \text{ on } M, \tag{19}$$

where $u \in \mathcal{U}$ as above and for a vector field X on M its linearization is denoted by $TX = (X, DX)$. If $M = \mathbb{R}^d$ this means in coordinates: For $X_j = \sum_{k=1}^d \alpha_{kj}(x) \frac{\partial}{\partial x_k}$, we denote the Jacobians of the coefficient functions by $A_j(x) = (\frac{\partial \alpha_{kj}(x)}{\partial x_\ell})$ and we set $\alpha_j(x) = (\alpha_{1j}(x), \ldots, \alpha_{dj}(x))^T$. Then $TX_j(x, v) = (\alpha_j(x), A_j(x)v)$, and the system is described by a pair of coupled differential equations, given as

$$\dot{x} = \alpha_0(x) + \sum_{i=1}^{m} u_i(t) \alpha_i(x),\ \dot{v} = A_0(x)v + \sum_{i=1}^{m} u_i(t) A_i(x) v.$$

This induces the linearized system flow

$$T\Phi : \mathbb{R} \times \mathcal{U} \times TM \to \mathcal{U} \times TM,\ T\Phi_t(u, x, v) = (\theta_t u, \varphi(t, x, u), D\varphi(t, x, u)v). \tag{20}$$

The flow (20) is a linear flow on the vector bundle $\pi : \mathcal{U} \times TM \to \mathcal{U} \times M$, where π denotes the usual projection. Hence the theory described above can be applied immediately with $\mathcal{V} = \mathcal{U} \times TM$, $B = \mathcal{U} \times M$, $\Psi = T\Phi$. For the results on the Morse spectrum, we can use the theory from [18] to describe the compact chain recurrent components $L \subset \mathcal{U} \times M$ in control theoretic terms. They are determined by their projection E to M which are the chain control sets, i.e., the maximal subsets E such that for all $x, y \in E$ and all $\varepsilon, T > 0$ there are $n \in \mathbb{N}$, times $t_0, ..., t_{n-1} \geq T$, points $x_0 = x, ...x_{n-1} = y$ and controls $u_0, ..., u_{n-1} \in \mathcal{U}$ with $\mathrm{d}(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon$ for all i. Similarly, the chain recurrent components $\mathcal{M} \subset \mathcal{U} \times \mathbb{P}M$ correspond to chain control sets ${}_{\mathbb{P}}E$ of the system

$$\frac{d}{dt} \mathbb{P}x(t) = \mathbb{P}X_0(\mathbb{P}x(t)) + \sum_{i=1}^{m} u_i(t) \mathbb{P}X_i(\mathbb{P}x(t)) =: \mathbb{P}X(\mathbb{P}x, u) \text{ on } \mathbb{P}M, \tag{21}$$

where $\mathbb{P}M$ is the projective bundle over M, $\mathbb{P}x \in \mathbb{P}M$, and the vector fields $\mathbb{P}X_j$ are the projections of TX_j onto $\mathbb{P}M$. Thus we can talk about the spectral intervals $\Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E)$ of chain control sets of (21) and about the Morse spectrum $\Sigma_{\mathrm{Mo}}(E) = \bigcup_{i=1}^{\ell} \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i)$ of the chain control sets E of (4). The decomposition into invariant subbundles $\mathcal{V}_i$, $i = 1, ..., \ell$, from Theorem 3.3 are nothing but the lifts of the ${}_{\mathbb{P}}E_i$ to $\mathcal{U} \times \mathbb{P}M$, extended to $\mathcal{U} \times TM$. To make the connection with the corresponding control sets, we formulate the following conditions for the projection (21) of the linearized system (21):

$$\dim \mathcal{LA}\{\mathbb{P}X_0 + \Sigma u_i \mathbb{P}X_i, u \in U\}(x, v) = 2 \cdot \dim M - 1 \text{ for all } (x, v) \in \mathbb{P}M. \tag{22}$$

Consider a family of control ranges depending on a parameter ρ and given by

$$U^{\rho} := \rho \cdot U,\ \rho \geq 0,$$

and denote the corresponding set of control functions by $\mathcal{U}^{\rho}$. Furthermore, let $\mathbb{P}\varphi$ denote the solution of (21) and $\mathcal{O}^{+,\rho'}(p)$ the positive orbit of $p \in \mathbb{P}M$ under the control range $U^{\rho'}$. The inner-pair condition is:

$$\begin{array}{c}\text{For all } \rho, \rho' \in [0, \rho^*) \text{ with } \rho < \rho' \text{ and all chain control sets } {}_{\mathbb{P}}E_i^{\rho} \\ \text{of (21) every } (u,p) \in {}_{\mathbb{P}}\mathcal{E}_i^{\rho} \subset \mathcal{U}^{\rho} \times \mathbb{P}M \text{ satisfies: There exists} \\ T = T(u,p) > 0 \text{ such that } \mathbb{P}\varphi(T,p,u) \in \operatorname{int} \mathcal{O}^{+,\rho'}(p).\end{array} \tag{23}$$

Under this condition the chain control sets of (4) and of (21) are the closures of control sets with nonvoid interiors for almost all $\rho > 0$. For these ρ-values the statements above hold for the corresponding control sets. We will show in the next section that under these conditions also the Morse and the Lyapunov spectrum of (20) have a particularly simple form.

For stochastic perturbation systems the situation is somewhat more complicated. Consider the system dynamics (4) with perturbation model (3) such that P is a θ–invariant probability measure on $\mathcal{U}$. The natural spectral concept in this situation is the Oseledets spectrum, compare Definition 3.11, for which we need an invariant measure of the base flow, i.e. of the system flow (5). Note that, in general, this flow will have a multitude of invariant measures.

In this set–up, the Oseledets construction yields a spectrum for each Φ–invariant measure μ (with marginal P on $\mathcal{U}$).

