Chain Recurrence, Growth Rates and Ergodic Limits

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Abstract: Averages of functionals along trajectories are studied by evaluating the averages along chains. This yields results for the possible limits and, in particular, for ergodic limits. Applications to Lyapunov exponents and to concepts of rotation numbers of linear Hamiltonian flows and of general linear flows are given.

Key Words: Rotation numbers, Lyapunov exponents, Morse spectrum

1 Introduction

The purpose of this paper is to expose a topological construction to obtain characteristics of dynamical systems via ergodic limits obtained by averages along trajectories. We replace limits along trajectories by limits along $(\varepsilon, T)-$chains as $\varepsilon \to 0$ and $T \to \infty$. Then the numbers which are defined by an ergodic limit procedure are replaced by intervals, each of them corresponding to a maximal chain transitive set, i.e., a chain recurrent component. Thus as a part of the construction the chain recurrent components of the relevant flow have to be determined.

In particular, we consider linear flows on vector bundles. Then, for Lyapunov exponents, the relevant flow is the induced flow on the projective bundle and the construction yields the Morse spectrum as introduced in [4]. For rotation numbers of linear Hamiltonian flows in $\mathbb{R}^{2n}$ (see, e.g., [7]) the induced flow on the $n-$dimensional Lagrange subspaces is the relevant one. For rotation numbers in the sense of L.A.B. San Martin [11] (for the stochastic case see also L. Arnold [1]) the induced flow on the bundle of oriented $2-$planes has to be considered. In all cases the corresponding maximal chain transitive sets are described, since they determine the numbers of intervals of Lyapunov exponents and of rotation numbers.

The contents of the paper are as follows: Section 2 presents the construction of limit intervals corresponding to chain transitive sets. Section 3 discusses rotation numbers for linear Hamiltonian flows and Section 4 discusses rotation numbers of oriented planes for arbitrary linear flows.

2 Growth Rates and Chain Transitivity

This section describes a general construction for flows on fiber bundles of metric spaces that relates growth rates along trajectories and, in particular, ergodic limits, to evaluations along chains.

In the following a (locally trivial) fiber bundle with typical fiber E is considered which is given by a continuous map $\pi : X \to B$ of metric spaces and a finite open cover (U_α) of B together with homeomorphisms

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times E,\ z \mapsto (\varphi_\alpha^1(z), \varphi_\alpha^2(z)),$$

satisfying $\varphi_\alpha^1(z) = \pi(z)$. We will always assume that the base space B is compact.

Let Φ be a (continuous) fiber preserving flow on the fiber bundle $\pi : X \to B$ given by $\Phi_t(x) = \Phi(t, x)$. By local triviality, Φ induces a flow $\pi\Phi$ on the base space which, for simplicity, is denoted by $b \cdot t,\ b \in B$ and $t \in \mathbb{R}$. Furthermore, let $f : X \to \mathbb{R}^m$ be continuous. We are interested in the growth rates along trajectories for $t \to \infty$ given by

$$\frac{1}{t}\left[f(\Phi(t,x)) - f(x)\right]. \tag{1}$$

We impose the following two basic assumptions:

The map

$$f_B : \mathbb{R} \times B \to \mathbb{R}^m,\ (t,b) \mapsto f(\Phi(t,x)) - f(x) \text{ with } \pi x = b, \tag{2}$$

is well defined, and the growth rates are uniformly bounded, i.e.,

$$C := \sup_{x\in X} \limsup_{t\to\infty} \frac{1}{t} \left| f(\Phi(t,x)) - f(x) \right| < \infty, \tag{3}$$

with some norm $|\cdot|$ in $\mathbb{R}^m$. Assumption (2) means that the difference $f(\Phi(t,x)) - f(x)$ is independent of the choice of x with $\pi x = b$. In view of (2), assumption (3) is equivalent to

$$C = \sup_{b\in B} \limsup_{t\to\infty} \frac{1}{t} \left| f_B(t,b) \right| < \infty.$$

Assumption (2) immediately implies that cluster points for $t \to \infty$ of (1) are independent of the element x in the fiber over πx.

Remark 2.1 *For a linear flow Φ on a vector bundle $\mathcal{V} \to \Omega$ with compact metric base space Ω write $\|v\| = \|(\omega,x)\| = \|x\|$. Then the Lyapunov exponent of v is given by the limit as $t \to \infty$ (if it exists) of $\frac{1}{t}\log\|\Phi(t,v)\|$. Clearly, $\mathcal{V}$ is not compact, while the projective bundle $\mathbb{P}\mathcal{V} \to \Omega$ is. With*

$$f : \mathcal{V} \to \mathbb{R},\ f(v) := \log\|v\|.$$

one sees that

$$f(\Phi(t,v)) - f(v) = \log\|\Phi(t,v)\| - \log\|v\|$$

only depends on t and $\mathbb{P}v \in \mathbb{P}\mathcal{V}$. Thus condition (2) is satisfied if we consider Φ as the fiber preserving flow in the fiber bundle $\mathcal{V} \to \mathbb{P}\mathcal{V}$ with base space $\mathbb{P}\mathcal{V}$. Furthermore, condition (3) is satisfied, since there are constants K,$\alpha > 0$ with

$$1/Ke^{-\alpha t}\|v\| \le \|\Phi(t,v)\| \le Ke^{\alpha t}\|v\|,\ t \in \mathbb{R}.$$

This implies for t large enough

$$\frac{1}{t}\left[\log\|\Phi(t,v)\| - \log\|v\|\right] \le \frac{1}{t}\log K + \alpha \le \alpha + 1,$$

and an analogous lower bound is valid.

Remark 2.2 *The general problem above includes ergodic limits on a compact metric space B. In fact, consider for a flow $b \mapsto b \cdot t : B \to B$ and a continuous functional $g : B \to \mathbb{R}$ ergodic limits for $t \to \infty$ of*

$$\frac{1}{t} \int_0^t g(b \cdot \tau) \, d\tau. \tag{4}$$

Define on the fiber bundle $\mathbb{R} \times B \to B$ the fiber preserving flow $\Phi_t(s, b) = (s - t, b \cdot t)$ and the functional

$$f : \mathbb{R} \times B \to \mathbb{R}, \; f(s, b) = \int_s^0 g(b \cdot \tau) \, d\tau.$$

Then for $x = (s, b)$

$$\begin{aligned} f(\Phi_t(x)) - f(x) &= f(s - t, b \cdot t) - f(s, b) \\ &= \int_{s-t}^0 g(b \cdot (t + \tau)) \, d\tau - \int_s^0 g(b \cdot \tau) \, d\tau \\ &= \int_s^t g(b \cdot \tau) \, d\tau - \int_s^0 g(b \cdot \tau) \, d\tau \\ &= \int_0^t g(b \cdot \tau) \, d\tau. \end{aligned}$$

Thus this difference is independent of the element (s, b) in the fiber over $b \in B$, as required in (2); furthermore, also (3) is satisfied, since g is bounded on the compact space B.

