The PABTEC Algorithm for Passivity-Preserving Model Reduction of Circuit Equations

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***Abstract*— We present a passivity-preserving balanced truncation model reduction method for circuit equations (PABTEC). This method is based on balancing the solutions of the projected Lur'e equations and admit computable error bounds. We show how the topological structure of circuit equations can be exploited to reduce the computational complexity of the presented model reduction method.**

I. INTRODUCTION

With decreasing structural size and increasing complexity of modern integrated circuits, there is a growing demand for new modelling techniques and simulation algorithms for circuit design that make use of the structure and properties of the underlying problem. The numerical treatment of complex circuit models containing hundreds of millions of equations and variables is extremely expensive with respect to both computing time and memory requirements. Therefore, the reduction of model complexity or model order reduction is of great importance.

Electronic circuits often contain large linear RLC subnetworks that consist of resistors, inductors and capacitors only. Such subnetworks are used to model interconnects, transmission lines and pin packages. Using a modified nodal analysis (MNA), linear RLC circuits can be modelled by a linear system of differential-algebraic equations (DAEs)

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t), \end{array} \tag{1}$$

where

$$\begin{aligned} E &= \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T. \end{aligned} \tag{2}$$

Here $A_{\mathcal{C}} \in \mathbb{R}^{n_\eta, n_{\mathcal{C}}}$, $A_{\mathcal{L}} \in \mathbb{R}^{n_\eta, n_{\mathcal{L}}}$, $A_{\mathcal{R}} \in \mathbb{R}^{n_\eta, n_{\mathcal{R}}}$, $A_{\mathcal{V}} \in \mathbb{R}^{n_\eta, n_{\mathcal{V}}}$ and $A_{\mathcal{I}} \in \mathbb{R}^{n_\eta, n_{\mathcal{I}}}$ are incidence matrices describing the circuit topology, and $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{C}$ are resistance, inductance and capacitance matrices, respectively. We will assume that

- the matrix $A_{\mathcal{V}}$ has full column rank;
- the matrix $[\,A_{\mathcal{C}},\, A_{\mathcal{L}},\, A_{\mathcal{R}},\, A_{\mathcal{V}}]$ has full row rank;
- the matrices $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$ are symmetric, positive definite.

The first two conditions mean that the circuit does not contain loops of voltage sources and cutsets of current sources. These conditions together with positive definiteness of the element matrices guarantee that the pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \not\equiv 0$. Moreover, system (1), (2) is *passive*, i.e., it does not generate energy, and *reciprocal*, i.e., its transfer function $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ satisfies the symmetry relation $\boldsymbol{G}(s) = S_{\rm ext}\boldsymbol{G}(s)^T S_{\rm ext}$ with an external signature $S_{\rm ext} = \mathrm{diag}(I_{n_{\mathcal{I}}}, -I_{n_{\mathcal{V}}})$, see [1]. Passivity is an important system property in circuit design. It is well known in network theory [2] that system (1) is passive if and only if its transfer function $\boldsymbol{G}$ *positive real* meaning that $\boldsymbol{G}$ is analytic in the open right half-plane $\mathbb{C}_+$ and $\boldsymbol{G}(s) + \boldsymbol{G}^T(\overline{s})$ is positive semidefinite for all $s \in \mathbb{C}_+$.

A general idea of model reduction is to approximate the large-scale system (1) by a reduced-order model

$$\begin{array}{rcl} \widetilde{E}\,\dot{\widetilde{x}}(t) & = & \widetilde{A}\,\widetilde{x}(t) + \widetilde{B}\,u(t), \\ \widetilde{y}(t) & = & \widetilde{C}\,\widetilde{x}(t), \end{array} \tag{3}$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{\ell,\ell}$, $\widetilde{B} \in \mathbb{R}^{\ell,m}$, $\widetilde{C} \in \mathbb{R}^{m,\ell}$ and $\ell \ll n$. It is required that the approximate system (3) captures the input-output behavior of (1) to a required accuracy and preserves passivity and reciprocity.

For linear systems, a variety of passivity-preserving methods exist. These are interpolation-based methods like PRIMA [3], SPRIM [4] and spectral zero interpolation [5], [6] and also balancing-related methods [7], [8], [9], [10]. Interpolatory model reduction methods are closely related to rational Krylov subspace methods. Despite the successful application of these methods in circuit simulation, they provide good local approximations only and so far, there exist no global error bounds. Another drawback of Krylov subspace methods is the *ad hoc* choice of interpolation points that strongly influence the approximation quality. Recently, an optimal point selection strategy based on tangential interpolation has been developed [11], [12] that provides an optimal $\mathbb{H}_2$ approximation.