Proposition 4.1 *Consider the system flow (5) and let μ be an ergodic, Φ–invariant probability measure. Then there exists a chain control set E of (4) such that $\Sigma_{\mathrm{Os}}(\mu) \subset \Sigma_{Mo}(E)$.*

The proof of this result follows from the definition of $\Sigma_{Mo}(E)$ above, since the support of an ergodic measure is chain recurrent. For nondegenerate perturbation processes we obtain a stronger conclusion, which holds, in particular, for the Markov diffusion model (9). Let $\{\eta(t), t \in \mathbb{R}\}$ be a stationary stochastic process with trajectory space $\mathcal{U}$ and θ–invariant measure P. Denote by supp $P_{\eta(0)}$ the support of the distribution of $\eta(0)$ in U. We use the following assumption (compare Arnold/Kliemann [4, p. 16])

$$\begin{array}{c}\text{There exists } y_0 \in \operatorname{supp} P_{\eta(0)} \text{ such that for all } \delta > 0 \text{ and all continuous} \\ u : [0,T] \to U \text{ with } u(0) = y_0 \text{ we have } P\{\max_{0 \leq t \leq T} |\eta(t) - u(t)| < \delta\} > 0.\end{array} \tag{24}$$

Proposition 4.2 *Consider the system flow (5) under the Lie algebra rank condition*

$$\dim \mathcal{LA}\{X_0 + \Sigma u_i X_i,\ u \in U\}(x) = \dim M \textit{ for all } x \in M. \tag{25}$$

and assume that the chain control sets of (4) are closures of some control sets. Let μ be an ergodic, Φ–invariant measure with marginal P on $\mathcal{U}$ satisfying (24). Then there exists an invariant control set C such that $\Sigma_{\mathrm{Os}}(\mu) \subset \Sigma_{Mo}(C)$.

P r o o f. Every control set is contained in a unique chain control set. These sets are closed, and hence the assumption of this proposition implies that all invariant control sets of (4) are isolated. Then $\operatorname{supp}\mu \subset \mathcal{C}$, where $\mathcal{C}$ is the lift of some (closed) invariant control set. Hence $\Sigma_{\mathrm{Os}}(\mu) \subset \Sigma_{Mo}(C)$. □

In the Markov diffusion case one can also bystep Oseledets' theorem and consider the Lyapunov exponents of the solutions under the family of Markov measures. More precisely, consider the Markov diffusion model (9) with linearization (19). For the background noise η we assume the Lie algebra rank condition (7) and that the function $f : N \to U$ satisfies the conditions of [17, Lemma 3.17]. Instead of the rank condition (22) we need an assumption for the pair process $(\eta, \mathbb{P}x)$:

$$\dim \mathcal{LA} \left\{ \begin{pmatrix} Y_0 + \sum w_j Y_j \\ \mathbb{P}X \end{pmatrix}, w \in \mathbb{R}^\ell \right\} \begin{pmatrix} \eta \\ \mathbb{P}x \end{pmatrix} = \dim N + 2 \cdot \dim M - 1 \qquad (26)$$

for all $\binom{\eta}{\mathbb{P}x} \in N \times \mathbb{P}M$.

Proposition 4.3 *Consider the Markov diffusion model (9) under the Lie algebra rank conditions (7) and (26). Let $C_1, ..., C_k$ be the invariant control sets of (4) in M.*
(i) Over each C_i there are at most finitely many invariant control sets ${}_{\mathbb{P}}C_{ij}$, $j = 1, ..., \ell(i)$ of the system (22) in $\mathbb{P}M$. Each ${}_{\mathbb{P}}C_{ij}$ is contained in a unique chain control set ${}_{\mathbb{P}}E_{ij}$ of (22).
(ii) Denote by $P_{(x,v)}$ the measure induced by the Markov diffusion process $(\eta_t^*, Tx(t))$ with stationary η–component and fixed initial value $(x, v) \in TM$ in $C(\mathbb{R}^+, N \times TM) = \Omega$. For $\omega \in \Omega$ and $(x, v) \in TM$, $v \neq 0$ define the Lyapunov exponent as above by

$$\lambda(x, v, \omega) = \limsup_{t\to\infty} \frac{1}{t} \log |D\varphi(t, x, \omega)v|.$$

Then $\lambda(x, v, \omega) \in \bigcup_{i,j} \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_{ij})$ for each $(x, v) \in TM$, $v \neq 0$, $P_{(x,v)}$–a.s.
(iii) If the control system (4) is real analytic, then ${}_{\mathbb{P}}C_{ij}$ is unique over C_i.

P r o o f. Part (i) is Proposition 6.2.8 and Part (iii) is [18, Theorem 6.2.10 in]. Furthermore, the process $\mathbb{P}\varphi(t, (x, v), \omega)$ on $\mathbb{P}M$ enters $\bigcup_{i,j} {}_{\mathbb{P}}C_{ij}$ in finite time $P_{(x,v)}$–almost surely and each of these sets is invariant for the process. The result now follows from Theorem 3.2 (i). □

In fact, more can be said in the real analytic case, where over each invariant control set C_i in M there is a unique invariant control set ${}_{\mathbb{P}}C_i$ in $\mathbb{P}M$ for $i = 1, \ldots, k$. If for $x \in M$ there is a unique $i \in \{1, ..., k\}$ with $x \in \mathbf{A}(C_i)$ then $\lambda(x, v, \omega) = \lambda^i \in \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i)$ for all $v \in T_xM$, $v \neq 0$, $P_{(x,v)}$–a.s., where ${}_{\mathbb{P}}E_i$ is the unique chain control set in $\mathbb{P}M$ containing ${}_{\mathbb{P}}C_i$. This is proved via a control theoretic argument in complete analogy to Arnold/Kliemann/Oeljeklaus [6]. It turns out that λ^i is the top Lyapunov exponent of the Oseledets spectrum over C_i, corresponding to the Φ–invariant measure over $N \times C_i$, which is constructed from the invariant Markov measure, see Crauel [21]. This does not mean that the Oseledets spectrum, with its associated ω–dependent Oseledets spaces, consists only of this value.

So far, we have considered the spectrum for regular perturbed systems, i.e. systems that satisfy the Lie algebra rank condition (21). We now turn to a discussion for singular systems with common fixed point $x^* \in M$, i.e.

$$X_0(x^*) = \ldots = X_m(x^*) = 0. \qquad (27)$$

In this case the linearization (19) at x^* yields a linear system

$$\dot{v} = A_0 v + \sum_{i=1}^{m} u_i(t) A_i v = A(u)v, \text{ where } A_j = (D_x X_j)(x^*) \qquad (28)$$

and the corresponding system flow is of the form

$$T\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d,\ T\Phi_t(u, v) = (\theta_t u, \psi(t, v, u)) \tag{29}$$

where $\psi(\cdot, v, u)$ are the solution of (28) for $u \in \mathcal{U}$, $u \in \mathbb{R}^d$. All the results described above remain true for the (trivial) linear flow (29) by disregarding all the assumptions about the M–component and using $\mathcal{U} \times \{x^*\}$ as the base space. In fact, there is a vast literature on the spectral theory of the system (28), (29), compare, e.g. [18, Chapter 7] for the topological theory and Arnold [2] for the stochastic theory, together with the literature mentioned in these books.

