Controllability for Systems with Slowly Varying Parameters

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ABSTRACT. For systems with slowly varying parameters the controllability behavior is studied and the relation to the control sets for the systems with frozen parameters is clarified.

1. Introduction

This paper studies control systems with parameters which are slowly evolving according to a differential equation. We show that the controllability behavior is determined by the corresponding family of systems with frozen parameters.

More precisely, we consider the following family of systems on $\mathbb{R}^d$ depending on a parameter y:

$$(1.1) \qquad \dot{x}(t) = f(x(t), y, u(t)), \ u \in \mathcal{U},$$

where $\mathcal{U} = \{u : \mathbb{R} \to U \subset \mathbb{R}^m, \text{ locally integrable}\}$, the control range U is a subset of $\mathbb{R}^m$, and $f(y, \cdot, u)$ are smooth vector fields on $\mathbb{R}^d$. Throughout we assume that unique solutions $\varphi(t, x_0, y, u)$, $t \in \mathbb{R}$, exist for all x_0, u, and y. We model slowly varying y by requiring that

$$(1.2) \qquad \dot{y}(t) = \varepsilon g(y(t))$$

for a smooth vector field g on a Riemannian manifold M and $\varepsilon > 0$.

Here we are concerned with the controllability behavior of the fast subsystem (1.1). This is complementary to other contributions to the area of singularly perturbed systems concentrating on the behavior of the slow subsystem; then averaging is applied in order to describe the influence of the fast subsystem on the slow subsystem; see, for example, Kokotovic, Khalil, O'Reilly [12], Khalil [11] and Artstein [1], Artstein, Gaitsgory [2], Vigodner [16], Grammel [7]. Our approach (correcting and extending Colonius/Kliemann [5]; see Remark 1 at the end of Section 3) is, in particular, motivated by bifurcation problems, where for different (constant) parameters control sets may be born; see, e.g., Grünvogel [9], [10]. Our results show that for slow, dynamic parameter perturbations this picture remains valid.

In Section 2, we collect some preliminaries and specify our technical assumptions. Section 3 contains the controllability results. In particular, we introduce the notion of control(ability) bundles and show that they are equivalent to families of control sets for the system with frozen parameters. Finally, Section 4 briefly discusses two examples.

2. Notation and Preliminaries

In this section, we recall some notation and state our basic assumptions.

Consider for $\varepsilon \geq 0$ the systems in $\mathbb{R}^d \times M$

$$(2.1) \qquad \dot{x}(t) = f(x(t), y(t), u(t)),$$
$$\dot{y}(t) = \varepsilon g(y(t)),$$

where $u \in \mathcal{U}$. Partly, we will restrict our attention to the special case of control-affine systems where

$$(2.2) \qquad f(x, y, u) = f_0(x, y) + \sum_{i=1}^{m} u_i(t) f_i(x, y)$$

with smooth vector fields f_i; here we assume that the control range U is convex and compact. This guarantees that $\mathcal{U}$ is a weak* compact subset of $L_\infty(\mathbb{R}, \mathbb{R}^m)$ and that the trajectories depend continuously on u. Using the notation $z = (x, y)$ and $F^\varepsilon = (f, \varepsilon g)$ we can write system (2.1) as

$$(2.3) \qquad \dot{z}(t) = F^\varepsilon(z(t), u(t)), \quad z \in \mathbb{R}^d \times M.$$

We assume that for all $u \in U$ the vector fields $(f(\cdot, u), \varepsilon g(\cdot))$ on $\mathbb{R}^d \times M$ are smooth and that unique global solutions exist for all $u \in \mathcal{U}$ and $\varepsilon \geq 0$. Prescribing an initial value $z_0 = (x_0, y_0)$ at time $t = 0$ and a control function u the solution in $\mathbb{R}^d \times M$ can be rewritten as

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x^\varepsilon(t, x_0, y_0, u) \\ y^\varepsilon(t, y_0) \end{pmatrix}.$$

Observe that for $\varepsilon = 0$ we obtain the solutions of (1.1), i.e., $x^0(t, x_0, y_0, u) = \varphi(t, x_0, y_0, u)$, $t \in \mathbb{R}$. The corresponding reachable sets are denoted by $\mathcal{O}^{\varepsilon, \pm}(z_0)$. The reachable sets for the x-component which depend on the initial value of the y-component are

$$\mathcal{O}^{y_0, \varepsilon, +}_{\leq T}(x_0) = \{ x^\varepsilon(t, x_0, y_0, u), \ 0 \leq t \leq T \text{ and } u \in \mathcal{U} \}.$$