In this paper, we consider the PAssivity-preserving Balanced Truncation model reduction method for Electrical Circuits (PABTEC) developed first in [13]. Exploiting the circuit topological structure, we present an improvement to this method that further reduces the numerical effort in

This work is supported in part by the DFG Research Center MATHEON in Berlin and by the Research Network SYRENE - *System Reduction for Nanoscale IC Design* funded by the German Federal Ministry of Education and Science (BMBF), Grant No. 03STPAE3.

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computing the reduced-order model. Besides preservation of passivity and reciprocity, the PABTEC method provides also computable error bounds.

Throughout the paper, $\mathbb{R}^{n,m}$ denotes the space of $n \times m$ real matrices. The open right half-plane is denoted by $\mathbb{C}_+$ and j is the imaginary unit. The matrix A^T denotes the transpose of A. An identity matrix of order n is denoted by I_n or simply by I. We denote by $\mathrm{im}(A)$ and $\ker(A)$ the image and the kernel of the matrix A, respectively. A matrix $Z \in \mathbb{R}^{n,k}$ is called a basis matrix for a subspace $\mathcal{Z} \subset \mathbb{R}^n$ if Z has full column rank and $\mathrm{im}(Z) = \mathcal{Z}$. A matrix $Z' \in \mathbb{R}^{n,n-k}$ is called a complementary matrix to Z if $[Z, Z']$ is nonsingular. Further, for symmetric matrices X and Y, we write $X > Y$ ($X \geq Y$) if $X - Y$ is positive definite (semidefinite).

II. PASSIVITY-PRESERVING BALANCED TRUNCATION

In this section, we describe the PABTEC method that is based on bounded real balanced truncation applied to a Moebius-transformed system.

For a square transfer function $\boldsymbol{G}$ with $\det(I+\boldsymbol{G}(s)) \not\equiv 0$, a *Moebius transformation* is defined as

$$\hat{\boldsymbol{G}}(s) = \mathcal{M}(\boldsymbol{G})(s) = \big(I - \boldsymbol{G}(s)\big)\big(I + \boldsymbol{G}(s)\big)^{-1}.$$

One can show that $\boldsymbol{G}$ is positive real if and only if the Moebius-transformed function $\hat{\boldsymbol{G}}$ is bounded real, i.e., $\hat{\boldsymbol{G}}$ is analytic in $\mathbb{C}_+$ and $I - \hat{\boldsymbol{G}}(s)\hat{\boldsymbol{G}}^T(\overline{s})$ is positive semidefinite for all $s \in \mathbb{C}_+$, see [2].

For the transfer function $\boldsymbol{G}(s) = C(sE-A)^{-1}B$ of the passive system (1), we first determine $\hat{\boldsymbol{G}}(s) = \mathcal{M}(\boldsymbol{G})(s)$ which is bounded real. This function can be represented as $\hat{\boldsymbol{G}}(s) = \hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B} + I$ with

$$\hat{E} = E, \quad \hat{A} = A - BC, \quad \hat{B} = -\sqrt{2}B = -\hat{C}^T.$$

Then using the bounded real balanced truncation method [8], [10], $\hat{\boldsymbol{G}}(s)$ can be approximated by a bounded real function $\hat{\boldsymbol{G}}_r(s) = \hat{C}_r(s\hat{E}_r - \hat{A}_r)^{-1}\hat{B}_r + I$ of lower dimension. Finally, a back transformation

$$\tilde{\boldsymbol{G}}(s) = \mathcal{M}^{-1}(\hat{\boldsymbol{G}}_r)(s) = \big(I - \hat{\boldsymbol{G}}_r(s)\big)\big(I + \hat{\boldsymbol{G}}_r(s)\big)^{-1}$$

will gives the positive real function that can be realized as $\tilde{\boldsymbol{G}}(s) = \tilde{C}(s\tilde{E}-\tilde{A})^{-1}\tilde{B}$ with

$$\tilde{E} = \hat{E}_r, \;\; \tilde{A} = \hat{A}_r - \frac{1}{2}\hat{B}_r\hat{C}_r, \;\; \tilde{B} = -\frac{\sqrt{2}}{2}\hat{B}_r, \;\; \tilde{C} = \frac{\sqrt{2}}{2}\hat{C}_r.$$