We would like to mention only one additional result that relates the top Morse exponent to moment Lyapunov exponents of Markov diffusion systems. Consider the Markov diffusion system (9) with singular point $x^* \in M$ under the nondegeneracy condition (7), and assume the Lie algebra rank condition (26) for the projected system of (28) on $\mathbb{P}^{d-1}$, the projective space in $\mathbb{R}^d$. Denote by P_v the measure induced by the Markov diffusion process $(\eta_t^*, v(t))$ as in Proposition 4.3(ii). The moment Lyapunov exponents are then defined as

$$g(v, p) = \limsup_{t\to\infty} \frac{1}{t} \log(\mathbf{E}_v|\psi(t, v, \omega)|^p) \text{ for } p \geq 0. \tag{30}$$

Under the above assumptions it turns out that $g(v, p) = g(p)$ for all $v \neq 0$, compare Arnold/Kliemann/Oeljeklaus [6]. We obtain the following result.

Proposition 4.4 *Consider the Markov diffusion system (9) with singular point $x^* \in M$ as in (27). Under the assumptions above the linearized system (28) satisfies $\lim_{p\to\infty} \frac{g(p)}{p} = \kappa$, where κ is the largest Morse exponent of (28).*

Note that this result holds for all background noises satisfying (7) if the function $f : N \to U$ fulfills the conditions of [17, Lemma 3.17]. We refer to Arnold/Kliemann [5] and Arnold/Kliemann/Oeljeklaus [6] for a proof and further details on the Markov diffusion case.

Remark 4.5 *(On numerical methods).* The results presented in this section require the numerical computation of the intervals of the Morse spectrum, and of the Oseledets spectrum in the stochastic case. An optimal control approach for the Morse spectrum of low dimensional systems was developed by Grüne [26]. For the numerical computation of the Oseledets spectrum we refer to Dieci/van Vleck [40] and Beyn/Lust [11] and, in the stochastic case, to algorithms developed by Talay [51], [52].

5 The Spectrum of Parameter Dependent Perturbed Systems

This section serves mainly two purposes: On the one hand we clarify the relation between the different spectral concepts for perturbed systems under the inner pair condition (23). This will be done for systems as introduced in Section 4, but now depending on a parameter. On the other hand, we discuss the behavior of a system as its Lyapunov exponents pass through zero. For this analysis we will concentrate on systems with a singular fixed point, since this is the set–up of most stochastic bifurcation studies.

Our starting point is a family of perturbed systems, parametrized by $(\alpha, \rho) \in I \times [0, \infty)$,

$$\dot{x} = X_0(x, \alpha) + \sum_{i-1}^{m} u_i(t) X_i(x, \alpha),\ u \in \mathcal{U}^\rho,\ \alpha \in I \subset \mathbb{R}^p,\ \rho \geq 0 \tag{31}$$

with associated system flow

$$\Phi^{\alpha,\rho}:\mathbb{R}\times\mathcal{U}^{\rho}\times M\to\mathcal{U}^{\rho}\times M,\ \Phi_t(u,x)=(\theta_t u,\varphi(t,x,u)). \tag{32}$$

For each $(\alpha,\rho)\in I\times[0,\infty)$ the system (31) has a linearization as in (19), with corresponding flow $T\Phi^{\alpha,\rho}$ as in (20). The flow of the projected system on $\mathbb{P}M$, compare (21), will be denoted by $\mathbb{P}\Phi^{\alpha,\rho}$. Our first result is a roughness statement for the Morse spectrum, which holds for general linear flows on vector bundles, compare [18, Theorem 5.3.9].

Theorem 5.1 *Consider the system flow (32)$^{\alpha,\rho}$ such that the vector fields in (31)$^{\alpha,\rho}$ depend continuously on α. For some $(\alpha^0,\rho^0)\in I\times[0,\infty)$ let $\mathcal{M}^0\subset\mathbb{P}(\mathcal{U}^\rho\times TM)$ be a chain recurrent component of $\mathbb{P}\Phi^{\alpha^0,\rho^0}$. Assume that for every neighborhood $\mathcal{W}$ of $\mathcal{M}^0$ there exists a neighborhood V of (α^0,ρ^0) such that for all $(\alpha,\rho)\in V$ there is a compact, $\mathbb{P}\Phi^{\alpha,\rho}$–invariant set $\mathcal{M}^{\alpha,\rho}\subset\mathcal{W}$. Then*

$$\limsup_{(\alpha,\rho)\to(\alpha^0,\rho^0)}\Sigma_{\mathrm{Mo}}(\mathcal{M}^{\alpha,\rho},\Phi^{\alpha,\rho})\subset\Sigma_{\mathrm{Mo}}(\mathcal{M}^0,\Phi^{\alpha^0,\rho^0}).$$

In particular, if $(\alpha^k,\rho^k)\to(\alpha^0,\rho^0)$ and if the system flow Φ^{α^0,ρ^0} has a finest Morse decomposition on $\mathcal{U}^{\rho_0}\times M$, then

$$\{\lambda\in\mathbb{R},\ there\ are\ \lambda^k\in\Sigma_{\mathrm{Mo}}(\Phi^{\alpha^k,\rho^k})\ \textit{with}\ \lambda^k\to\lambda\}\subset\Sigma_{\mathrm{Mo}}(\Phi^{\alpha^0,\rho^0}).$$

In other words, the Morse spectrum depends upper semicontinuously on parameters. Compare e.g. Coppel [20] for a similar statement concerning the dichotomy spectrum.

For the following discussion we fix the parameter α and drop it from our notation. We now use explicitly the control structure of the projected perturbed system (21). We first consider the regular case, i.e. we assume the Lie algebra rank condition (22) for all $\rho>0$, as well as the inner pair condition (23). The control structure of the flow $\mathbb{P}\Phi^\rho$ allows us to introduce the Floquet spectrum as a part of the regular Lyapunov spectrum, compare (14) and (15).