In order to describe the limiting behavior of (1) as $t \to \infty$, it is helpful to relax this problem: instead of studying directly the evaluation along trajectories, we study the evaluation along $(\varepsilon, T)-$chains, and then let $\varepsilon \to 0, \; T \to \infty$.

For an $(\varepsilon, T)-$chain ζ in B given by $n \in \mathbb{N}, b_0 = b, b_1, ..., b_n \in B, T_0, ..., T_{n-1} > T$ with

$$\mathrm{d}(b_i \cdot T_i, b_{i+1}) < \varepsilon \text{ for all } i = 0, ..., n - 1,$$

pick arbitrary points x_i in X with $\pi x_i = b_i$. Then

$$f(\zeta) := \left(\sum_{i=0}^{n-1} T_i \right)^{-1} \sum_{i=0}^{n-1} [f(\Phi(T_i, x_i)) - f(x_i)] \tag{5}$$

is independent of the choice of the x_i. An easy consequence of (3) is that for all $(\varepsilon, T)-$chains with T large enough

$$|f(\zeta)| < C + 1. \tag{6}$$

Recall that a set M is chain transitive, if for any two points $a, b \in M$ and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)-$chain from a to b. For a compact invariant chain transitive set $M \subset B$ define

$$F(M) = \left\{ \lambda \in \mathbb{R}^m, \quad \begin{array}{c} \text{there are } \varepsilon^k \to 0,\ T^k \to \infty \text{ and} \\ (\varepsilon^k, T^k) - \text{chains } \zeta^k \text{ in } M \text{ with } f(\zeta^k) \to \lambda \end{array} \right\}. \tag{7}$$

It is our aim to describe the set $F(M)$ and its relation to the asymptotic growth rates (1) as $t \to \infty$.

We note a number of observations.

Lemma 2.3 *The map f_B defined by (2) is continuous.*

Proof. By local triviality of the fiber bundle, there exists for $b \in B$ a local continuous section $\sigma : U \to X$ defined on a neighborhood U of b with $\sigma \circ \pi = \mathrm{id}_U$. Thus for $t_n \to t,\ b_n \to b$ one has with $x_n = \sigma(b_n)$

$$f_B(t_n, b_n) = f(\Phi(t_n, x_n)) - f(x_n) \to f(\Phi(t, \sigma(b))) - f(\sigma(b)) = f_B(t, b),$$

as claimed. ■

The following lemma (Lemma B.2.23 in [5]) gives a uniform upper bound for the time needed to connect any two points in a chain transitive set. For convenience of the reader we sketch the proof.

Lemma 2.4 *Let M be a compact invariant chain transitive set and fix ε, $T > 0$. Then there exists $\bar{T}(\varepsilon, T) > 0$ such that for all x, $y \in M$ there is an $(\varepsilon, T)-$chain from x to y with total length $\leq \bar{T}(\varepsilon, T)$.*

Proof. By assumption, one finds for all x, $y \in M$ an $(\frac{\varepsilon}{2}, T)-$chain in M from x to y. Using continuous dependence on initial values and compactness, one finds finitely many $(\varepsilon, T)-$chains connecting every $x \in M$ with a fixed $z \in M$. One also finds finitely many (modulo their endpoints) $(\varepsilon, T)-$chains connecting z with arbitrary elements $y \in M$. Thus one ends up with finitely many $(\varepsilon, T)-$chains connecting all points in M. The maximum of their total lengths is the desired upper bound $\bar{T}(\varepsilon, T)$. ■

The growth rates of concatenated chains are a convex combination of the individual growth rates.

Lemma 2.5 *Let ξ, ζ be $(\varepsilon, T)-$chains in B of total lengths σ and τ, respectively, such that the initial point of ζ coincides with the final point of ξ. Then for the concatenated chain $\zeta \circ \xi$ one has*

$$f(\zeta \circ \xi) = \frac{\sigma}{\sigma + \tau} f(\xi) + \frac{\tau}{\sigma + \tau} f(\zeta).$$

Proof. Let the chains ξ and ζ be given by $\pi x_0, ..., \pi x_k$ and $\pi y_0 = \pi x_k, ...,$ πy_n, with times S_0, ..., S_{k-1} and T_0, ..., T_{n-1}, respectively. Thus the total times are $\sigma = \sum_{i=0}^{k-1} S_i$ and $\tau = \sum_{i=0}^{n-1} T_i$ and

$$\begin{aligned}
& f(\zeta \circ \xi) \\
&= [\sigma + \tau]^{-1} \left[\frac{\sigma}{\sigma} \sum_{i=0}^{k-1} [f(\Phi(S_i, x_i)) - f(x_i)] + \frac{\tau}{\tau} \sum_{i=0}^{n-1} [f(\Phi(T_i, y_i)) - f(y_i)] \right] \\
&= [\sigma + \tau]^{-1} [\sigma f(\xi) + \tau f(\zeta)] \\
&= \frac{\sigma}{\sigma + \tau} f(\xi) + \frac{\tau}{\sigma + \tau} f(\zeta).
\end{aligned}$$

■

The following proposition shows that it is sufficient to consider periodic chains.