Consider the dual projected Lur'e equations

$$\begin{aligned} &E X\hat{A}^T + \hat{A}XE^T + 2P_lBB^TP_l^T = -2K_cK_c^T, \\ &EXC^T - P_lBM_0^T = -K_cJ_c^T, \\ &J_cJ_c^T = I - M_0M_0^T, \qquad X = P_rXP_r^T, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &E^TY\hat{A} + \hat{A}^TYE + 2P_r^TC^TCP_r = -2K_o^TK_o, \\ &-E^TYB + P_r^TC^TM_0 = -2K_o^TJ_o, \\ &J_o^TJ_o = I - M_0^TM_0, \qquad Y = P_l^TYP_l, \end{aligned} \tag{5}$$

where $\hat{A} = A - BC$, P_r and P_l are the spectral projectors onto the right and left deflating subspaces of the pencil $\lambda E - \hat{A}$ corresponding to the finite eigenvalues and

$$M_0 = \lim_{s\to\infty} \hat{\boldsymbol{G}}(s) = I - 2\lim_{s\to\infty} C(sE-\hat{A})^{-1}B.$$

For the passive MNA system (1), (2) these equations are solvable for $X \in \mathbb{R}^{n,n}$, $K_c \in \mathbb{R}^{n,m}$, $J_c \in \mathbb{R}^{m,m}$ and $Y \in \mathbb{R}^{n,n}$, $K_o \in \mathbb{R}^{m,n}$, $J_o \in \mathbb{R}^{m,m}$, respectively, see [13]. Moreover, there exist the extremal solutions that satisfy

$$0 \leq X_{\min} \leq X \leq X_{\max}, \quad 0 \leq Y_{\min} \leq Y \leq Y_{\max}$$

for all symmetric solutions X and Y of (4) and (5), respectively. The minimal solutions $X_{\min}$ and $Y_{\min}$ are called the *bounded real controllability* and *observability Gramians* of the Moebius-transformed system $\hat{\boldsymbol{G}}$.

In the bounded real balanced truncation method, we determine the Cholesky factors R and L of $X_{\min} = RR^T$ and $Y_{\min} = LL^T$, respectively, and compute the singular value decomposition

$$L^TER = [\,U_1, U_2\,]\begin{bmatrix}\Pi_1 & \\ & \Pi_2\end{bmatrix}[\,V_1, V_2\,]^T,$$

where the matrices $[U_1, U_2]$ and $[V_1, V_2]$ have orthonormal columns,

$$\begin{aligned}\Pi_1 &= \mathrm{diag}(\pi_1 I_{l_1}, \ldots, \pi_r I_{l_r}), \\ \Pi_2 &= \mathrm{diag}(\pi_{r+1} I_{l_{r+1}}, \ldots, \pi_q I_{l_q})\end{aligned}$$

with $\pi_1 > \ldots > \pi_r > \pi_{r+1} > \ldots > \pi_q$. The values π_j are called the *characteristic values* of $\hat{\boldsymbol{G}}$. A reduced-order model for $\hat{\boldsymbol{G}}$ can be computed by projection onto the left and right subspaces corresponding to the dominant characteristic values. We obtain $\hat{\boldsymbol{G}}_r(s) = \hat{C}_r(s\hat{E}_r - \hat{A}_r)^{-1}\hat{B}_r + I$ with

$$\hat{E}_r = \begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}, \qquad \hat{A}_r = \begin{bmatrix} W^T(A-BC)T & 0 \\ 0 & I\end{bmatrix},$$

$$\hat{B}_r = \begin{bmatrix} -\sqrt{2}\,W^TB \\ B_\infty \end{bmatrix}, \qquad \hat{C}_r = \begin{bmatrix}\sqrt{2}\,CT, & C_\infty\end{bmatrix},$$

where $W = L\,U_1 \Pi_1^{-1/2}$, $T = R\,V_1 \Pi_1^{-1/2}$, and the matrices B_∞ and C_∞ are chosen such that $I - M_0 = C_\infty B_\infty$.

Using the structure of circuit equations, the model reduction procedure presented above can be made more efficient and accurate. Since the MNA matrices in (2) satisfy

$$E^T = S_{\rm int}\,E\,S_{\rm int}, \;\; A^T = S_{\rm int}\,A\,S_{\rm int}, \;\; B^T = S_{\rm ext}\,C\,S_{\rm int},$$

where

$$\begin{aligned} S_{\rm int} &= \mathrm{diag}(I_{n_\eta}, -I_{n_{\mathcal{L}}}, -I_{n_{\mathcal{V}}}), \\ S_{\rm ext} &= \mathrm{diag}(I_{n_{\mathcal{I}}}, -I_{n_{\mathcal{V}}}), \end{aligned}$$

we find that $P_l = S_{\rm int}\,P_r^T S_{\rm int}$ and

$$Y_{\min} = S_{\rm int}\,X_{\min}\,S_{\rm int} = S_{\rm int}RR^TS_{\rm int}^T = LL^T.$$

Thus, for the linear circuit equations (1), (2), it is enough to compute only one projector and solve only one projected Lur'e equation. Another projector and also the solution of the dual Lur'e equation are given for free. Furthermore, we can show that $L^TER = R^TS_{\rm int}ER$ is symmetric. Then the characteristic values π_j can be computed from an eigenvalue

Algorithm 1. Passivity-preserving balanced truncation for electrical circuits (PABTEC).