Definition 5.2 Let $_{\mathbb{P}}D$ be a control set with nonvoid interior of the projected system (21) in $\mathbb{P}M$. The Floquet spectrum over $_{\mathbb{P}}D$ is defined as

$$\Sigma_{Fl}({}_{\mathbb{P}}D)=\left\{\begin{array}{c}\lambda(u,x),\,(u,\mathbb{P}x)\in\mathcal{U}\times int_{\mathbb{P}}D,\,u\text{ is piecewise constant}\\ \text{and }\tau-\text{periodic for some }\tau\geq0\text{ such that }\mathbb{P}\varphi(\tau,\mathbb{P}x,u)=\mathbb{P}x\end{array}\right\},$$

and the Floquet spectrum over a control set $D\subset M$ of (4) is

$$\Sigma_{Fl}(D)=\cup\{\Sigma_{Fl}({}_{\mathbb{P}}D),\,{}_{\mathbb{P}}D\text{ as above with }\mathbb{P}\pi({}_{\mathbb{P}}D)\subset D\}.$$

In order to obtain sharp results, we also define the Lyapunov spectrum over control sets in $\mathbb{P}M$ as

$$\Sigma_{Ly}({}_{\mathbb{P}}D)=\{\lambda(u,x),(u,x)\in\mathcal{U}\times TM\text{ with }\varphi(t,\mathbb{P}x,u)\in\mathrm{cl}_{\mathbb{P}}D\text{ for all }t\geq0\}.$$

and over a control set $D\subset M$ as

$$\Sigma_{Ly}(D)=\{\lambda(u,x),(u,x)\in(\mathcal{U}\times TM)\backslash Z,\varphi(t,\pi x,u)\in\mathrm{cl}D\text{ for all }t\geq0\}.$$

We first state a result on the control sets of the projective system (21), see [18, Theorem 6.1.3].

Theorem 5.3 *Fix $0 < \rho^* \leq \infty$ and consider for $\rho \in (0, \rho^*)$ the projective systems (21)$^\rho$ under the Lie algebra rank condition (22) for all $\rho > 0$ and the inner pair condition. Let E^0 be a chain recurrent component of the uncontrolled system $\dot{x} = X_0(x)$ on M, and let E^ρ be the unique chain control set of (31)$^\rho$ with $E^0 \subset E^\rho$. Then the following assertions hold.*

(i) *For all $\rho \in [0, \rho^*)$ and for every chain recurrent component ${}_{\mathbb{P}}E_i^0$, $1 \leq i \leq \ell(0) \leq d$ of the equation $\frac{d}{dt}\mathbb{P}x(t) = \mathbb{P}X_0(\mathbb{P}x(t))$ on $\mathbb{P}M$ there are chain control sets ${}_{\mathbb{P}}E_i^\rho$ with ${}_{\mathbb{P}}E_i^0 \subset {}_{\mathbb{P}}E_i^\rho$ and $\mathbb{P}\pi({}_{\mathbb{P}}E_i^\rho) = E^\rho$. There are no further chain control sets with $\mathbb{P}\pi({}_{\mathbb{P}}E^\rho) \cap E^\rho \neq \emptyset$. The number $\ell(\rho)$ of chain control sets is decreasing in ρ and satisfies $1 \leq \ell(\rho) \leq d$.*

(ii) *For all $\rho \in (0, \rho^*]$ there are unique control sets D^ρ of (31)$^\rho$ with $E^0 \subset \mathrm{int} D^\rho$, and control sets ${}_{\mathbb{P}}D_i^\rho$ of (21) with ${}_{\mathbb{P}}E_i^0 \subset \mathrm{int}_{\mathbb{P}} D_i^\rho$ and $\mathbb{P}\pi({}_{\mathbb{P}}D_i^\rho) = D^\rho$. For all but at most countably many ρ–values and for all $i = 1 \ldots \ell(\rho)$ we have $\mathrm{cl} D^\rho = E^\rho$ and $\mathrm{cl}_{\mathbb{P}} D_i^\rho = {}_{\mathbb{P}}E_i^\rho$.*

The theorem describes the chain control sets, over which the Morse spectrum of the system is defined (compare the remarks after (21)), and the control sets, over which the Floquet spectrum is defined. In particular, Theorem 3.2 yields the following result.

Corollary 5.4 *Under the assumptions of Theorem 5.3, for all $\rho \in [0, \rho^*)$ and for each $i = 1, ..., \ell(\rho)$ the Morse spectrum has the form*

$$\Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i^\rho) = [\kappa^*({}_{\mathbb{P}}E_i^\rho), \kappa({}_{\mathbb{P}}E_i^\rho)]$$

with $\kappa^*({}_{\mathbb{P}}E_i^\rho) < \kappa^*({}_{\mathbb{P}}E_j^\rho)$ and $\kappa({}_{\mathbb{P}}E_i^\rho) < \kappa({}_{\mathbb{P}}E_j^\rho)$ iff ${}_{\mathbb{P}}E_i^\rho \preceq {}_{\mathbb{P}}E_j^\rho$.

This allows us to define, for each chain recurrent component E^0 of the uncontrolled system on M, set-valued spectral maps for $i = 1, ..., \ell(0)$

$$\rho \mapsto \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i^\rho) \text{ and } \rho \mapsto cl\Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho), \tag{33}$$

which are both increasing in ρ. The main result on the ρ–dependence of the spectra of the perturbed system (33)$^\rho$ is that the spectra agree 'almost always', compare [18, Theorem 6.1.3].