Proposition 2.6 *Let $M \subset B$ be a compact invariant chain transitive set. Then*

$$F(M) = \left\{ \begin{array}{c} \lambda \in \mathbb{R}^m, \text{ there are } \varepsilon^k \to 0,\, T^k \to \infty \text{ and periodic} \\ (\varepsilon^k, T^k)-\text{chains } \zeta^k \text{ in } M \text{ with } f(\zeta^k) \to \lambda \text{ as } k \to \infty \end{array} \right\}.$$

Proof. Let $\lambda \in F(M)$ and fix $\varepsilon, T > 0$. It suffices to prove that for every $\delta > 0$ there exists a periodic $(\varepsilon, T)-$chain ζ' with $|\lambda - f(\zeta')| < \delta$. By Lemma 2.4 there exists $\bar{T}(\varepsilon, T) > 0$ such that for all $\pi x, \pi y \in M$ there is an $(\varepsilon, T)-$chain ξ in M from πx to πy with total time $\leq \bar{T}(\varepsilon, T)$. For $S > T$ choose an $(\varepsilon, S)-$chain ζ with $|\lambda - f(\zeta)| < \frac{\delta}{2}$ given by, say, $\pi x_0, ..., \pi x_m$ with times S_0, ..., $S_{m-1} > S$ and with total time $\sigma = \sum_{i=0}^{m-1} S_i$. Concatenate this with an $(\varepsilon, T)-$chain ξ from πx_n to πx_0 with points $\pi y_0 = \pi x_n, ..., \pi\, y_m =$ πx_0, with times T_0, ..., $T_{m-1} > T$ and total time $\tau = \sum_{i=0}^{m-1} T_i \leq \bar{T}(\varepsilon, T)$. The periodic $(\varepsilon, T)-$chain $\zeta' = \xi \circ \zeta$ has the desired approximation property:

Since the chain ξ depends on ζ, also τ depends on ζ. However, the total length of ξ is bounded as $\tau = \tau(\xi) \le \bar{T}(\varepsilon, T)$. Lemma 2.5 implies

$$|f(\zeta) - f(\xi \circ \zeta)| = \left| f(\zeta) - \frac{\sigma}{\sigma+\tau} f(\zeta) - \frac{\tau}{\sigma+\tau} f(\xi) \right|$$
$$\le \left[1 - \frac{\sigma}{\sigma+\tau}\right] |f(\zeta)| + \frac{\tau}{\sigma+\tau} |f(\xi)|$$

By (6) there is a uniform bound for $|f(\xi)|$ and $|f(\zeta)|$ for all considered chains ξ and ζ. Since τ remains bounded for chains ζ with total length σ tending to ∞, the right hand side tends to 0 as $S \to \infty$. ■

The next theorem describes the set of asymptotic growth rates; part (ii) shows that the cluster points for chains in the chain recurrent set comprise all asymptotic growth rates for arbitrary initial points. Together with Proposition 2.6 it is our main result on growth rates.

Theorem 2.7 *(i) Assume that $M \subset B$ is a compact invariant set and that the flow $b \mapsto b \cdot t$ restricted to M is chain transitive. Then the set $F(M)$ is compact and convex.*
(ii) For all $x \in X$ and all sequences $t_k \to \infty$ we have

$$\lim_{k\to\infty} \frac{1}{t_k} f(\Phi(t_k, x)) \in F(\omega(\pi x)) \subset F(M),$$

if the limit exists; here M is the maximal chain transitive set containing $\omega(\pi x) \subset B$.

Proof. (i) The proof is based on a 'mixing' of growth rates. It is clear that $F(M)$ is closed; it is also bounded by (6), hence compact. Thus it suffices to show that for all $\lambda \in \mathrm{co}F(M)$, all $\delta > 0$, and all ε, $T > 0$ there is a periodic $(\varepsilon, T)-$chain ζ in M with

$$|f(\zeta) - \lambda| < \delta. \tag{8}$$

So let $\lambda = \sum_{i=0}^{N} \alpha^i \lambda^i$ with $\lambda^i \in F(M)$ and $\alpha^i > 0$, $\sum \alpha^i = 1$. For fixed $\delta > 0$ and ε, $T > 0$, there are periodic $(\varepsilon, T)-$chains ζ^i in M with

$$\left| f(\zeta^i) - \lambda^i \right| < \delta.$$

Denote the initial (and final) point of ζ^i by πx^i. By chain transitivity there are $(\varepsilon, T)-$chains ξ^i from πx^i to πx^{i+1} and ξ^N from πx^n to πx^0. For $k \in$

$\mathbb{N}$ let $\zeta^{i,k}$ be the $k-$fold concatenation of ζ^i. Then for $k^1, ..., k^N \in \mathbb{N}$ the concatenation $\zeta(k^1, ..., k^N) = \xi^N \circ \zeta^{N,k^N} \circ ... \circ \xi^1 \circ \zeta^{1,k^1}$ is a periodic $(\varepsilon, T)-$chain in M. Recall from Lemma 2.5 that $f-$values of concatenated chains are convex combinations of the individual $f-$values. Hence one finds numbers $k^1, ..., k^N \in \mathbb{N}$ such that $\left| f(\zeta(k^1, ..., k^N)) - \lambda \right| < \delta$. This proves (8).

(ii) Here we start from an *arbitrary* initial point $x \in X$ and have to show that the corresponding limit points can be approximated by chains in $\omega(\pi x)$ Recall that $\omega-$limit sets are connected and contained in the chain recurrent set. Hence the maximal chain transitive set $M \subset B$ containing $\omega(\pi x)$ is well defined for $x \in X$ and the inclusion is obvious. Thus it suffices to show the following:

Let $\lambda(x)$ be a cluster point of $\frac{1}{t} f(\Phi(t, x))$ for $t \to \infty$. For all $\delta > 0$ and all $\varepsilon > 0$, $T > 1$ there exists a periodic $(\varepsilon, T)-$chain ζ in $\omega(\pi x)$ with

$$|f(\zeta) - \lambda(x)| < \delta. \tag{9}$$

Fix $\delta > 0$, $\varepsilon > 0$, and $T > 1$. By uniform continuity of $\pi\Phi$ on the compact set $[0, 2T] \times B$, one finds $\delta_1 = \delta_1(\delta, \varepsilon, T) > 0$ such that for all $\pi y, \pi z \in B$ it follows from $d(\pi y, \pi z) < \delta_1$ that

$$d(\pi y \cdot t, \pi z \cdot t)) < \frac{\varepsilon}{3} \tag{10}$$

Invoking Lemma 2.3 one can also require that

$$|f(\Phi(t, y)) - f(y) - [f(\Phi(t, z)) - f(z)]| = |f_B(t, \pi y) - f_B(t, \pi z)| < \frac{\delta}{4} \tag{11}$$

holds for all $t \in [0, 2T]$. Furthermore, one may assume that

$$d(\pi x \cdot t, \omega(\pi x)) < \delta_1 \text{ for all } t > 0. \tag{12}$$

By definition there are $t_n \to \infty$ with

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{t_n} f(\Phi(t_n, x)),$$

where one can assume without loss of generality that $\pi x \cdot t_n$ converges for $t_n \to \infty$. Now fix $N \in \mathbb{N}$ such that for all $n \geq N$

$$d(\pi x \cdot t_n, \pi x \cdot t_N) < \frac{\varepsilon}{3}. \tag{13}$$

Setting $\sigma_n = t_n - t_N$ we have

$$\lambda(x) = \lambda(\Phi(t_N, x)) = \lim_{n\to\infty} \frac{1}{\sigma_n} f(\Phi(\sigma_n, \Phi(t_N, x))).$$