For the system with $\varepsilon = 0$ we also abbreviate

$$\mathcal{O}^{y, +}(x) = \mathcal{O}^{y, 0, +}(x_0).$$

We remark that by continuous dependence on parameters one has, uniformly on bounded t-intervals,

$$(2.4) \qquad x^\varepsilon(t, x_0, y_0, u) \to x^0(t, x_0, y_0, u) \text{ for } \varepsilon \to 0.$$

For controls $u^i \in U$ and times $\tau^i > 0$, $i = 1, ..., d$, define a piecewise constant control $u^\tau \in \mathcal{U}$ by

$$(2.5) \qquad u^\tau(t) = u^i \text{ for } t \in [\tau^0 + \tau^1 + \ldots + \tau^{i-1}, \tau^0 + \tau^1 + \ldots + \tau^i].$$

We write $F^\varepsilon(u) = F^\varepsilon(\cdot, u)$ and denote by $e^{tF^\varepsilon(u)}$ the solution maps at time t of $\dot{z} = F^\varepsilon(z, u)$, $u \in U$. We obtain

$$(2.6) \qquad z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\tau^d F^\varepsilon(u^d)} \ldots e^{\tau^1 F^\varepsilon(u^1)} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
$$= \begin{pmatrix} x^\varepsilon(t, x_0, y_0, u^\tau) \\ y^\varepsilon(t, y_0) \end{pmatrix}.$$

Frequently, we will assume the following accessibility rank condition for system (1.1) in $\mathbb{R}^d$ with frozen $y \in M$. For the Lie algebra

$$\mathcal{L}^y = \mathcal{LA}\{f(y, u),\ u \in U\}$$

generated by the vector fields $f(y, u) := f(\cdot, y, u)$, we require

$$\dim \Delta_{\mathcal{L}^y}(x) = d, \tag{2.7}$$

where $\Delta_{\mathcal{L}^y}$ denotes the subspace of the tangent space generated by the vector fields in $\mathcal{L}^y$. This implies that there exist a time $T > 0$ and (constant) controls $u^1, \ldots, u^d \in U$ such that the map

$$(0, T)^d \subset \mathbb{R}^d \to \mathbb{R}^d$$

$$\tau := (\tau^1, \ldots, \tau^d) \mapsto e^{\tau^d\, f(y, u^d)} \ldots e^{\tau^1\, f(y, u^1)} x_0$$

has full rank at every point. Thus it follows that the orbits $\mathcal{O}^{y, \pm}_{\leq T}(x)$ have nonvoid interiors for all $T > 0$.

Next we recall the notion of control sets, maximal sets of complete controllability (compare [**4**]).

DEFINITION 1. *A subset $D^y \subset \mathbb{R}^d$ is a control set for system (1.1), if D^y is a maximal set with the properties that for all $x \in D^y$ one has $D^y \subset \mathrm{cl}\,\mathcal{O}^{y,+}(x)$ and there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D^y$ for all $t \geq 0$.*

Note that local accessibility implies exact controllability in the interior of a control set, i.e., $\mathrm{int}\,D^y \subset \mathcal{O}^{y,+}(x)$ for all $x \in D^y$; furthermore $\mathrm{cl}\,\mathrm{int}\,D^y = \mathrm{cl}\,D^y$. We will need the following result from [**4**, Theorem 3.2.28] concerning a lower semicontinuity property of control sets depending on a parameter.

THEOREM 1. *Let D^{y_0} be a control set of (1.1) for $y = y_0$ and consider a compact subset $Q \subset \mathrm{int}\,D^{y_0}$ such that for all points $x \in Q$ system (1.1) satisfies the accessibility rank condition (2.7). Then there exists $\delta > 0$ such that for all $y \in M$ with $\mathrm{d}(y, y_0) < \delta$ there is a unique control set D^y of (1.1) with $Q \subset \mathrm{int}\,D^y$.*

Concerning the slow system (1.2), we note that in the slow time $s = \varepsilon t$ it is described by

$$\frac{d}{ds}\tilde{y}(s) = g(\tilde{y}(s)),\ u \in \mathcal{U}, \tag{2.8}$$

where we write $\tilde{y}(s) = \tilde{y}(s, y_0) = y^\varepsilon(s/\varepsilon)$. We refer to (2.8) as the base system and write

$$\mathcal{O}^+(y_0) = \{\tilde{y}(s, y_0),\ s \geq 0\}.$$

3. Control Sets for Slowly Varying Parameters

In this section we analyze the behavior of the control sets D^y as y becomes slowly varying.

First we specify our notion of subsets of complete controllability for the singularly perturbed system (2.1).