Given passive $\boldsymbol{G}=(\,E,\,A,\,B,\,C\,)$, compute a reduced-order model $\tilde{\boldsymbol{G}}=(\,\tilde{E},\,\tilde{A},\,\tilde{B},\,\tilde{C}\,)$.

1) Compute the Cholesky factor R of the minimal solution $X_{\min}=RR^T$ of the projected Lur'e equation (4).
2) Compute the eigenvalue decomposition

$$R^T S_{\rm int} E R = [\,U_1,\,U_2\,]\begin{bmatrix}\Lambda_1 & \\ & \Lambda_2\end{bmatrix}[\,V_1,\,V_2\,]^T,$$

where $[U_1,U_2]$ and $[V_1,V_2]$ have orthonormal columns, $\Lambda_1=\mathrm{diag}(\lambda_1,\ldots,\lambda_{\ell_f})$, $\Lambda_2=\mathrm{diag}(\lambda_{\ell_f+1},\ldots,\lambda_{n_f})$.
3) Compute the eigenvalue decomposition

$$(I-M_0)S_{\rm ext}=U_0\Lambda_0U_0^T,$$

where U_0 is orthogonal and $\Lambda_0=\mathrm{diag}(\hat{\lambda}_1,\ldots,\hat{\lambda}_m)$.
4) Compute the reduced-order system

$$\begin{aligned}
\tilde{E}&=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},\\
\tilde{A}&=\frac{1}{2}\begin{bmatrix} 2\,W^TA\,T & \sqrt{2}\,W^TBC_\infty\\ -\sqrt{2}\,B_\infty C\,T & 2\,I-B_\infty C_\infty\end{bmatrix},\\
\tilde{B}&=\begin{bmatrix} W^TB\\ -B_\infty/\sqrt{2}\end{bmatrix},\\
\tilde{C}&=\begin{bmatrix} C\,T, & C_\infty/\sqrt{2}\end{bmatrix},
\end{aligned}\tag{6}$$

where

$$\begin{aligned}
&B_\infty=S_0|\Lambda_0|^{1/2}U_0^TS_{\rm ext},\quad C_\infty=U_0|\Lambda_0|^{1/2},\\
&W=S_{\rm int}R\,U_1|\Lambda_1|^{-1/2},\quad T=R\,U_1S_1|\Lambda_1|^{-1/2},\\
&S_0=\mathrm{diag}(\mathrm{sign}(\hat{\lambda}_1),\ldots,\mathrm{sign}(\hat{\lambda}_m)),\\
&|\Lambda_0|=\mathrm{diag}(|\hat{\lambda}_1|,\ldots,|\hat{\lambda}_m|),\\
&S_1=\mathrm{diag}(\mathrm{sign}(\lambda_1),\ldots,\mathrm{sign}(\lambda_{\ell_f}))\\
&|\Lambda_1|=\mathrm{diag}(|\lambda_1|,\ldots,|\lambda_{\ell_f}|).
\end{aligned}$$

decomposition of $R^TS_{\rm int}ER$ instead of a more expensive singular value decomposition. Finally, using the symmetry of $(I-M_0)S_{\rm ext}$, we can determine B_∞ and C_∞ from the eigenvalue decomposition of $(I-M_0)S_{\rm ext}$. We summarize the PABTEC method in Algorithm 1.

One can show the reduced-order model computed by the PABTEC method preserves not only passivity but also reciprocity. Moreover, we have the following error bound

$$\|\widetilde{\boldsymbol{G}}-\boldsymbol{G}\|_{\mathbb{H}_\infty}\le\frac{\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}^2(\pi_{r+1}+\ldots+\pi_q)}{1-\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1}+\ldots+\pi_q)},\tag{7}$$

provided $\|I+\boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1}+\ldots+\pi_q)<1$, see [10] for details.

III. TOPOLOGICAL ANALYSIS

Using the topological structure of circuit equations, the matrix M_0 and the projector P_r can be computed in explicit form as given in the following theorem.