Choose n large enough such that with $T_0 = \sigma_n$ we have $T_0 > 2T$ and

$$\left| \lambda(x) - \frac{1}{T_0} f(\Phi(T_0, \Phi(t_N, x))) \right| < \frac{\delta}{2}. \tag{14}$$

Clearly, (12) remains valid, with $\Phi(t_N, x)$ instead of x. Hence writing x instead of $\Phi(t_N, x)$ in (13) and (14), we obtain in addition to (12)

$$d(\pi x \cdot T_0, \pi x) < \frac{\varepsilon}{3} \tag{15}$$

and

$$\left| \lambda(x) - \frac{1}{T_0} f(\Phi(T_0, x)) \right| < \frac{\delta}{2}. \tag{16}$$

We partition the interval $[0, T_0]$ into pieces of length τ_j with $T \le \tau_j < 2T$ for $j = 0, ..., l-1$. Thus

$$T_0 = \sum_{j=0}^{l-1} \tau_j \text{ and } T_0 \ge l. \tag{17}$$

Set $y_0 = x$ and $y_{j+1} = \Phi(\tau_j, y_j)$ for $j = 0, ..., l-1$. Then $\Phi(T_0, x) = y_l$ and

$$\begin{aligned} \frac{1}{T_0} f(\Phi(T_0, x)) &= \frac{1}{T_0} \sum_{j=0}^{l-1} [f(y_{j+1}) - f(y_j)] \qquad (18) \\ &= \left(\sum_{j=0}^{l-1} \tau_j \right)^{-1} \sum_{j=0}^{l-1} [f(\Phi(\tau_j, y_j)) - f(y_j)]. \end{aligned}$$

Define an $(\frac{\varepsilon}{3}, T)-$chain $\tilde{\zeta}$ in X by $l \in \mathbb{N}$, $\tau_0, ..., \tau_{l-1} \ge T$, $\pi y_0, ..., \pi y_{l-1}, \pi y_0 \in X$, noting that by (15) we have

$$d(\pi y_{l-1} \cdot \tau_{l-1}, \pi y_0) < \frac{\varepsilon}{3}. \tag{19}$$

Using (18) we obtain

$$f(\tilde{\zeta}) = \frac{1}{T_0} f(\Phi(T_0, x)). \tag{20}$$

However, the chain $\tilde{\zeta}$ is not necessarily contained in $\omega(\pi x)$. In order to obtain an appropriate chain ζ in $\omega(\pi x)$, we use (10) and (12): For $y_j = \Phi(\tau_j, y_{j-1})$, $j = 0, ..., l-1$, we find points $z_0, .., z_{l-1},\ z_l = z_0$ with $\pi z_i \in \omega(\pi x)$ with

$$d(\pi y_j \cdot t, \pi z_j \cdot t) < \frac{\varepsilon}{3} \text{ for } t \in [0, 2T]\,.$$

Hence we obtain for $j = 0, ..., l-1$

$$\begin{aligned}
&d(\pi z_j \cdot \tau_j, \pi z_{j+1}) \\
&\leq d(\pi z_j \cdot \tau_j, \pi y_j \cdot \tau_j) + d(\pi y_j \cdot \tau_j, \pi y_{j+1})) + d(\pi y_{j+1}, \pi z_{j+1})) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

where for $j = l-1$ we have used (19). Thus $l \in \mathbb{N}$, $\tau_1, ..., \tau_{l-1} \geq T$, and $\pi z_0, ..., \pi z_{l-1}$, $\pi z_l = \pi z_0 \in X$ define a periodic $(\varepsilon, T)-$chain in $\omega(\pi x)$. For the chain ζ the estimates (16) and (20) yield

$$\begin{aligned}
|\lambda(x) - f(\zeta)| &\leq \left|\lambda(x) - \tfrac{1}{T_0} f(\Phi(T_0, x))\right| + \left|f(\tilde{\zeta}) - f(\zeta)\right| \\
&< \tfrac{\delta}{2} + \tfrac{1}{T_0} \sum_{j=0}^{l-1} \left\{ f(\Phi(\tau_j, y_j)) - f(y_j) - [f(\Phi(\tau_j, z_j) - f(z_j)] \right\}.
\end{aligned}$$

Hence (11), (12), and (17) yield

$$|\lambda(x) - f(\zeta)| < \frac{\delta}{2} + \frac{1}{T_0} l \frac{\delta}{2} < \delta.$$

This proves (9) and concludes the proof of the theorem. ■

The special case of the average functional (4) has the advantage that one can also use methods from ergodic theory for its analysis leading to the following results.

Theorem 2.8 *Consider the average functional (4) given by*

$$\frac{1}{t} \int_0^t g(\Phi_s(x))\, ds,$$

where $g : B \to \mathbb{R}^m$ is continuous.
(i) Let ν be an ergodic measure and let M be the maximal chain transitive set containing the topological support of ν. Then

$$\nu(f) := \int f \, d\nu \in F(M).$$

(ii) Every extremal point of $F(M)$ is attained for an appropriate ergodic measure.

Proof. (i) follows from the fact

$$\int f \; d\nu = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\Phi_t(x)) \; dx$$

for almost all $\pi x \in \operatorname{supp}\nu$. The proof of (ii) follows the arguments in Johnson, Palmer and Sell [9]. ■

Remark 2.9 *For Lyapunov exponents of linear flows as discussed in Remark 2.1, we get back the main results on the Morse spectrum from [4] which is a special case of the construction above. Since by Selgrade's Theorem there are at most $d = \dim \mathcal{V}$ maximal chain transitive sets in the projective bundle $\mathbb{P}\mathcal{V}$, the Morse spectrum consists of at most d (possibly overlapping) compact intervals, each of them corresponding to a maximal chain transitive set in the projective bundle. If we use the integral representation for Lyapunov exponents (which is based on an embedding of the (continuous) linear flow Ψ into a subflow of a smooth linear flow), we also recover the results on the ergodic representation of the boundary points of the spectral intervals.*

Remark 2.10 *An analogous construction gives the Morse spectrum on flag bundles and on Grassmann bundles (instead of the projective bundle) as presented in [6].*

3 Rotation Numbers for Linear Hamiltonian Systems

In this section we will apply the general results above to linear Hamiltonian flows and rotation numbers. For the definition of the rotation number we follow the exposition in Fabbri, Johnson and Nunez [7, Section 2] without, however, specifying an ergodic measure on the base.