Theorem 5.5 *Under the assumptions of Theorem 5.3 consider the spectral maps (33). For each $i = 1, ..., \ell(0)$ the first map is right continuous and the second one is left continuous. The sets of continuity points in $(0, \rho^*)$ of the two maps agree, and 0 is a continuity point. There are at most countably many points of discontinuity and at each continuity point ρ we have $cl\Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho) = \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i^\rho)$. In particular, if ρ is a continuity point for all $i = 1, ..., \ell(\rho)$, then*

$$cl\Sigma_{Fl}(D^\rho) = \bigcup_{i=1}^{\ell(\rho)} cl\Sigma_{Fl}({}_{\mathbb{P}}D_i^\rho) = \Sigma_{Ly}(D^\rho) = \bigcup_{i=1}^{\ell(\rho)} \Sigma_{\mathrm{Mo}}({}_{\mathbb{P}}E_i^\rho) = \Sigma_{\mathrm{Mo}}(E^\rho).$$

The maps $\rho \mapsto \kappa^*({}_{\mathbb{P}}E_1^\rho)$ and $\rho \mapsto \kappa({}_{\mathbb{P}}E_{\ell(0)}^\rho)$ are continuous. Furthermore, taking the union over all chain control sets E^ρ and over all control sets D^ρ of (31)$^\rho$, respectively, one obtains at common continuity points

$$\mathrm{cl}\Sigma_{Fl}^\rho = \bigcup \mathrm{cl}\Sigma_{Fl}(D^\rho) = \Sigma_{Ly}^\rho = \bigcup \Sigma_{Ly}(D^\rho) = \bigcup \Sigma_{\mathrm{Mo}}(E^\rho) = \Sigma_{\mathrm{Mo}}^\rho.$$

The Lyapunov spectrum is continuous at the (common) continuity points of the Floquet and the Morse spectrum. At the discontinuity points the Lyapunov spectrum need not be left nor right continuous.

We now discuss the Oseledets spectrum of a stochastic perturbation model, i.e. we consider for the system flow $(32)^\rho$ the sets $\mathcal{P}(\Phi^\rho)$ and $\mathcal{P}^\rho(\Phi^\rho)$ of Φ^ρ–invariant (and ergodic) probability measures. In general, we only know that $\Sigma^\rho_{Fl} \subset \bigcup\{\Sigma_{\mathrm{Os}}(\mu), \mu \in \mathcal{P}(\Phi^\rho)\} \subset \Sigma^\rho_{Ly}$, compare Theorem 3.12. At the continuity points of the spectral maps (33) one can say more:

Corollary 5.6 *Under the assumptions of Theorem 5.3 let ρ be a continuity point of the maps (33) for chain recurrent component E^0 of the uncontrolled system $\dot{x} = X_0(x)$ on M. Then the following equalities hold:*

$$\mathrm{cl}\Sigma_{Fl}(D^\rho) = \mathrm{cl}\bigcup \Sigma_{\mathrm{Os}}(\mu) = \Sigma_{Ly}(D^\rho) = \Sigma_{\mathrm{Mo}}(E^\rho),$$

where the union is taken over all $\mu \in \mathcal{P}^\rho(\Phi^\rho)$ with $\mathrm{supp}\mu \subset \mathcal{E}^\rho$, the lift of the chain control set $E^\rho \subset M$ to $\mathcal{U} \times M$.

This result follows from Theorem 3.12 and the fact that periodic trajectories in $\mathcal{U}^\rho \times \mathbb{P}M$ project onto ergodic Φ^ρ–invariant measures in $\mathcal{U} \times M$. Since all such measures have support in some chain recurrent component of Φ^ρ, the equality in Corollary 5.6 also holds for the entire spectra of the system flow Φ^ρ. If a specific θ–invariant measure P is given on $\mathcal{U}$, then these equalities need not hold. Their validity depends on the nondegeneracy of P.

We now return to the singular situation, i.e. to perturbed systems (31) with common fixed point $x^* \in M$, as described in (27) – (29). All the results above remain true for the linearization at x^* by considering $\mathcal{U}^\rho \times \{x^*\}$ as the space of the linearized flow, compare (29). For this case, which has played a major role in stochastic bifurcation theory, we will formulate an invariant manifold theorem that serves for the local stability analysis of the system (31) around x^*. Similar theorems exist for general linear flows on vector bundles, compare Bronstein/Chernii [12] or [18, Section 5.6].

Consider the projective flow $\mathbb{P}\Phi_t : \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U} \times \mathbb{P}^{d-1}$ of the linearized system (29). Let $E_1, \ldots, E_\ell$ be the chain control sets on $\mathbb{P}^{d-1}$ with lifts $\mathcal{E}_1, \ldots, \mathcal{E}_\ell$ to $\mathcal{U} \times \mathbb{P}^{d-1}$. These lifts are the chain recurrent components of $\mathbb{P}\Phi$. We define for each $u \in \mathcal{U}$

$$\mathcal{V}_j(u) = \{x \in \mathbb{R}^d,\, x \neq 0 \text{ implies } \mathbb{P}(u, x) \in \mathcal{E}_j\},$$

compare Theorem 3.3(i). Then $\mathcal{U} \times \mathbb{R}^d = \bigoplus_{i=1}^{\ell} \mathcal{V}_j$ and for each $u \in \mathcal{U}$ the space $\mathcal{V}_j(u)$ is a (time varying) linear subspace of $\mathbb{R}^d$, which is invariant under the flow $T\Phi$. To each $\mathcal{V}_j$ is associated one interval $\Sigma_{\mathrm{Mo}}(E_j)$ of the Morse spectrum of (29). Consider a decomposition $\mathcal{U} \times \mathbb{R}^d = \bigoplus_{j=1}^{n} \mathcal{V}_j \oplus \bigoplus_{j=n+1}^{\ell} \mathcal{V}_j$. If $\max \Sigma_{\mathrm{Mo}}(\mathcal{V}_n) < 0$, then $\mathcal{V}_n^+ = \bigoplus_{j=1}^{n} \mathcal{V}_j$ is a stable subbundle, and if $\min \Sigma_{\mathrm{Mo}}(\mathcal{V}_{n+1}) > 0$, then $\mathcal{V}_n^- = \bigoplus_{n=n+1}^{\ell} \mathcal{V}_j$ is unstable.