Theorem 1: Let E, A, B and C be as in (2). Then the matrix $M_0=I-2\lim\limits_{s\to\infty}C(sE-A+BC)^{-1}B$ and the projector P_r onto the right deflating subspace of the pencil $\lambda E-(A-BC)$ corresponding to the finite eigenvalues along the right deflating subspace corresponding to the eigenvalue at infinity are given by

$$M_0=\begin{bmatrix} I-2A_{\mathcal{I}}^TZH_0^{-1}Z^TA_{\mathcal{I}} & 2A_{\mathcal{I}}^TZH_0^{-1}Z^TA_{\mathcal{V}}\\ -2A_{\mathcal{V}}^TZH_0^{-1}Z^TA_{\mathcal{I}} & -I+2A_{\mathcal{V}}^TZH_0^{-1}Z^TA_{\mathcal{V}}\end{bmatrix},\tag{8}$$

$$P_r=\begin{bmatrix} H_5(H_4H_2-I) & H_5H_4A_{\mathcal{L}}H_6 & 0\\ 0 & H_6 & 0\\ -A_{\mathcal{V}}^T(H_4H_2-I) & -A_{\mathcal{V}}^TH_4A_{\mathcal{L}}H_6 & 0\end{bmatrix},\tag{9}$$

where

$$\begin{aligned}
&H_0=Z^T(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T+A_{\mathcal{V}}A_{\mathcal{V}}^T)Z,\\
&H_1=Z_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^TZ_{\mathcal{CRIV}},\\
&H_2=A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T+A_{\mathcal{V}}A_{\mathcal{V}}^T\\
&\qquad +A_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^TZ_{\mathcal{CRIV}}H_1^{-1}Z_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T,\\
&H_3=Z_{\mathcal{C}}^TH_2Z_{\mathcal{C}},\\
&H_4=Z_{\mathcal{C}}H_3^{-1}Z_{\mathcal{C}}^T,\\
&H_5=Z_{\mathcal{CRIV}}H_1^{-1}Z_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^T-I,\\
&H_6=I-\mathcal{L}^{-1}A_{\mathcal{L}}^TZ_{\mathcal{CRIV}}H_1^{-1}Z_{\mathcal{CRIV}}^TA_{\mathcal{L}},\\
&Z=Z_{\mathcal{C}}Z_{\mathcal{RIV}-\mathcal{C}}',\\
&Z_{\mathcal{C}}\ \text{ is a basis matrix for } \ker(A_{\mathcal{C}}^T),\\
&Z_{\mathcal{RIV}-\mathcal{C}}\ \text{ is a basis matrix for } \ker([\,A_{\mathcal{R}},\,A_{\mathcal{I}},\,A_{\mathcal{V}}\,]^TZ_{\mathcal{C}}),\\
&Z_{\mathcal{RIV}-\mathcal{C}}'\ \text{ is a complementary matrix to } Z_{\mathcal{RIV}-\mathcal{C}},\\
&Z_{\mathcal{CRIV}}\ \text{ is a basis matrix for } \ker([\,A_{\mathcal{C}},\,A_{\mathcal{R}},\,A_{\mathcal{I}},\,A_{\mathcal{V}}\,]^T).
\end{aligned}\tag{10}$$

Proof: It has been shown in [13] that

$$M_0=I+B_0^TA_0^{-1}B_0$$

with

$$A_0=\begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T-A_{\mathcal{I}}A_{\mathcal{I}}^T & -A_{\mathcal{C}} & -A_{\mathcal{V}}\\ A_{\mathcal{C}}^T & 0 & 0\\ A_{\mathcal{V}}^T & 0 & -I\end{bmatrix},$$

$$B_0=\sqrt{2}\begin{bmatrix} A_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & I\end{bmatrix}.$$

Let

$$\begin{bmatrix} X_{11} & X_{12}\\ X_{21} & X_{22}\\ X_{31} & X_{32}\end{bmatrix}=-\frac{1}{\sqrt{2}}A_0^{-1}B_0.$$

Then the matrices X_{ij} satisfy the equations

$$(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T)X_{11}+A_{\mathcal{C}}X_{21}+A_{\mathcal{V}}X_{31}=A_{\mathcal{I}},\tag{11}$$

$$-A_{\mathcal{C}}^TX_{11}=0,\tag{12}$$

$$-A_{\mathcal{V}}^TX_{11}+X_{31}=0,\tag{13}$$

and

$$(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T+A_{\mathcal{I}}A_{\mathcal{I}}^T)X_{12}+A_{\mathcal{C}}X_{22}+A_{\mathcal{V}}X_{32}=0,\tag{14}$$

$$-A_{\mathcal{C}}^TX_{12}=0,\tag{15}$$

$$-A_{\mathcal{V}}^TX_{12}+X_{32}=I.\tag{16}$$

Substituting X_{31} from (13) in (11), we obtain

$$(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T + A_{\mathcal{V}}A_{\mathcal{V}}^T)X_{11} + A_{\mathcal{C}}X_{21} = A_{\mathcal{I}}. \quad (17)$$