DEFINITION 2. *Consider the singularly perturbed control system (2.1). For an open subset $B \subset M$, a family of subsets $B^y \subset \mathbb{R}^d$, $y \in B$, with nonvoid interiors is called a control bundle if*

(i) for all $y_0 \in B$ one has $\mathrm{cl\,int}\, B^{y_0} = \mathrm{cl}\, B^{y_0}$ and for all $x \in \mathrm{int}\, B^{y_0}$ there is $\delta > 0$ such that $\mathrm{d}(y, y_0) < \delta$ implies $x \in \mathrm{int}\, B^y$;
(ii) for every $y_0 \in B$ and every compact subset $Q \subset \mathrm{int}\, B^{y_0}$ there are a neighborhood $N_0(y_0)$ and numbers $\varepsilon_0 > 0$ and $T_0 > 0$ such that for every $y_1 \in N_0(y_0) \cap \mathcal{O}^+(y_0)$, all $x_0, x_1 \in Q$ and for every $0 < \varepsilon < \varepsilon_0$ there are a control $u \in \mathcal{U}$ and a time $0 < t < T_0$ with $x^\varepsilon(t, x_0, y_1, u) = x_1$;
(iii) the sets B^y, $y \in B$, are maximal (with respect to inclusion) with these properties.
We refer to the sets B^y as the fibers of the control bundle.

The maximality property (iii) means that for every family of subsets $\tilde{B}^y$, $y \in B$, satisfying properties (i) and (ii) and $B^y \subset \tilde{B}^y$ for all $y \in B$, it follows that $B^y = \tilde{B}^y$ for all $y \in B$. The decisive property in this definition is (ii). It states a complete controllability property in compact subsets Q contained in a single fiber B^{y_0} which holds as y varies. We would like to emphasize that here the times t are uniformly bounded, independently of $0 < \varepsilon < \varepsilon_0$.

The next theorem shows complete controllability from one fiber B^{y_0} to another fiber B^{y_1} in large times, where the point reached by the slow system during this time from y_0 is close to y_1. Thus, as announced, control bundles may be viewed as subsets of approximate controllability for the system with slowly varying parameters.

THEOREM 2. *Let $B^y \subset \mathbb{R}^d$, $y \in B \subset M$, be a control bundle. Let $y_0 \in B$ and $y_1 = \tilde{y}(S, y_0)$, $S \geq 0$, with $\tilde{y}(s, y_0) \in B$ for all $s \in [0, S]$. Then for all $x^0 \in B^{y_0}$, $x^1 \in \mathrm{int}\, B^{y_1}$ and all $\tau_0 > 0$ there is $\overline{\varepsilon}_0 > 0$ such that for all $0 < \varepsilon < \overline{\varepsilon}_0$ there are a control $u \in \mathcal{U}$ and a time $T > 0$ such that $0 < S - \varepsilon T < \tau_0$ and $x^\varepsilon(T, x^0, y_0, u) = x^1$.*

Note that here the distance between the point reached by the slow system, i.e., $y^\varepsilon(T, y_0) = \tilde{y}(\varepsilon T, y_0)$, and $y_1 = y^\varepsilon(S/\varepsilon, y_0) = \tilde{y}(S, y_0)$ can be made arbitrarily small by choosing $\tau_0 > 0$ small.

PROOF. Consider $y_0 \in B$, $y_1 = \tilde{y}(S, y_0)$ and $x^0 \in B^{y_0}$, $x^1 \in \mathrm{int}\, B^{y_1}$. Fix $\varepsilon > 0$ and recall that $\tilde{y}(S, y_0) = y^\varepsilon(S/\varepsilon, y_0)$. By property (i) of control bundles, there exists for every $\sigma \in [0, S]$ a number $\delta > 0$ such that for $\tilde{y}(\sigma) = \tilde{y}(\sigma, y_0)$

$$\bigcap_{\sigma \in [s-\delta, s+\delta] \cap [0, S]} \mathrm{int}\, B^{\tilde{y}(\sigma)} \neq \emptyset.$$

Choose δ small enough such that the neighborhood $N_0(\tilde{y}(s))$ as in property (ii) of control sets contains every $y(\sigma)$, $\sigma \in [s-\delta, s+\delta]$. By compactness, there are finitely many $s_0 = 0, s_1, ..., s_m = S$ and $\delta_0, ..., \delta_m > 0$ such that $\bigcup_{i=0}^m [s_i - \delta_i, s_i + \delta_i] \supset [0, S]$. Then we find points