Theorem 5.7 *Consider the perturbed system (31) with singular point x^*. Let $\mathcal{U} \times \mathbb{R}^d = \mathcal{V}^+ \oplus \mathcal{V}^-$ be a decomposition as above with $\kappa^+ = \max \Sigma_{\mathrm{Mo}}(\mathcal{V}^+) < 0$. Then there are $\delta > 0$ and a map $S^+ : \{(u, x) \in \mathcal{V}^+, |x| < \delta\} \to \mathcal{U} \times \mathbb{R}^d$ of the form $S^+(u, x) = (u, s^+(u, x))$ with the following properties:*

(i) *For every $\alpha > \kappa^+$ and every $(u, y) \in \mathcal{W}^+ = \mathrm{im}S^+$ on has $\lim_{t\to\infty} e^{-\alpha t}(\varphi(t, y, u) - x^*) = 0$. We call $\mathcal{W}^+$ a local stable manifold corresponding to the subbundle $\mathcal{V}^+$.*

(ii) *The map S^+ is a bundle isomorphism onto its image $\mathcal{W}^+$; in particular for every $u \in \mathcal{U}$ the fibers $\mathcal{W}_u^+ = \{y \in \mathbb{R}^d,\ (u,y) \in \mathcal{W}^+\}$ are topological manifolds and their dimension equals the dimension of $\mathcal{V}^+$*

(iii) *The local stable manifold $\mathcal{W}^+$ is positively invariant under the system flow Φ, i.e. for $(u,y) \in \mathcal{W}^+$ one has $(\theta_t u, \varphi(t,y,u)) \in \mathcal{W}^+$ for all $t \geq 0$.*

(iv) *The distance between $\mathcal{W}^+$ and $\mathcal{V}^+$ can be made arbitrarily small in the Lipschitz sense, i.e. for all $h > 0$ there is $\delta > 0$ such that $\mathcal{W}^+$ is contained in the cone $K(\mathcal{V}^+, h)$ of angle h around $\mathcal{V}^+$ given by $K(\mathcal{V}^+, h) = \{(u, x^+ + x^-),\ (u,x^+) \in \mathcal{V}^+ \ and\ (u,x^-) \in \mathcal{V}^-$ with $|x^-| \leq h|x^+|\}$.*

Unstable manifolds are obtained in complete analogy if $\kappa^- = \min \Sigma_{\mathrm{Mo}}(\mathcal{V}^-) > 0$. Note that the δ in Theorem 5.7 does not depend on u, and hence Theorem 5.5 yields the following consequence.

Corollary 5.8 *Assume that the Lie algebra rank condition (22) holds for the projected system on $\mathbb{P}^{d-1}$. If $\kappa = \sup \Sigma_{Fl}(x^*) < 0$, then there exists $\sigma > 0$ such that for $y \in M$ with $d(y, x^*) < \sigma$ and all $u \in \mathcal{U}$ it follows that $y \in \mathcal{W}_u^+$.*

This result implies, in particular, that in the stable case there are no Φ–invariant measures in the σ–neighborhood of x^* besides the Dirac measure at x^*. Now the obvious question arises about the structure of the system flow Φ around x^*, if zero lies in the interior of the maximal spectral interval. One cannot expect a uniform behavior as in Theorem 5.7, because for any $\varepsilon > 0$ there exist (u_i, x_i), $i = 1, 2$, with $u_i \in \mathcal{U}$ and x_i in the ε–neighborhood $B(x^*, \varepsilon)$ of x^* such that $\varphi(t_1, x_1, u_1) \to x^*$ for $t \to \infty$ and $\varphi(t, x_2, u_2) \to x^*$ for $t \to -\infty$. We expect that in this case there exists a multitude of Φ–invariant measures μ with $\pi_M \mathrm{supp}\mu \cap B(x^*, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$, indicating some kind of bifurcation phenomenon as κ passes through zero. This should be reflected in the bifurcation of homoclinic orbits (for some $u \in \mathcal{U}$) from x^*.

We conclude this section with some comments on the two parameter systems (31)$^{\alpha,\rho}$ with $\alpha \in I \subset \mathbb{R}^p$, $\rho \geq 0$ and on ideas of bifurcation theory in the context of the spectral theory developed so far. The unperturbed version of (31)$^{\alpha,\rho}$ is given by

$$\dot{x} = X_0(x, \alpha) \text{ in } M,\ \alpha \in I \subset \mathbb{R}^p. \tag{34}$$

Let us assume that (34) has a continuous family $\{\mathcal{M}_\alpha, \alpha \in I\}$ of Morse sets, e.g., of fixed points. Generically, the perturbation vector fields will turn the systems (34) into regular systems, i.e. the set $(X_1, X_2) \in \mathfrak{X}(M)^2$ such that the Lie algebra rank condition for accessibility holds for $\dot{x} = X_0(x) + \sum_{i=1}^2 u_i(t) X_i(x)$, is open and dense in $\mathfrak{X}(M)^2$, where $\mathfrak{X}(M)$ is the space for all C^∞ vector fields on M, see Sussmann [49]. (If dim $M = 1$, then one perturbation vector field X_1 suffices.) Hence the family $\{\mathcal{M}_\alpha, \alpha \in I\}$ will be imbedded into a family of control sets of (31) under the inner pair condition. This is, in particular, the case for additive perturbations, i.e., the vector fields $X_1, \ldots, X_m$ are constant. In the regular situation, a (bifurcation) study of the systems (31) can be based on the concepts from Section 3. and the spectra over the control sets in M, see e.g. Grünvogel [28], Colonius/Kliemann [19], and Colonius et al. [13] for some results in this direction. At this moment, it is not clear to us how the spectral point of view is related to the global (bifurcation) picture of (31), compare the comments after Corollary 5.8 above.

In the stochastic white noise case, Crauel/Flandoli [22] observe the destruction of a pitchfork bifurcation (in the unperturbed system) through additive noise, which makes the system regular. Similarly, Arnold et al. [3] observe the same effect for a Hopf bifurcation subject to parametric noise that results in a regular system. Stepping outside of the context of perturbation flows, one can use properties of Markov invariant measures to formulate bifurcation concepts for regular systems. The paper Liang/Namachchivaya [41] is one example for such an approach.

For the singular case one has to require that for all $\alpha \in I$ the family $\{\mathcal{M}_\alpha, \alpha \in I\}$ is singular, e.g., in the case of a fixed point that (27) holds for all α. Then the local stability behavior of $\Phi^{\alpha,\rho}$ around $\{\mathcal{M}_\alpha, \alpha \in I\}$ is described by the spectra over the $\mathcal{M}_\alpha$ and one can expect that bifurcation scenarios are reflected in these spectra. That this is, in fact, true in the stochastic perturbation context was observed in many papers, e.g. [9], [10], [7], and [23], compare also [32] for an approach based on smooth dynamics and ergodic theory. If the perturbation model is a mixing flow as in (3) or a nonergodic process, one has to consider the entire spectral intervals and not just the top exponent from the ergodic Oseledets spectrum. At this moment it is not clear how the behavior of the system flow $\Phi^{\alpha,\rho}$ changes, locally around a singular set, as its top spectral interval passes through zero.