Furthermore, it follows from equation (12) that columns of X_{11} belong to $\ker(A_{\mathcal{C}}^T) = \mathrm{im}(Z_{\mathcal{C}})$, i.e., $X_{11} = Z_{\mathcal{C}}Y_1$ for some matrix Y_1. Substituting X_{11} in (17) and multiplying this equation from the left by $Z_{\mathcal{C}}^T$, we get

$$Z_{\mathcal{C}}^T(A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T + A_{\mathcal{V}}A_{\mathcal{V}}^T)Z_{\mathcal{C}}Y_1 = Z_{\mathcal{C}}^T A_{\mathcal{I}}. \quad (18)$$

Let $Y_1 = Z_{\mathcal{RIV}-\mathcal{C}}Y_{11} + Z'_{\mathcal{RIV}-\mathcal{C}}Y_{12}$. Then a multiplication of (18) from the left by $(Z'_{\mathcal{RIV}-\mathcal{C}})^T$ yields $H_0Y_{12} = Z^TA_{\mathcal{I}}$ with H_0 and Z as in (10). Since H_0 is nonsingular, we have

$$\begin{aligned} X_{11} &= Z_{\mathcal{C}}Z_{\mathcal{RIV}-\mathcal{C}}Y_{11} + ZH_0^{-1}Z^TA_{\mathcal{I}}, \\ X_{31} &= A_{\mathcal{V}}^T(Z_{\mathcal{C}}Z_{\mathcal{RIV}-\mathcal{C}}Y_{11} + ZH_0^{-1}Z^TA_{\mathcal{I}}) \\ &= A_{\mathcal{V}}^T ZH_0^{-1}Z^TA_{\mathcal{I}}. \end{aligned}$$

Analogously, we find from equations (14)–(16) that

$$\begin{aligned} X_{12} &= Z_{\mathcal{C}}Z_{\mathcal{RIV}-\mathcal{C}}Y_{21} - ZH_0^{-1}Z^TA_{\mathcal{V}}, \\ X_{32} &= I - A_{\mathcal{V}}^T ZH_0^{-1}Z^TA_{\mathcal{V}} \end{aligned}$$

with some matrix Y_{21}. Finally, substituting the matrices X_{11}, X_{31}, X_{12} and X_{32} in

$$M_0 = I + B_0^TA_0^{-1}B_0 = \begin{bmatrix} I - 2A_{\mathcal{I}}^TX_{11} & -2A_{\mathcal{I}}^TX_{12} \\ -2X_{31} & I - 2X_{32} \end{bmatrix}$$

and taking into account that $A_{\mathcal{I}}^TZ_{\mathcal{C}}Z_{\mathcal{RIV}-\mathcal{C}} = 0$, we obtain the expression (8) for M_0.

In order to prove (9), we first show that

$$Z_{\mathcal{CRIV}}H_1^{-1}Z_{\mathcal{CRIV}}^T = Q_{\mathcal{CRIV}}\hat{H}_1^{-1}Q_{\mathcal{CRIV}}^T,$$

where

$\hat{H}_1 = P_{\mathcal{CRIV}}^TP_{\mathcal{CRIV}} + Q_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^TQ_{\mathcal{CRIV}}$,
$Q_{\mathcal{CRIV}}$ is a projector onto $\ker([\,A_{\mathcal{C}},\, A_{\mathcal{R}},\, A_{\mathcal{I}},\, A_{\mathcal{V}}\,]^T)$,
$P_{\mathcal{CRIV}} = I - Q_{\mathcal{CRIV}}$.

Since $\mathrm{im}(Z_{\mathcal{CRIV}}) = \mathrm{im}(Q_{\mathcal{CRIV}})$, the projector $Q_{\mathcal{CRIV}}$ can be represented as $Q_{\mathcal{CRIV}} = Z_{\mathcal{CRIV}}\hat{Z}^T$ with $\hat{Z}^TZ_{\mathcal{CRIV}} = I$. Then

$$\begin{aligned} I &= \hat{Z}^TZ_{\mathcal{CRIV}} = \hat{Z}^T\hat{H}_1^{-1}\hat{H}_1Q_{\mathcal{CRIV}}Z_{\mathcal{CRIV}} \\ &= \hat{Z}^T\hat{H}_1^{-1}Q_{\mathcal{CRIV}}^T\hat{H}_1Z_{\mathcal{CRIV}} \\ &= \hat{Z}^T\hat{H}_1^{-1}\hat{Z}Z_{\mathcal{CRIV}}^T\hat{H}_1Z_{\mathcal{CRIV}}. \end{aligned}$$

Hence, $(Z_{\mathcal{CRIV}}^T\hat{H}_1Z_{\mathcal{CRIV}})^{-1} = \hat{Z}^T\hat{H}_1^{-1}\hat{Z}$. On the other hand, we have

$$(Z_{\mathcal{CRIV}}^T\hat{H}_1Z_{\mathcal{CRIV}})^{-1} = (Z_{\mathcal{CRIV}}^TA_{\mathcal{L}}\mathcal{L}^{-1}A_{\mathcal{L}}^TZ_{\mathcal{CRIV}})^{-1} = H_1^{-1}.$$