$x_0 = x^0 \in \mathrm{int}\, B^{y_0}$, $x_1 \in \mathrm{int}\, B^{\tilde{y}(s_1)}, \ldots, x_{m-1} \in \mathrm{int}\, B^{\tilde{y}(s_{m-1})}$, $x_m = x^1 \in \mathrm{int}\, B^{y_1}$

with $x_i \in \bigcap_{\sigma \in [s_i, s_{i+1}]} \mathrm{int}\, B^{\tilde{y}(\sigma)}$ for all i. We may assume that for all

$$s_0' = 0, s_1', ..., s_{m-1}', s_m' = S$$

in neighborhoods of $s_0 = 0, s_1, ..., s_{m-1}, s_m = S$, respectively, one has

$$\bigcup_{i=1}^m (s_{i-1}', s_i') \supset [0, S] \text{ and } x_i \in \bigcap_{\sigma \in [s_i', s_{i+1}']} \mathrm{int}\, B^{\tilde{y}(\sigma)},$$

and that the neighborhoods $N_0(\tilde{y}(s_i))$ are also neighborhoods of $\tilde{y}(s_i')$ containing all $y(\sigma)$, $\sigma \in [s_{i-1}, s_{i+1}]$.

Now we iteratively use property (ii): for every i there are a neighborhood $N_0(\tilde{y}(s_i))$ and $\varepsilon_i > 0$ and $T_i > 0$ such that for every $y' \in N_0(\tilde{y}(s_i)) \cap \mathcal{O}^+(\tilde{y}(s_i))$ and $x_i \in \operatorname{int} B^{\tilde{y}(s_i)}$ and for all $0 < \varepsilon < \varepsilon_i$ there are a control $u \in \mathcal{U}$ and a time $0 < t < T_i$ with $x^\varepsilon(t, x_i, y', u) = x_i$.

For $i = 0$ we find $u_0^0 \in \mathcal{U}$ and t_0^0 with $0 < t_0^0 < T_0$ such that $x^\varepsilon(t_0^0, x_0, y_0, u_0^0) = x_1$. Observe that after time $t_0^0 > 0$ the y-system is in $y^\varepsilon(t_0^0, y_0)$. We proceed to find controls $u_0^j \in \mathcal{U}$ and times t_0^j with $0 < t_0^j < T_0$ such that $x^\varepsilon(t_0^j, x_1, y^\varepsilon(t_0^{j-1}), u_0^j) = x_1$. Denote by J_0 the largest J with

$$\sum_{j=0}^{J} t_0^j < s_1/\varepsilon.$$

Note that for $\varepsilon > 0$, small enough,

$$s_1' := \varepsilon \sum_{j=0}^{J_0} t_0^j$$

can be chosen in any neighborhood of s_1, since

$$s_1 - s_1' \le \varepsilon t_0^{J_0+1} < \varepsilon T_0.$$

Thus, again by property (ii) of control bundles, we can for $\varepsilon > 0$, small enough, choose a control $u_1^0 \in \mathcal{U}$ and a time t_1^0 with $0 < t_1^0 < T_0$ such that $x^\varepsilon(t_1^0, x_1, y^\varepsilon(s_1'), u_1^0) = x_2$.

Proceeding iteratively for all i we arrive at $x^\varepsilon(s_m'/\varepsilon) = x_m = x^1$; here

$$s_m' := \sum_{i=0}^{m-1} s_i' + \varepsilon \sum_{j=0}^{J_{m-1}} t_{m-1}^j < s_m = S.$$

Then

$$0 \le s_m - s_m' \le \varepsilon t_{m-1}^{J_{m-1}+1} < \varepsilon T_m.$$

With $T = s_m'/\varepsilon$ this concludes the proof, since $s_m - s_m' < \varepsilon T_{m-1}$ can be made arbitrarily small by choosing $\varepsilon > 0$ small (note that the bound T_{m-1} does not depend on ε). $\square$

Next we will relate control bundles to the control sets for frozen parameter y. We need some preparations and note the following easy consequence of the accessibility rank condition.

LEMMA 1. *Suppose that accessibility rank condition (2.7) is satisfied for some point $(x_0, y_0) \in \mathbb{R}^d \times M$. Then there are neighborhoods $N_1(x_0)$ and $N_1(y_0)$ and $\varepsilon_1 = \varepsilon_1(x_0, y_0) > 0$ such that the map*

$$[0, \varepsilon_1) \times (0, T)^d \times N_1(x_0) \times N_1(y_0) \to \mathbb{R}^d \ : (\varepsilon, \tau^1, \dots, \tau^d, x, y) \mapsto x^\varepsilon(\sum_{i=1}^{d} \tau^i, x, y, u^\tau),$$

is continuously differentiable and the derivative with respect to $\tau = (\tau^1, ..., \tau^d)$ has full rank.