Consider a family of linear Hamiltonian systems

$$J\dot{z} = H(\xi \cdot t)z, \; \xi \in \Omega. \tag{21}$$

Here

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with the identity matrix I_n, Ω is a compact metric space with continuous flow σ abbreviated by $\sigma(\xi,t) = \xi \cdot t$; and H is a continuous $2n \times 2n$ matrix-valued function on Ω with values $H(\xi)$ in the real symmetric $2n \times 2n$ matrices (equivalently, $J^{-1}H \in \mathfrak{sp}(n,\mathbb{R})$, the algebra of infinitesimally symplectic matrices). The solutions induce a skew product flow on $\Omega \times \mathcal{L}_{\mathbb{R}}$ where $\mathcal{L}_{\mathbb{R}}$ is the space of all real $n-$dimensional Lagrangian planes of $\mathbb{R}^{2n}$. This space is a compact manifold of dimension $n(n+1)/2$. Let $U(t,\omega)$ be the fundamental solution of (21). Then $U(t,\omega)l_0 \in \mathcal{L}_{\mathbb{R}}$ for $l_0 \in \mathcal{L}_{\mathbb{R}}$. Hence we obtain a linear skew product flow Φ on $\mathcal{K}_{\mathbb{R}} = \Omega \times \mathcal{L}_{\mathbb{R}}$ given by

$$\Phi(t,\omega,l_0) = (\omega \cdot t, U(t,\omega)l_0), \; t \in \mathbb{R}. \tag{22}$$

Recall that the space $\mathcal{L}_{\mathbb{R}}$ of Lagrange planes can be identified with $U(n)/O(n)$. Following V.I. Arnold [2] define

$$\mathrm{Det}^2 : U(n)/O(n) \to \mathbb{S}^1, \; \mathrm{Det}^2(u \cdot O(n)) = -(\det u)^2.$$

Note that this is independent of the representation by u, since $u \cdot O(n) = u' \cdot O(n)$ implies $u = u'O$ for an orthogonal matrix O, and hence

$$(\det u)^2 = [\det(u'O)]^2 = [\det u' \det O]^2 = (\det u')^2.$$

Then, finally, for an ergodic measure μ on Ω define the *rotation number* α by

$$\alpha(\mu) = \frac{1}{2t} \lim_{t\to\infty} \arg \mathrm{Det}^2 U(t,\omega)l_0, \tag{23}$$

where arg is any argument of a complex number. Then $\mu-$almost everywhere this limit exists and is constant. (Note that the argument here is independent of the branch, but follows one of them; thus we may choose $\arg \mathrm{Det}^2 l_0$ as the principal value in $[0,2\pi)$.) Furthermore, the limit in (23) can also be written in the form (4) as an ergodic limit.

The limit in (23) fits into the framework of Section 2. Define the continuous function

$$f : X = \Omega \times \mathcal{L}_{\mathbb{R}} \to \mathbb{R}, \; f(\omega,l_0) = \arg \mathrm{Det}^2 l_0$$

Our previous results yield the following.

Corollary 3.1 *Consider a nonautonomous linear Hamiltonian differential system of the form (21) and the associated linear Hamiltonian skew product* *flow on $\mathcal{K}_{\mathbb{R}} = \Omega \times \mathcal{L}_{\mathbb{R}}$ given by (22). Then for every compact chain transitive set M of this flow the set of chain rotation numbers over M is a compact interval,*

$$\rho(M) = [\alpha^*(M), \alpha(M)] .$$

For every ergodic invariant measure ν on $\mathcal{K}_{\mathbb{R}}$ the corresponding rotation number satisfies

$$\alpha(\nu) = \lim_{t \to \infty} \arg \frac{1}{2t} \mathrm{Det}^2 U(t, \omega) l_0 \in \rho(M),$$

where M is the maximal chain transitive set containing the support of ν. Furthermore, the boundary points $\alpha^*(M)$ and $\alpha(M)$ are attained for certain ergodic invariant measures on $\mathcal{K}_{\mathbb{R}}$.

Remark 3.2 *In order to analyze rotation numbers corresponding to ergodic measures μ on the base space Ω, we have to lift them to ergodic measures ν on $\Omega \times \mathcal{L}_{\mathbb{R}}$ projecting down to μ. This is always possible, however, there may be several possibilities for ν (depending, in particular, on the choice of the maximal chain transitive set containing the support of ν).*

The next step is the classification of the maximal chain transitive sets in $\mathcal{K}_{\mathbb{R}}$. If the flow on Ω is chain transitive and *locally transitive* this has been done by Braga Barros and San Martin [3], who show in particular that their number is bounded by 2^n. Here is their setting:

The space $\mathcal{L}_{\mathbb{R}}$ of n-dimensional Lagrangian subspaces of $\mathbb{R}^{2n}$ is a compact manifold embedded in the Grassmannian $\mathrm{Gr}_n(2n)$ of n-dimensional subspaces of $\mathbb{R}^{2n}$. The group $\mathrm{Sp}(n, \mathbb{R})$ acts transitively on $\mathcal{L}_{\mathbb{R}}$. We cite the following definition.

Definition 3.3 *The local group of local homeomorphisms of a metric space X is locally transitive with parameters $c,\ \rho > 0$, if for every $x \in X$ and every y in the ball $\mathbf{B}(x, \rho)$ of radius ρ around x there is a local homeomorphism ξ of X with $\xi(x) = y$ and $\mathrm{d}(\xi, \mathrm{id}) = \sup \mathrm{d}(\xi(z), z) \leq c\mathrm{d}(x, y)$ where the supremum is taken over all elements in the domain of definition of ξ.*

Under the assumption of local transitivity the chain recurrent components of Φ can be described by fixed point sets (in $\mathcal{L}_{\mathbb{R}}$) of diagonalizable matrices in $\mathrm{Sp}(n, \mathbb{R})$. Consider a diagonal matrix of the form

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$$

with $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ a $n \times n$ diagonal matrix with $\lambda_i > 0$, $i = 1, \ldots, n$. Such a matrix belongs to $\text{Sp}\,(n, \mathbb{R})$. Put

$$\mathcal{O}\,(\Lambda) = \{gDg^{-1} : g \in \text{Sp}\,(n, \mathbb{R})\}$$

for the adjoint orbit of D. To have this orbit we can choose $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$ (this is because there are matrices $g \in \text{Sp}\,(n, \mathbb{R})$ such that the conjugation gDg^{-1} just permutes the entries of Λ). The chain recurrent components of Φ are described in terms of $\mathcal{O}\,(\Lambda)$ as follows [3].