Thus,

$$\begin{aligned} Z_{\mathcal{CRIV}}H_1^{-1}Z_{\mathcal{CRIV}}^T &= Z_{\mathcal{CRIV}}(Z_{\mathcal{CRIV}}^T\hat{H}_1Z_{\mathcal{CRIV}})^{-1}Z_{\mathcal{CRIV}}^T \\ &= Z_{\mathcal{CRIV}}\hat{Z}^T\hat{H}_1^{-1}\hat{Z}Z_{\mathcal{CRIV}}^T \\ &= Q_{\mathcal{CRIV}}\hat{H}_1^{-1}Q_{\mathcal{CRIV}}^T. \end{aligned}$$

Analogously, we can show that $Z_{\mathcal{C}}H_3^{-1}Z_{\mathcal{C}}^T = Q_{\mathcal{C}}\hat{H}_3^{-1}Q_{\mathcal{C}}^T$, where $\hat{H}_3 = A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T + Q_{\mathcal{C}}^TH_2Q_{\mathcal{C}}$ and $Q_{\mathcal{C}}$ is a projector onto $\ker(A_{\mathcal{C}}^T)$. Thus, the matrices H_2, H_4, H_5 and H_6 in (10) coincide with those in [13], where the representation (9) for the projector P_r has been proved. ■

Note that the matrices H_i in (10) are more efficient to compute than those presented in [13]. Indeed, H_1 and H_3 have smaller dimension and they are often much better conditioned than $\hat{H}_1$ and $\hat{H}_3$ used in [13]. The basis matrices $Z_{\mathcal{C}}$ and $Z_{\mathcal{CRIV}}$ can be computed by analyzing the corresponding subgraphs of the given network graph as described in [14]. For example, the matrix $Z_{\mathcal{C}}$ can be constructed in the form

$$Z_{\mathcal{C}} = \Pi_{\mathcal{C}}\begin{bmatrix} \mathbb{1}_{k_1} & & \\ & \ddots & \\ & & \mathbb{1}_{k_r} \\ & 0 & \end{bmatrix},$$

where $\mathbb{1}_{k_i} = [1, \dots, 1]^T \in \mathbb{R}^{k_i}$, $i = 1, \dots, s$, and $\Pi_{\mathcal{C}}$ is a permutation matrix, by searching the components of connectivity [15] in the C-subgraph consisting of the capacitive branches only. As a consequence, the nonzero columns of $Z_{\mathcal{C}}^T[A_{\mathcal{R}},\, A_{\mathcal{I}},\, A_{\mathcal{V}}\,]$ form an incidence matrix, and, hence, $Z_{\mathcal{RIV}-\mathcal{C}}$ can also be determined from the associated graph as described above. In this case, the complementary matrix $Z'_{\mathcal{RIV}-\mathcal{C}}$ required for M_0 is just a selector matrix constructed from the identity matrix by removing some columns. One can see that the resulting basis matrices and also the matrices H_2, H_3, H_5 and H_6 are sparse. Of course, the projector P_r will never be constructed explicitly. Instead, we use projector-vector products required in the numerical solution of the Lur'e equation.

IV. COMPUTING THE GRAMIANS

In order to compute the Gramian $X_{\min}$ we have to solve the projected Lur'e equation (4). If $D_0 = I - M_0M_0^T$ is nonsingular, then this equation is equivalent to the projected Riccati equation

$$\begin{aligned} &(A-BC)XE^T + EX(A-BC)^T + 2P_lBB^TP_l^T \\ &\quad + 2(EXC^T - P_lBM_0^T)D_0^{-1}(EXC^T - P_lBM_0^T)^T = 0, \\ &X = P_rXP_r^T. \end{aligned} \quad (19)$$

that can be solved via Newton's method. This method was first developed for standard Riccati equations ($E = I$) [16], [17] and then extended in [10], [18] to projected Riccati equations. In each Newton iteration, we have to solve the projected Lyapunov equations of the form

$$EXF^T + FXE^T = -P_lGG^TP_l^T, \;\; X = P_rXP_r^T \quad (20)$$

with given matrices E, F, G, the projector P_r as in (9) and $P_l = S_{\mathrm{int}}P_r^TS_{\mathrm{int}}$. Such equations can be solved using the generalized alternating direction implicit (ADI) method [19]. Low-rank version of this method provides low-rank Cholesky factors of the solution of (20) that allow, finally, to determine an approximate solution of the projected Riccati equation (19) in factored form $X_{\min} \approx \tilde{R}\tilde{R}^T$ with $\tilde{R} \in \mathbb{R}^{n,k}$ and $k \ll n$, see [10] for details.