Next we consider controllability for positive $\varepsilon > 0$ between two points in a control set in $\mathbb{R}^d$.

PROPOSITION 1. *Suppose that accessibility rank condition (2.7) is satisfied at points x_0, $x_1 \in \mathrm{int}\, D^{y_0}$, where D^{y_0} is a control set for some $y_0 \in M$. Then there exist neighborhoods $N(y_0)$, $N(x_0)$, and $N(x_1)$, a number $\varepsilon_0 > 0$ and a time $T_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $y \in N(y_0)$, $x \in N(x_0)$ and $\bar{x} \in N(x_1)$ we have*

$$x^\varepsilon(T, x, y, u) = \bar{x}$$

for some time $0 < T < T_0$ and some control $u \in \mathcal{U}$.

PROOF. Applying Lemma 1 we find neighborhoods $N_1(y_0)$ and $N_1(x_0)$ and a number $\varepsilon_1 = \varepsilon_1(x_0, y_0) > 0$ such that the map

$$[0, \varepsilon_1) \times N_1(x_0) \times N_1(y_0) \times (0, T)^d : (\varepsilon, x, y, \tau) \mapsto x^\varepsilon\left(\sum_{i=1}^d \tau^i, x, y, u^\tau\right)$$

has partial derivative with respect to τ of full rank. The implicit function theorem implies that (for possibly smaller ε_1 and $N_1(x_0) \times N_1(y_0)$) there are a point $\xi_0 \in \mathrm{int}\, D^{y_0}$ and a neighborhood $N(\xi_0)$ such that for all $0 < \varepsilon < \varepsilon_1$, all $y \in N_1(y_0)$, all $x \in N_1(x_0)$, and all $\xi \in N(\xi_0)$ there is $\tau_0 = \tau_0(\varepsilon, x, y, \xi) \in (0, T)^d$ such that

$$\xi = x^\varepsilon(\sigma_0, x, y, u^{\tau_0}) \text{ with } \sigma_0 = \sum_{i=1}^d \tau_0^i.$$

Observe that in system (1.2) we obtain $y^\varepsilon(\sigma_0, y)$.

Applying Lemma 1 to the point x_1 backwards in time, we find neighborhoods $N_2(x_1)$ and $N_2(y_0)$ and $\varepsilon_2 = \varepsilon_2(x_1, y_0) > 0$ such that the map

$$[0, \varepsilon_2) \times N_2(x_1) \times N_2(y_0) \times (0, T)^d \to \mathbb{R}^d : (\varepsilon, x, y, \tau) \mapsto x^\varepsilon\left(-\sum_{i=1}^d \tau^i, x, y, u^\tau\right)$$

has partial derivative with respect to τ of full rank. Again the implicit function theorem implies that (for possibly smaller ε_2 and $N_2(x_1) \times N_2(y_0)$) there are a point $\zeta_0 \in \mathrm{int}\, D^{y_0}$ and a neighborhood $N(\zeta_0)$ such that for all $0 < \varepsilon < \varepsilon_2$, all $\bar{x} \in N_2(x_1)$, all $y \in N_2(y_0)$, and all $\zeta \in N(\zeta_0)$ there is $\tau_1 = \tau_1(\varepsilon, \bar{x}, y, \zeta)$ with

$$\zeta = x^\varepsilon(-\sigma_1, \bar{x}, y, u^\tau) \text{ where } \sigma_1 = \sum_{i=1}^d \tau_1^i.$$

Since ξ_0, $\zeta_0 \in \mathrm{int}\, D^{y_0}$, we can find a control $v \in \mathcal{U}$ and a time $R > 0$ such that $\varphi(R, \xi_0, y_0, v) = \zeta_0$. Then there is $\varepsilon_3 \leq \min(\varepsilon_1, \varepsilon_2)$ such that for $N(\zeta_0)$ small enough and all $0 < \varepsilon < \varepsilon_3$ the neighborhood $\{x^\varepsilon(R, \xi, y^\varepsilon(\sigma_0, y_0, u^{\tau_0}), v), \ \xi \in N(\xi_0)\}$ is contained in $N(\zeta_0)$. Applying these controls we reach the points

$$\eta_x = x^\varepsilon\big(R, x^\varepsilon(\sigma_0, x, y, u^{\tau_0}), y^\varepsilon(\sigma_0, y), v\big)$$
$$\eta_y = y^\varepsilon\big(R, y^\varepsilon(\sigma_0, y)\big).$$