6 Robust Stability and Feedback Stabilizability

For (deterministic) perturbation systems the spectrum consists of intervals. Hence it is of particular interest to study the change of the system behavior as the upper or the lower interval boundaries pass through zero. For the top spectral interval this is related to the concepts of robust stability and of feedback stabilizability, respectively. We will briefly explain the ideas.

Consider the perturbation system (31) with singular fixed point x^* as in (27).

Definition 6.1 The stability radius of the linearized system (28) is defined as

$$r = \inf\{\rho \geq 0, \text{there exists } u \in \mathcal{U}^\rho \text{ such that } \dot{v} = A(u)v \text{ is not exponentially stable}\},$$

and the stability radius of the underlying nonlinear system satisfying (26) at x^* is

$$r_{n\ell} = \inf\{\rho \geq 0, \text{there exists } u \in \mathcal{U}^\rho \text{ such that } x^* \text{ is not loc. asympt. stable for (31)}\}.$$

The maximal Lyapunov exponent of the system (28) is denoted by $\kappa(\rho) = \max \Sigma^\rho_{Ly}$.

Theorem 6.2 *For the perturbation system (31)$^\rho$ with singular fixed point $x^* \in M$ the inequalities*

$$\sup\{\rho \geq 0, \kappa(\rho) < 0\} \leq r_{n\ell} \leq r = \inf\{\rho \geq 0, \kappa(\rho) > 0\}$$

hold. In particular, if the increasing, continuous function $\kappa(\rho)$ is strictly increasing at $\rho = r$, then $r_{n\ell} = r$.

For the Markov diffusion model (9) we obtain a relation to the stability radius by considering its moment Lyapunov exponents. Let $f^\rho : N \to U^\rho$ be a family of surjective maps that satisfy the assumptions of [17, Lemma 3.17]..

Corollary 6.3 *Consider the linearization $\dot{v} = A(f^\rho(\eta^*_t))v$ of the Markov diffusion model (9) at the singular point x^* and its moment Lyapunov exponents $g^\rho(p)$, $p \geq 0$ as in (30). Then $g^\rho(p) < 0$ for all $p \geq 0$ iff $\rho < r$.*

This result follows from Theorem 6.2 with Proposition 4.4.

Let $[\tilde{\kappa}(\rho), \kappa(\rho)]$ be the top Morse spectral interval of the linearized system (28). Then Theorem 6.2 and Corollary 6.3 characterize the passing through zero of $\kappa(\rho)$ in terms of robust stability. It turns out that $\tilde{\kappa}(\rho)$ passes through zero as (28) and the nonlinear system $(31)^\rho$ change from feedback stabilizability (and asymptotic null controllability) to stability. Since these concepts are not related to perturbation systems, but rather to fundamental control theoretic questions, we refer the reader to Grüne [26], [27], Wang [53] and Colonius/Kliemann [18, Chapter 12] for the technical details.

References

[1] Arnold, L. *Stochastic Differential Equations: Theory and Applications*. Wiley, 1974. Original German language edition 1973 by Oldenbourg Verlag, München; English edition reprinted by Krieger, Malaba (Florida), 1992.

[2] Arnold, L. *Random Dynamical Systems*. Springer-Verlag, 1998.

[3] Arnold, L., et al. "The Stochastic Brusselator: Parametric Noise Destroys Hopf Bifurcation." *Stochastic Dynamics* edited by H. Crauel and V. M. Gundlach, Springer-Verlag, 1998.

[4] Arnold, L. and W. Kliemann. "Qualitative theory of linear stochastic systems." *Probabilistic Analysis and Related Topics, Vol 3* edited by A.T. Bharucha-Reid, 1–79, Academic Press, 1983.

[5] Arnold, L. and W. Kliemann. "Large deviations of linear stochastic differential equations." *Stochastic Differential Systems* edited by J.H. Engelbrecht and W. Schmidt, 117–151, Springer-Verlag, 1987.

[6] Arnold, L., W. Kliemann, and E. Oeljeklaus "Lyapunov exponents of linear stochastic systems." *Lyapunov Exponents* LN Mathematics Vol. 1186, edited by L. Arnold and V. Wihstutz, 85–125, Springer Verlag, 1986.

[7] Arnold, L., N. Sri Namachchivaya, and K.R. Schenk-Hoppé "Toward an Understanding of Stochastic Hopf Bifurcation," *Intern. J. Bifur. Chaos Appl. Sci. Engrg.*, *6*:1947–1975 (1996).

[8] Aulbach, B. and Th. Wanner. "Integral manifolds for Carathéodory type differential equations in Banach spaces." *Six Lectures on Dynamical Systems* edited by B. Aulbach and F. Colonius, 45–119, World Scientific, 1996.

[9] Baxendale, P. "A stochastic Hopf bifurcation," *Probab. Theory Relat. Fields*, *99*:581–616 (1994).

[10] Baxendale, P. H. "Stability Along Trajectories at a Stochastic Bifurcation Point." *Stochastic Dynamics* edited by H. Crauel and V. M. Gundlach, Springer-Verlag, 1998.

[11] Beyn, W.-J. and A. Lust, "A hybrid method for calculating Lyapunov exponents." Universität Bielefeld, 2007.

[12] Bronstein, I.U. and V.F. Chernii. "Linear extensions satisfying Perron's condition I," *Differential Equations*, *14*:1234–1243 (1978).

[13] Colonius, F., R. Fabbri, R. Johnson, and M. Spadini "Bifurcation Phenomena in Control Flows," *Topological Methods in Nonlinear Analysis*, *30*:87–111 (2007).

[14] Colonius, F., T. Gayer, and W. Kliemann "Near Invariance for Markov Diffusion Systems," *SIAM J. Applied Dyn. Systems*, *7*:79–107 (2008).

[15] Colonius, F. and W. Kliemann. "Some aspects of control systems as dynamical systems," *J. Dyn. Diff. Equations*, *5*:469–494 (1993).

[16] Colonius, F. and W. Kliemann. "The Morse spectrum of linear flows on vector bundles," *Trans. Amer. Math. Soc.*, *348*:4355–4388 (1996).