Theorem 3.4 *Assume that the base flow on Ω is chain transitive and that the local group of local homeomorphisms on Ω is locally transitive. Then there exist $\Lambda_\Phi \in \text{Sp}\,(n, \mathbb{R})$ and a map $D_\Phi : \Omega \to \mathcal{O}\,(\Lambda_\Phi)$ such that any chain recurrent component has the form $\bigcup_{x\in\Omega} \text{fix}D_\Phi\,(\omega)$ where $\text{fix}D_\Phi\,(\omega)$ is a fixed point set of $D_\Phi\,(\omega)$ in $\mathcal{L}_\mathbb{R}$.*

The fixed point set which enter in each union is taken in a compatible way. The following example in the regular situation clarifies the meaning of compatible. Suppose that for a given flow Φ

$$\Lambda_\Phi = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$$

is such that $\lambda_1 > \cdots > \lambda_n \geq 1$. Denote by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be the basis which diagonalizes $D_\Phi(\omega)$. There are 2^n fixed points of $D_\Phi(\omega)$ in $\mathcal{L}_\mathbb{R}$, namely, the n-dimensional Lagrangian subspaces spanned by basic elements (that is, $\text{span}\{e_{i_1}, \ldots, e_{i_k}\} \cup \{f_{j_1}, \ldots, f_{j_{n-k}}\}$ with $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_{n-k}\} = \emptyset$). Thus there are 2^n chain transitive components of the flow in $\mathcal{L}_\mathbb{R}$. To see how they are built above the base space Ω note that Λ_Φ has just one attractor, namely $\text{att}\,(\Lambda_\Phi) = \text{span}\{e_1, \ldots, e_n\}$. The same way $D_\Phi\,(\omega)$ has also one attractor $\text{att}\,(D_\Phi\,(\omega))$. This way,

$$\bigcup_{\omega\in\Omega} \text{att}\,(D_\Phi\,(\omega))$$

is a chain transitive component, which turns out to be an attractor of Φ. Also, Λ_Φ (and each $D_\Phi\,(\omega)$) has just one repeller ($\text{span}\{f_1, \ldots, f_n\}$) fixed-point and these repellers are combined together to give a repeller component of Φ.

Analogously, the other fixed points of $D_{\Phi}(\omega) \in \mathcal{O}(\Lambda_{\Phi})$, $\omega \in \Omega$, can be labelled (in terms of bases like $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$) in a consistent way to give all the 2^n components.

In case Λ_{Φ} is not regular (that is, there is repetition of eigenvalues) the picture is similar. The main difference is that a repetition of the eigenvalues forces a collapsing of the components.

This implies the following result.

Corollary 3.5 *Consider a nonautonomous linear Hamiltonian differential system (21) and the associated linear Hamiltonian skew product flow $\mathcal{K}_{\mathbb{R}}$ given by (22). Assume that the base flow on Ω is chain transitive and that the local group of local homeomorphisms on Ω is locally transitive. Then there are at most 2^n compact intervals of rotation numbers, each of them corresponding to a chain recurrent component M of the flow Φ on $\mathcal{K}_{\mathbb{R}}$.*

Since the rotation number is specified by the behavior of a Lagrange plane, the corresponding Lyapunov exponent also has to be specified for Lagrange planes. Thus one has to look at the growth rates in the corresponding exterior product space. For the Morse spectrum this was analyzed in [6]. One can either study these Lyapunov exponents separately; or, and this appears to be more adequate, study the exponential growth rates (of the Lagrange planes) and the rotation numbers *simultaneously*. Thus we define the continuous map

$$f : X = \mathcal{K}_{\mathbb{R}} = \Omega \times \mathcal{L}_{\mathbb{R}} \to \mathbb{R}^2,\ \ f(\omega, l_0) = (\log |l_0|, \arg \mathrm{Det}^2 l_0). \tag{24}$$

Conditions (2) and (3) are satisfied, and we obtain the following result.

Corollary 3.6 *Consider a nonautonomous linear Hamiltonian differential system of the form (21) and the associated linear Hamiltonian skew product flow on $\mathcal{K}_{\mathbb{R}}$ given by (22). Assume that the base flow on Ω is chain transitive and that the local group of local homeomorphisms on Ω is locally transitive. Then, for each of the at most 2^n chain recurrent component M_i of the flow Φ on $\mathcal{K}_{\mathbb{R}}$, the set*

$$F(M_i) = \left\{ (\lambda, \alpha) \in \mathbb{R}^2,\ \ \begin{array}{c} \textit{there are } \varepsilon^k \to 0,\ T^k \to \infty \textit{ and} \\ (\varepsilon^k, T^k) - \textit{chains } \zeta^k \textit{ with } f(\zeta^k) \to (\lambda, \alpha) \end{array} \right\}$$

is a compact and convex set. For each ergodic measure ν on the base Ω there is a chain recurrent component M in $\mathcal{K}_{\mathbb{R}}$ such that the corresponding

Lyapunov exponent and the rotation number

$$\lambda(\nu) = \lim_{t\to\infty} \frac{1}{t} \log |\Phi(t,\omega)l_0| \,,\ \alpha(\nu) = \lim_{t\to\infty} \arg \frac{1}{2t} \mathrm{Det}^2 U(t,\omega)l_0,$$

respectively, satisfy

$$(\lambda(\nu), \alpha(\nu)) \in F(M).$$

Remark 3.7 *It may appear more natural to consider f in (24) as a map into the complex numbers $\mathbb{C}$ where the Lyapunov exponent is the real part and the rotation number is the imaginary part. Thus $F(M_i)$ is a compact convex subset of $\mathbb{C}$.*

Remark 3.8 *Braga Barros and San Martin [3] show that local transitivity holds, e.g., if X is a compact Riemannian manifold. It is not clear if local transitivity holds for the closure of an almost periodic function or for the set $\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U$ for almost all $t \in \mathbb{R}\}$ of functions with values in a compact and convex set $U \subset \mathbb{R}^m$, endowed with the weak* topology on L_∞. The latter space is of relevance in control theory (cp. [5]).*

Remark 3.9 *If we omit the assumption of local transitivity, we conjecture that the methods from Salamon/Zehnder [10] combined with [6] (concerning general linear flows on vector bundles) can be modified so that they give existence of a finest Morse decomposition for the flow on Lagrange planes. Thus it would follow that the number of maximal chain transitive sets is finite.*

Remark 3.10 *A possible application of the framework developed above concerns autonomous linear Hamiltonian systems of the form*

$$J\dot{z} = H_0 z,$$

and their nonautonomous perturbations

$$J\dot{z} = [H_0 + \textstyle\sum_{i=1}^m H_i u_i(t)]z,\ u(t) \in \rho U,$$

where $\rho \geq 0$ is a parameter and U is a compact and convex subset of $\mathbb{R}^m$. Then we can take as the base flow the shift on the space $\mathcal{U}$ of perturbation (or control) functions. One will expect semicontinuity results of rotation numbers and Lyapunov exponents in dependence on ρ. Under an inner-pair condition (cp. Gayer [8] for methods to verify it) also continuous dependence on ρ may be expected.