We may choose $\varepsilon_0 \leq \varepsilon_3$ and the neighborhoods $N(x_0)$ and $N(y_0)$ small enough such that for all $0 < \varepsilon < \varepsilon_0$ and all $y \in N(y_0)$, all $x \in N(x_0)$, and all $\bar{x} \in N_2(x_1)$

$$x^\varepsilon(\sigma_1, \eta_x, \eta_y, u^{\tau_1}) = \bar{x}.$$

Putting things together we conclude that there is a time $T_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and for every $y \in N(y_0)$ every point in $N(x_0)$ can be steered to every point $\bar{x}$ in $N_2(x_1)$ for some time $t < T_0$, i.e., $x^\varepsilon(t, x, y, u) = \bar{x}$ for some control $u \in \mathcal{U}$. $\qquad\square$

Next we show that these properties hold uniformly in compact subsets in the interior of control sets.

PROPOSITION 2. *Let D^{y_0} be a control set with nonvoid interior of system (1.1) for a parameter $y_0 \in M$. Suppose that accessibility rank condition (2.7) is satisfied for y_0 and for all $x \in D^{y_0}$, and let $Q \subset \operatorname{int} D^{y_0}$ be compact. Then there exist a neighborhood $N(y_0)$, a number $\varepsilon_0 > 0$ and a time $T_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, all $y \in N(y_0)$, and all $x_0, x_1 \in Q$ we find a time $0 < T < T_0$ and a control $u \in \mathcal{U}$ such that $x^{\varepsilon}(T, x_0, y, u) = x_1$.*

PROOF. Cover the compact set $Q \times Q$ by the open sets $N(x_0) \times N(x_1)$, $x_0, x_1 \in Q$, with corresponding neighborhoods $N(y_0) = N(y_0; x_0, x_1)$ and times $T_0(x_0, x_1) > 0$ and $\varepsilon_0(x_0, x_1) > 0$, as constructed in Proposition 1. By compactness of Q finitely many of these open sets are sufficient and we can take T_0 as the maximum of the corresponding $T_0(x_0, x_1)$ and ε_0 as the minimum of the $\varepsilon_0(x_0, x_1)$ and $N(y_0)$ as the intersection of the $N(y_0; x_0, x_1)$.   □

The next theorem presents the announced equivalence between the control sets D^y for the system (1.1) with frozen parameter y and the control bundles for the perturbed system (2.1). It is our main motivation for the consideration of control bundles.

THEOREM 3. *Consider for the control-affine system (1.1), (2.2) a family of subsets F^y, $y \in Y$, with nonvoid interiors in $\mathbb{R}^d$, where $Y \subset M$ is open. Assume that for all $y \in Y$ there is $\delta > 0$ with $\operatorname{int} F^y \cap \operatorname{int} F^{y'} \neq \emptyset$ for all y' with $\operatorname{d}(y, y') < \delta$ and that accessibility rank condition (2.7) holds for all $x \in F^y$ and all $y \in Y$. Then the following properties are equivalent:*
(i) The sets $\{F^y, \ y \in Y\}$ form a control bundle for the singularly perturbed system (2.1), (2.2).
(ii) For every $y \in Y$ the set F^y is a control set for system (1.1), (2.2) with parameter value y.

PROOF. (i) Suppose that the sets $\{F^y, y \in Y\}$ form a control bundle for system (2.1). In order to prove that F^{y_0} is a control set for system (1.1) with frozen parameter y_0, take $x_0 \in F^{y_0}$ and $x_1 \in \operatorname{int} F^{y_0}$. By property (i) of control bundles, we have that $x_1 \in \operatorname{int} F^y$ for all y in a neighborhood $N_0(y_0)$ of y_0. Let $S_1 > 0$ be small enough such that $y_1 := \tilde{y}(S_1, y_0) \in N_0(y_0) \cap \mathcal{O}^+(y_0)$ and hence $x_1 \in \operatorname{int} F^{y_1}$. Thus, by property (ii) of control bundles, there are $\varepsilon_0 > 0$ and $T_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there are a control $u^{\varepsilon} \in \mathcal{U}$ and a time $0 < t < T_0$ with $x^{\varepsilon}(t, x_0, y_0, u^{\varepsilon}) = x_1$. Since the set $\mathcal{U}$ is a compact metric space there is a converging subsequence u^{ε_k} in $\mathcal{U}$ for $\varepsilon_k \to 0$. Furthermore, the corresponding solutions converge uniformly on compact time intervals. Thus cluster points t and u for $\varepsilon_k \to 0$ satisfy

$$\varphi(t, x_0, y_0, u) = x^0(t, x_0, y_0, u) = x_1.$$

Thus that there exists a control set D^{y_0} with $\operatorname{int} F^{y_0} \subset D^{y_0}$. Hence the assumption $\operatorname{cl} \operatorname{int} F^{y_0} = \operatorname{cl} F^{y_0}$ implies that $F^{y_0} \subset D^{y_0}$. The proof of the converse inclusion will be deferred to (iii), below.