[17] Colonius, F. and W. Kliemann. "Continuous, Smooth, and Control Techniques for Stochastic Dynamics." *Stochastic Dynamics* edited by H. Crauel and M. Gundlach, 181–208, Springer-Verlag, 1999.

[18] Colonius, F. and W. Kliemann. *The Dynamics of Control*. Birkhäuser, Boston, 2000.

[19] Colonius, F. and W. Kliemann. "Bifurcations of control systems-A view from control flows." *New Trends in Nonlinear Dynamics and Control, and their applications 295*. Lecture Notes in Control and Information Sciences, edited by W. Kang, et al., 19–35, Springer-Verlag, 2003.
[20] Coppel, W.A. *Dichotomies in Stability Theory*. LN in Mathematics, Vol. 629, Springer-Verlag, 1978.
[21] Crauel, H. "Markov measures for random dynamical systems," *Stochastics and Stochastics Reports*, *37*:153–173 (1991).
[22] Crauel, H. and F. Flandoli. "Additive Noise Destroys a Pitchfork Bifurcation," *J. Dynamics Diff. Equations*, *10*:259–274 (1998).
[23] Crauel, H., et al. "Bifurcation of One-Dimensional Stochastic Differential Equations." *Stochastic Dynamics* edited by H. Crauel and V. M. Gundlach, Springer-Verlag, 1998.
[24] Ethier, S.N. and T.G. Kurtz. *Markov Processes*. Wiley, 1986.
[25] Gichman, I.I. and A.V. Skorochod. *The Theory of Stochastic Processes I, II, III*. Springer-Verlag, New York, 1973, 1974, 1975.
[26] Grüne, L. "Numerical Stabilization of Bilinear Control Systems," *SIAM J. Control Optim.*, *34*:2024–2050 (1996).
[27] Grüne, L. "Asymptotic Controllability and Exponential Stabilization of Nonlinear Control Systems at a Singular Point," *SIAM J. Control Optim.*, *36*:1585–1603 (1998).
[28] Grünvogel, S. "Lyapunov Exponents and Control Sets," *J. Diff. Equations*, *187*:201–225 (2003).
[29] Hahn, W. *Stability of Motion*. Springer-Verlag, 1967.
[30] Ichihara, K. and H. Kunita. "A classification of the second order degenerate elliptic operators and its probabilistic characterization," *Z. Wahrsch. verw. Gebiete*, *30/39*:235–254/81–84 (1977).
[31] Ikeda, N. and S. Watanabe. *Stochastic Differential Equations and Diffusions Processes*. North Holland, 1981.
[32] Johnson, R. "Some questions in random dynamical systems involving real noise processes." *Stochastic Dynamics* edited by H. Crauel and V. M. Gundlach, 147–180, Springer-Verlag, 1998.
[33] Karoubi, M. *K-Theory, An Introduction*. Springer-Verlag, 1978.
[34] Katok, A. and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
[35] Khasminskii, R. Z. *Stochastic Stability of Differential Equations*. Alphen an der Rijn: Sijthoff and Noordhoff, 1980. Russian original: Nauka, Moscow, 1969.
[36] Kunita, H. *Diffusion Processes and Control Systems*. University of Paris, 1974. Lecture notes.
[37] Kunita, H. "Supports of diffusion processes." *Proc. International Symposium on Stochastic Differential Equations*, edited by K. Ito. 163–185. Wiley, 1978.
[38] Kunita, H. *Stochastic Flows and Stochastic Differential Equations*. Cambridge University Press, 1990.
[39] Latushkin, Y. and A. Stepin. "Weighted translation operators and linear extensions of dynamical systems," *Russian Math. Surveys*, *46*:95–165 (1991).
[40] L.Dieci and E.S.van Vleck. "On the error in computing Lyapunov exponents by QR methods," *Numer.Math.*, *101(4)*:619–642 (2005).
[41] Liang, Yan and N. S. Namachchivaya. "P-Bifurcations in the Noisy Duffing-Van der Pol Equation." *Stochastic Dynamics* edited by H. Crauel and V. M. Gundlach, Springer-Verlag, 1998.
[42] Lyapunov, A. M. *The General Problem of the Stability of Motion*. Comm. Soc. Math. Kharkow (in Russian), 1892. Problème Géneral de la Stabilité de Mouvement, Ann. Fac. Sci. Univ. Toulouse 9 (1907), 203-474, reprinted in Ann. Math. Studies, 17, Princeton (1949), in English: Taylor & Francis, London, 1992.
[43] Robinson, C. "Stability theorems and hyperbolicity in dynamical systems," *Rocky Mountain J. Math.*, *7*:425–437 (1977).
[44] Sacker, R.J. and G.R. Sell. "A spectral theory for linear differential systems," *J. Diff. Equations*, *37*:320–358 (1988).
[45] Salamon, D. and E. Zehnder. "Flows on vector bundles and hyperbolic sets," *Trans. Amer. Math. Soc.*, *306*:623–649 (1988).

[46] Selgrade, J. "Isolated invariant sets for flows on vector bundles," *Trans. Amer. Math. Soc.*, *203*:259–390 (1975).

[47] Sell, G.R., "Lectures on Linear Differential Systems." School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 1975.

[48] Stroock, D.W. and S.R.S. Varadhan. "On the support of diffusion processes with applications to the strong maximum principle." *Proc. 6th Berkeley Symp. Math. Stat. Probab. Vol 3*. 333–359. 1972.

[49] Sussmann, H.J. "Some properties of vector fields that are not altered by perturbations," *J. Diff. Equations*, *20*:292–315 (1976).

[50] Szolnoki, D., "Berechnung von Viabilitätskernen." Diplomarbeit, Institut für Mathematik, Universität Augsburg, Augsburg, 1997.

[51] Talay, D. "Approximation of Upper Lyapunov Exponents of Bilinear Stochastic Differential Systems," *SIAM J. Numerical Analysis*, *28*:1141–1164 (1991).

[52] Talay, D. "Simulation and Numerical Analysis of Stochastic Differential Systems: A Review." *Probabilistic Methods in Applied Physics 451*. Lecture Notes in Physics, edited by P. Krée and W. Wedig, 54–96, Springer-Verlag, 1995.

[53] Wang, H. "Feedback Stabilization of Bilinear Control Systems," *SIAM J. Control Optim.*, *36*:1669–1684 (1998).