4 Rotation Numbers in Planes

An alternative generalization of rotation numbers has been proposed by L.A.B. San Martin [11]. It relies on the restriction of the flow to oriented planes. We follow here the presentation in L. Arnold [1].

Recall that the free operation $G \times E \to E$ of a finite group G on a Hausdorff space E defines a principal $G-$bundle (cp., e.g., tom Dieck [12], Beispiel 6.5)

$$p : E \to E/G.$$

For rotation numbers the Grassmann manifold $\mathbb{G}_2^+(d)$ of oriented $2-$planes has to be considered. It is a twofold covering of the Grassmann manifold $\mathbb{G}_2(d)$. It is helpful to consider also the Stiefel manifold $\mathrm{St}_2(d)$ of orthonormal $2-$frames which is a principal bundle over $\mathbb{G}_2(d)$ with structure group $SO(2,\mathbb{R})$. One also has the principal fiber bundle over the $(1,2)-$flags

$$\mathrm{St}_2(d) \to \mathbb{F}_{(1,2)}. \tag{25}$$

with structure group $G \cong \mathbb{Z}_2^2$, the Kleinian group of four elements. More explicitly, for a flag $(V_1 \subset V_2) \in \mathbb{F}_{(1,2)}$ an element $n = (u,v) \in \mathrm{St}_2(d)$ is given by $u \in V_1$ with $\|u\| = 1$ and $v \in V_2$ with $\|v\| = 1$ and $\langle u, v \rangle = 0$. These two vectors give rise to four orthonormal frames, namely $n_1 = n = (u,v), n_2 = (-u,v), n_3 = (-u,-v), n_4 = (u,-v)$. Here n_1 and n_3 have the same orientation, and n_2 and n_4 also have the same, but opposite, orientation. The action of $G = \{g_1, g_2, g_3, g_4\}$ is defined by $g_i n = n_i,\ i = 1, ..., 4$. Thus (25) is a fourfold covering.

This generalizes to vector bundles

$$\pi : \mathcal{V} = \mathbb{R}^d \times \Omega \to \Omega$$

with fibers $\mathbb{R}^d \times \{\omega\}$ where $\mathbb{R}^d$ is endowed with the Euclidean inner product. Along with $\mathcal{V}$ come the Grassmann bundles $\mathbb{G}_2(\mathcal{V}) = \mathbb{G}_2(d) \times \Omega$ and $\mathbb{G}_2^+(\mathcal{V}) = \mathbb{G}_2^+(d) \times \Omega$, the Stiefel bundle $\mathrm{St}_2(\mathcal{V}) = \mathrm{St}_2(d) \times \Omega$, and the flag bundle $\mathbb{F}_{(1,2)}(\mathcal{V}) = \mathbb{F}_{(1,2)}(d) \times \Omega$.

Remark 4.1 *Analogous definitions can be given for the Stiefel manifold over a Riemannian manifold (cp. Arnold [1]). Note that here the rotation number (to be defined below) depends on the Riemannian metric.*

Let Φ be a smooth linear flow Φ on $\mathcal{V}$. This induces smooth flows on the oriented $2-$planes $\mathbb{G}_2^+(\mathcal{V})$ and on the flag bundle $\mathbb{F}_{(1,2)}(\mathcal{V})$. It also induces a smooth flow on the Stiefel bundle which is defined as follows: Take a frame $n = (u, v)$ and $\omega \in \Omega$. Then for $t \in \mathbb{R}$ define the image at time t as the orthonormalized pair

$$\perp(\Phi_t u, \Phi_t v) = \left(\frac{\Phi_t u}{\|\Phi_t u\|}, \frac{\Phi_t v - \langle \Phi_t v, \Phi_t u \rangle \Phi_t u}{\|\Phi_t v - \langle \Phi_t v, \Phi_t u \rangle \Phi_t u\|} \right).$$

This defines a flow on $\mathrm{St}_2(\mathcal{V})$ which, as those on the Grassmann bundles and on the flag bundle, we also denote for simplicity by Φ. Then

$$\mathrm{St}_2(\mathcal{V}) \to \mathbb{F}_{(1,2)}(\mathcal{V})$$

is a fiber bundle with structure group $G = \mathbb{Z}_2^2$ and compact base space. The induced flows are equivariant with respect to G.

The next proposition relates the maximal chain transitive sets in the bundle and in the base.

Proposition 4.2 *Consider a principal $G-$fiber bundle $\pi : E \to B$ with finite structure group G, $|G| = l$, and assume that B is compact. Let Φ be a fiber preserving flow on E which is equivariant with respect to G. Then for every compact maximal chain transitive set M_B of the induced flow on the base B there are $k \in \{1, ..., l\}$ maximal chain transitive sets $M_i \subset E$ of Φ with $\pi(M_i) = M_B$ and every compact maximal chain transitive set of Φ is of this form.*

Proof. Consider a maximal chain transitive set $M_B \subset B$ and let $b \in M_B$. Fix $\varepsilon > 0$ and consider for b and $b' \in M_B$ the sets $\pi^{-1}(b) = \{e_1, ..., e_l\}$ and $\pi^{-1}(b') = \{e'_1, ..., e'_l\}$. Then for $\delta > 0$, small enough, a $\delta-$chain in M_B from b to b' gives rise to $\varepsilon-$chains from each of the e_i to some $e'_{j(i)}$. Since $\pi^{-1}(M_B)$ is compact invariant, it contains a maximal chain transitive set which then is invariant. Hence there is a maximal chain transitive set M_E in $\pi^{-1}(M_B)$ which projects to M_B. By uniform continuity, a chain transitive set in E projects down to a chain transitive set in B. Thus, by maximality of M_B, the set M_E is also a maximal chain transitive set in E (not just in $\pi^{-1}(M_B)$). For every $g \in G$ the set gM_E is also a maximal chain transitive set.