(ii) Conversely, consider a family $\{F^y, \ y \in Y\}$ such that F^y are control sets for system (1.1) with frozen parameter value $y \in Y$. By Theorem 1 and the properties of control sets, property (i) of control bundles is satisfied. For property (ii) let

$y_0 \in B$ and $x_0, x_1 \in \text{int } F^{y_0}$. We claim that there are a neighborhood $N_0(y_0)$ and $\varepsilon_0 > 0$ and $T_0 > 0$ such that for every $y_1 \in N_0(y_0) \cap \mathcal{O}^+(y_0)$, all $x_0, x_1 \in Q$ and for every $0 < \varepsilon < \varepsilon_0$ there are a control $u \in \mathcal{U}$ and a time $0 < t < T_0$ with $x^\varepsilon(t, x_0, y_1, u) = x_1$. By Proposition 1 there exist a neighborhood $N(y_0)$, a number $\varepsilon_0 > 0$ and a time $T_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, all $y \in N(y_0)$, and all $x_0, x_1 \in Q$ we find a time $0 < T < T_0$ and a control $u \in \mathcal{U}$ such that $x^\varepsilon(T, x_0, y, u) = x_1$. Hence the F^y, $y \in Y$, are contained in a control bundle. The proof of the converse inclusion will be given in (iv), below.

(iii) To finish the argument in part (i) of the proof we need to show that every F^y, $y \in Y$, is a control set. By (i) there exists a control set D^y containing F^y for all $y \in Y$. By (ii) the corresponding family of control sets is contained in a control bundle. By the maximality property this control bundle coincides with the original one. Thus the F^y are control sets.

(iv) To finish the argument in part (ii) of the proof we need to show that the family F^y, $y \in B$, is a control bundle. By (ii) we know that there exists a control bundle B^y, $y \in Y$, containing F^y, $y \in Y$. By (i) the B^y are contained in control sets. By the maximality property of control sets they coincide with the original ones. Hence the F^y form a control bundle.   □

REMARK 1. *Preliminary results on the behavior of control sets for slowly varying parameters appeared in [5]. There only the situation near a singular point as discussed in the second part of our Section 4 was considered for the special case of scalar parameters y. In the present paper, we had to change the definition of control bundles: By our Theorem 3 the existence of a control set family $\{D^y, \ y \in Y\}$ is equivalent to the properties indicated in Definition 2. Theorem 2 indicates that this implies certain controllability properties for the systems with slowly varying y. In [5] these properties were taken as a definition of control bundles and their equivalence to the control set property of the system with frozen y was claimed. The corresponding proof is not valid (though the technique used in the proof of Theorem 3 is very similar.)*

4. Examples

In this section we briefly describe two situations where the results above give information on the considered systems.

First we consider the following escape equation with periodic excitation

$$(4.1) \qquad \ddot{x} + \gamma\dot{x} + x - x^2 = F\sin(\varepsilon t) + u(t)$$

with linear viscous damping $\gamma > 0$ and amplitude F of forcing; ε is the excitation frequency and $u(t)$ is a noise term. Note that the system governs the escape from the cubic potential well of

$$V(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3.$$

In Soliman and Thompson [13] and Fischer, Guder and Kreuzer in [6] this equation has been analyzed, where u is considered as a stochastic perturbation. For bounded, deterministic u this system has been studied in detail by Szolnoki [14], [15].

We rewrite system (4.1) as a three dimensional control system to eliminate the second order derivative as well as the time dependency of the forcing term and

require $u(t) \in [-\rho, \rho]$ for all $t \in \mathbb{R}$, where $\rho \geq 0$. We obtain the control system

$$(4.2) \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x_2 \\ -\gamma x_2 - x_1 + x_1^2 \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} F \sin y + u(t).$$

It is appropriate to consider the equation for y as an equation on the unit sphere $M = \mathbb{S}^1$ parametrized by the angle in $[0, 2\pi)$. Clearly, this is a system of the form (2.1), (2.2). For $F = 0$, $\varepsilon = 0$ and $\rho = 0$, there are three equilibria. For $\rho > 0$ small, these equilibria are, as can be easily proven, in the interior of three different control sets D_i^y, $i = 1, 2, 3$ (since $F = 0$, they are identical for all $y \in \mathbb{S}^1$). It is also clear that the accessibility rank condition (2.7) is satisfied. Then, by [**4**, Theorem 4.2.28], there are for $F > 0$, small, $\varepsilon = 0$, and every $y \in \mathbb{S}^1$ three different control sets. Theorem 3 shows that these control sets yield a control bundle for the perturbed system (4.1).