The most expensive step in the ADI method is solving linear systems of the form $(E + \tau F)z = f$ with different

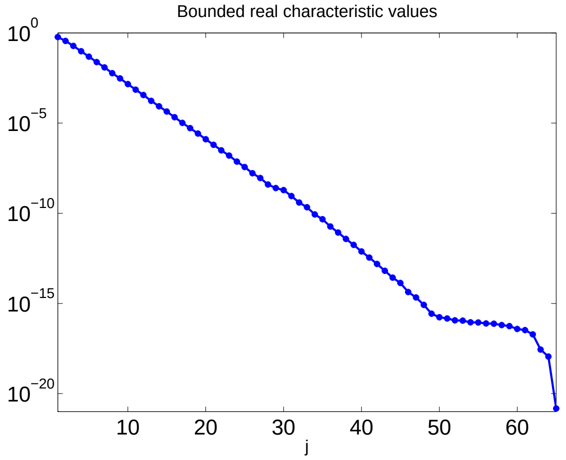

Fig. 1. The bounded real characteristic values of the Moebius-transformed system.

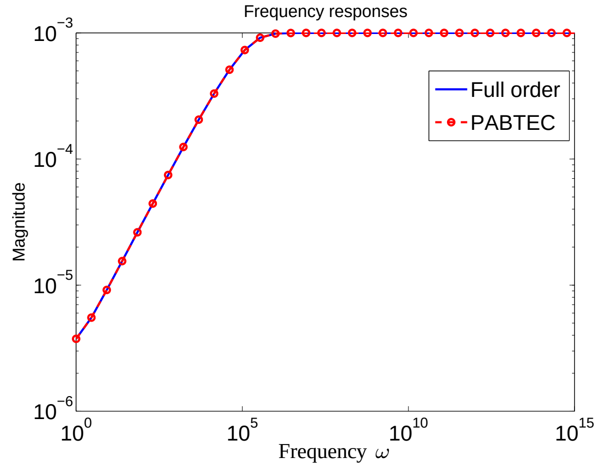

Fig. 2. The frequency responses of the original and the reduced-order models.

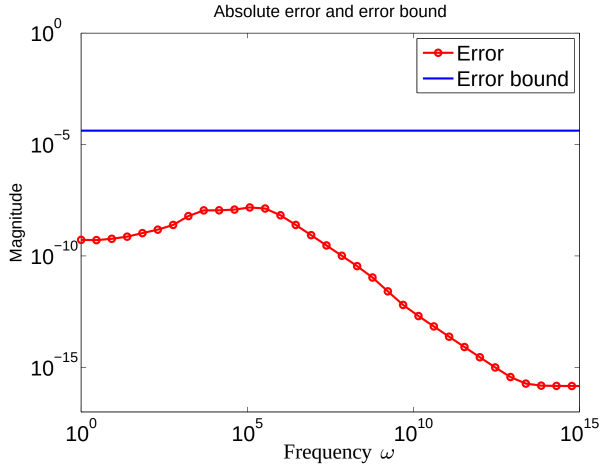

Fig. 3. The absolute error $|\widetilde{\boldsymbol{G}}(j\omega) - \boldsymbol{G}(j\omega)|$ and the error bound (7).

parameters τ. This can be done either by computing sparse LU factorization or by using Krylov subspace methods [20].

In case of singular $I - M_0 M_0^T$, small to medium-sized DAE systems can be transformed similarly to the standard state space case [21] to systems of smaller dimension for which the bounded real projected Riccati equations exist. For large-scale problems, the numerical solution of Lur'e equations requires further investigations.

V. NUMERICAL EXAMPLE

In this section, we present some results of numerical experiments to demonstrate the feasibility of the PABTEC method for large-scale circuit equations.

We consider a transmission line model consisting of a scalable number of RLC ladders. We have a reciprocal passive DAE system of order $n = 127\,869$ with a single input and a single output. The minimal solution of the projected Riccati equation (19) was approximated by a low-rank matrix $X_{\min} \approx \widetilde{R}\widetilde{R}^T$ with $\widetilde{R} \in \mathbb{R}^{n,84}$ using Newton's method. Figure 1 shows that the characteristic values decay rapidly, so we can expect a good approximation by a reduced-order model. The original system was approximated by a model of order $\ell = 24$. In Figure 2 we present the magnitude of the frequency responses $\boldsymbol{G}(j\omega)$ and $\tilde{\boldsymbol{G}}(j\omega)$ for a frequency range $\omega \in [1,\ 10^{15}]$. We also display in Figure 3 the absolute error $|\tilde{\boldsymbol{G}}(j\omega) - \boldsymbol{G}(j\omega)|$ and the error bound (7).

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