Every maximal chain transitive set maps down into a maximal chain transitive set in B. The arguments above show that it is of the form above and hence the projection is onto. Thus the assertion follows. ■

As a consequence of this result one can give estimates on the number of maximal chain transitive sets in Stiefel bundles and in oriented $2-$plane bundles.

Corollary 4.3 *Let Φ be a linear flow on a $d-$dimensional vector bundle $\pi : \mathcal{V} = \mathbb{R}^d \times \Omega \to \Omega$ with compact chain transitive base space Ω and let l be the number of chain recurrent components of the induced flow on the flag bundle $\mathbb{F}_{(1,2)}(\mathcal{V})$. Then $l \leq d(d-1)$ and the numbers of chain recurrent components of the induced flows on the Stiefel bundle $\mathrm{St}_2(\mathcal{V})$ and on the oriented Grassmann bundle $\mathbb{G}_2^+(\mathcal{V})$ are bounded above by $4l$ and $2l$, respectively.*

Proof. By [6], Theorem 5 the number of chain recurrent components in $\mathbb{F}_{(1,2)}(\mathcal{V})$ is at most $d(d-1)$. Thus the estimate in the Stiefel bundle follows from Proposition 4.2 and the fact that the structure group has four elements. By [6], Proposition 2 the number of chain recurrent components in $\mathbb{G}_2(\mathcal{V})$ is at most $d(d-1)$. Thus the estimate in $\mathbb{G}_2^+(\mathcal{V})$ follows from Proposition 4.2 and the fact that the structure group has two elements. ■

Remark 4.4 *If the base space Ω satisfies the local transitivity condition, see Definition 3.3, of Braga Barros/San Martin [3], then one will be able to give more precise estimates for l using the theory of semi-simple Lie groups.*

Now the rotation number (cp. [1]) can be defined in the following way.

Definition 4.5 *For a linear differential equation $\dot{x}(t) = A(t)x(t)$ in $\mathbb{R}^d$, with $d \geq 2$ and locally integrable $t \mapsto A(t)$, the rotation number of the oriented $2-$plane $p \in \mathbb{G}_2^+(d)$ is*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \langle A(t)\Phi_t^u n, \Phi_t^v n \rangle \, dt,$$

(provided the limit exists) where $n(t) = (\Phi_t^u n, \Phi_t^v n) = \Phi_t(n)$ is the flow induced in $\mathrm{St}_2(d)$ with arbitrary initial frame $n = (u, v) \in \mathrm{St}_2(d)$ in the fiber over p;

Note that the limit (if it exists) is independent of the choice of n; if an ergodic measure in the base space is given and additional assumptions are satisfied, existence of the limit can be guaranteed almost everywhere (cp. [1]). The chain construction introduced above leads us to the following definition.

Consider a smooth linear flow Φ on a vector bundle $\mathcal{V} = \mathbb{R}^d \times \Omega$ with compact base space Ω and let

$$\frac{d}{dt}\Phi_t(x,\omega) = A(\omega \cdot t)\Phi_t(x,\omega),\ (x,\omega) \in \mathbb{R}^d \times \Omega,$$

where $\omega \cdot t$ denotes the induced flow on the base. For a chain ζ in $\mathrm{St}_2(\mathcal{V})$ given by

$$(u_0, v_0), .., (u_n, v_n) \in \mathrm{St}_2(\mathcal{V}),\ T_0, ..., T_n > T.$$

define the rotation number of ζ as

$$\rho(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} \int_0^{T_i} \langle A(t)\Phi_t^u n_i, \Phi_t^v n_i\rangle \, dt.$$

Recall that the chain recurrent components in $\mathrm{St}_2(\mathcal{V})$ project down to the chain recurrent components in $\mathbb{G}_2^+(\mathcal{V})$.

Definition 4.6 *For a chain recurrent component M of the induced flow on the Grassmann bundle $\mathbb{G}_2^+(\mathcal{V})$ of oriented $2-$planes define the chain rotation numbers as*

$$\rho(M) = \left\{ \rho \in \mathbb{R}, \begin{array}{c} \text{there are } \varepsilon^k \to 0,\ T^k \to \infty \text{ and} \\ (\varepsilon^k, T^k)\text{−chains } \zeta^k \text{ with } \rho(\zeta^k) \to \lambda \end{array} \right\};$$

here the chains ζ^k are taken in the chain recurrent components in $\mathrm{St}_2(\mathcal{V})$ *which project down to M.*

The general theory above and the estimates on the numbers of chain recurrent components in $\mathbb{G}_2^+(\mathbb{V})$ imply the following result.

Theorem 4.7 *Let Φ be a smooth linear flow on a vector bundle $\mathbb{R}^d \times \Omega$ with compact and chain transitive base space Ω. Then, for every compact chain transitive set M of the induced flow on the Grassmann bundle $\mathbb{G}_2^+(\mathcal{V})$, the set of chain rotation numbers over M is a compact interval*

$$\rho(M) = [\rho^*(M), \rho(M)].$$

For every ergodic invariant measure ν on $\mathbb{G}_2^+(\mathcal{V})$ with support contained in the maximal chain transitive set M, the corresponding rotation number satisfies

$$\rho(\nu) = \lim_{t\to\infty} \frac{1}{T} \int_0^T \langle A(t)\Phi_t^u n, \Phi_t^v n\rangle \, dt \in \rho(M). \tag{26}$$

Furthermore, the boundary points $\rho^*(M)$ and $\rho(M)$ are attained for certain ergodic invariant measures. There are k intervals of rotation numbers with $k \in \{1, ..., 2d(d-1)\}$.

Proof. For maximal chain transitive sets in the Stiefel bundle $\mathrm{St}_2(\mathcal{V})$, Corollary 4.3 and Theorem 2.8 yield the assertions above. It remains to show that for every chain recurrent component in $\mathrm{St}_2(\mathcal{V})$ projecting down to M, one obtains the same interval I. Let ρ^* be a boundary point of I. Then ρ^* is attained in an ergodic measure on $\mathrm{St}_2(\mathcal{V})$. Hence there is an orthonormal frame in $\mathrm{St}_2(\mathcal{V})$, for which the limit in (26) exists and coincides with ρ^*. Then it follows (cp. [1]) that the limit does not change if we start in another orthonormal frame in the same fiber over $\mathbb{G}_2^+(\mathcal{V})$. Thus the boundary point ρ^* is also attained for each other chain recurrent component in $\mathrm{St}_2(\mathcal{V})$ projecting down to M. ∎

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