Another class of examples contains a singular point where the accessibility rank condition (2.7) is violated. We consider again a family of control-affine systems (1.1), (2.2) depending on a parameter y taking values in an interval $A = (a, b) \subset \mathbb{R}$. Suppose that $x^* \in \mathbb{R}^d$ is a singular point, i.e.,

$$(4.3) \qquad\qquad\qquad\qquad f_i(x^*, y) = 0$$

for all $i = 0, 1, ..., m$ and all considered y. Again we will replace y by a time dependent, slowly varying parameter, and our goal is to discuss the changes in the controllability behavior as y evolves. The linearized (bilinear) control system is

$$\dot{x}(t) = A_0(y)x(t) + \sum_{i=1}^{m} u_i(t)A_i(y)x(t)), \ u \in \mathcal{U},$$

where $A_i(y) := \frac{\partial}{\partial x} f_i(x^*, y)$ for all i with trajectories $x_{lin}(t, x_0, y, u)$. The corresponding Lyapunov exponents are given by

$$\lambda(u, x_0, y) = \limsup_{t \to \infty} \frac{1}{t} \log |x_{lin}(t, x_0, y, u)|,$$

and the Lyapunov spectra are

$$\Sigma^y = \{\lambda(u, x_0, y), \ x_0 \neq 0 \text{ and } u \in \mathcal{U}\}.$$

Again we model slowly varying y by requiring that $\dot{y}(t) = \varepsilon > 0$, small. In other words, we consider the system in $\mathbb{R}^d \times \mathbb{R}$

$$(4.4) \qquad\qquad \dot{x}(t) = f_0(x(t), y(t)) + \sum_{i=1}^{m} u_i(t)f_i(x(t), y(t)),$$

$$\dot{y}(t) = \varepsilon,$$

with $u \in \mathcal{U}$.

The following result due to Grünvogel [**9**, Theorem 8.1] shows that, for fixed y, control sets near the singular point are determined by the Lyapunov spectrum.

THEOREM 4. *Consider the control-affine systems (1.1), (2.2) with a singular point $x^* \in \mathbb{R}^d$ satisfying (4.3) and assume that the accessibility rank condition (2.7) holds for all $x \neq x^*$. Furthermore assume that*

(i) there are periodic control functions u^s and u^h such that for u^s the linearized system is exponentially stable, i.e., the corresponding Lyapunov (Floquet) exponents satisfy

$$0 > \lambda_1^s > ... > \lambda_d^s,$$

and for u^h the corresponding Lyapunov exponents satisfy

$$\lambda_1^h \geq ... \geq \lambda_k^h > 0 > \lambda_{k+1}^h > ... > \lambda_d^h;$$

(ii) all pairs $(u^h, x) \in \mathcal{U} \times \mathbb{R}^d$ with $x \neq x^*$ are strong inner pairs, i.e., $\varphi(t, x, y, u^h) \in \mathrm{int}\mathcal{O}^{y,+}(x)$ for all $t > 0$.

Then there exists a control set D^y with nonvoid interior such that $x^* \in \partial D^y$.

Using these results one observes in a number of control systems, e.g., in the Duffing-Van der Pol oscillator [**9**], that for some y-values the singular point x^* is exponentially stable for all controls, hence there are no control sets near x^*. Then, for increasing y-values, control sets D^y occur with $x^* \in \partial D^y$. For some upper y-value, they move away from x^*. These results refer to constant y-values only. Theorems 2 and 3 in the present paper show that also the systems with slowly increasing y-values have a similar controllability behavior.

REMARK 2. *Assumption (i) in Theorem 4 is in particular satisfied, if 0 is in the interior of the highest Floquet spectral interval and the corresponding subbundle is one-dimensional.*

REMARK 3. *Grünvogel [**9**] also shows that there are no control sets in a neighborhood of the origin if zero is not in the interior of the Morse spectrum of the linearized system. This also follows from a Hartman-Grobman Theorem for skew product flows; see Bronstein/Kopanskii [**3**]. One has to take into account that the spectral condition implies hyperbolicity, since the base space $\mathcal{U}$ is chain recurrent. Then the use of appropriate cut-off functions yields the desired local version.*

REMARK 4. *Using averaging techniques, Grammel/Shi [**8**] considered the stability behavior and the Lyapunov spectrum of bilinear control systems perturbed by a fast subsystem.*

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