Normal Forms for Control Systems at Singular Points

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December 11, 2002

Abstract

A normal form for open loop control systems is provided, based on their interpretation as skew product flows and on normal forms for nonautonomous differential equations.

1 Introduction

In this paper we derive normal forms for the following families of ordinary differential equations

$$\dot{y}(t) = f(y(t), u(t)) = f_0(y(t)) + \sum_{i=1}^{m} u_i(t) f_i(y(t)) \quad \text{in } \mathbb{R}^d \tag{1}$$
$$u \in \mathcal{U} = \{u : \mathbb{R} \to U \subset \mathbb{R}^m, \text{ locally integrable}\}.$$

The time-dependent term u which appears affinely in the system equation, may be interpreted as an (open loop) control or as a perturbation. We restrict our attention to the behavior near a singular point y^0, i.e., $f_i(y^0) = 0$ for $i = 0, 1, ..., m$. Normal forms for control systems, where the equivalence relation also allows for feedbacks, are a classical topic in control theory. In contrast to other work, the notion of normal forms developed here does not allow for feedbacks. Instead the admissible transformations have to depend continuously, in a sense specified below, on the control function u. The control system is viewed as a skew product flow over the base space of control functions endowed with the shift. This allows us to use recent results on normal forms for nonautonomous differential equations (Siegmund [6]). Then conjugacies eliminate all nonresonant terms in the Taylor expansion without changing the other terms up to the same order. There is also related work in the theory of random dynamical systems which can be considered as skew product flows, with an invariant measure on the base space; compare L. Arnold [2]. Our primary concern is the classification of changes in the controllability behavior. In particular, in the work of Grünvogel [4] bifurcations of control sets from a singular point have been studied. It is our hope that the normal form theory developed here will lead to a classification of this bifurcation behavior. Below we provide an example which illustrates this point.

In section 2, we formulate the normal form problem in our context and state some basic assumptions. Section 3 presents the normal form theorem and its proof, while section 4 discusses two examples.

2 Assumptions and Problem Formulation

In this section we collect some basic assumptions and notions and pose the normal form problem considered here.

We consider the control affine systems (1) and assume that the control range is a compact and convex set $U \subset \mathbb{R}^m$ containing 0. We also assume that $f_0, \ldots, f_m$ are C^k vector fields for a $k \geq 2$, and that for all $(y,u) \in \mathbb{R}^d \times \mathcal{U}$, equation (1) has a unique solution $\phi(t,y,u)$, $t \in \mathbb{R}$, with $\phi(0,y,u) = y$. Then (compare [3, Chapter 4]) the following skew product flow on $\mathbb{R}^d \times \mathcal{U}$,

$$\mathbf{\Psi} : \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d \times \mathcal{U}, \ \mathbf{\Psi}_t(y,u) := (\theta_t u, \phi(t,y,u)), \tag{2}$$

is associated with (1); here $\theta_t : \mathcal{U} \to \mathcal{U}$ is the shift $(\theta_t u)(s) = u(t+s)$, $s \in \mathbb{R}$, and $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m) = (L_1(\mathbb{R}, \mathbb{R}))^*$ is endowed with the weak* topology. Then $\mathcal{U}$ becomes a compact metrizable space with a corresponding metric. The flow Ψ, called the control flow, is continuous and, for fixed $u \in \mathcal{U}$, it is k times continuously differentiable with respect to y.

Let $y^0 \in \mathbb{R}^d$ be a singular point of (1), i.e., $f_i(y^0) = 0$ for all $i = 0, \ldots, m$. Our notion of conjugacies which, naturally, depend on u is specified in the following definition.

Definition 1 *Let $\phi : \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$ and $\psi : \mathbb{R} \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$ be two control systems with singular point $y^0 = 0$. Then ϕ and ψ (respectively their generators, which are systems of the form (1)) are said to be C^k* conjugate *if there exists a bundle mapping*

$$\mathbb{R}^d \times \mathcal{U} \ni (x,u) \mapsto (H(x,u), u) \in \mathbb{R}^d \times \mathcal{U}$$

which preserves the zero section $\{0\} \times \mathcal{U}$, such that

(i) $x \mapsto H(x,u)$ is a local C^k diffeomorphism (near $y^0 = 0$) for each fixed $u \in \mathcal{U}$ (with inverse denoted by $y \mapsto H(y,u)^{-1}$),

(ii) $(x,u) \mapsto H(x,u)$ *and* $(y,u) \mapsto H(y,u)^{-1}$ *are continuous,*

(iii) for all $t \in \mathbb{R}$*,* $x \in \mathbb{R}^d$ *and* $u \in \mathcal{U}$ *the conjugacy*

$$\psi(t,u,H(x,u)) = H(\theta_t u, \phi(t,x,u))$$

holds.

Remark 2 *Definition 1 is a semi local definition in the sense that the conjugacy is defined globally on the bundle* $\mathbb{R}^d \times \mathcal{U}$ *and is a fiberwise* C^k *diffeomorphism locally in the vicinity of the zero section. One could give a more technical definition of a local conjugacy which is defined on some subset of* $\mathbb{R}^d \times \mathcal{U}$*.*

Remark 3 *The conjugacy condition is equivalent to commutativity of the diagram*

$$\begin{array}{ccc} \mathbb{R}^d \times \mathcal{U} & \xrightarrow{\mathbf{\Phi}_t} & \mathbb{R}^d \times \mathcal{U} \\ \downarrow & & \downarrow \\ \mathbb{R}^d \times \mathcal{U} & \xrightarrow{\mathbf{\Psi}_t} & \mathbb{R}^d \times \mathcal{U} \end{array}$$

where the vertical errors indicate the conjugacy and Φ *and* Ψ *are the corresponding control flows.*

Next we discuss the Taylor expansions and the terms which are to be eliminated by conjugacies. The system linearized at y^0 has the form

$$\dot{x} = A(u(t))x(t) = [A_0 + \sum_{i=1}^{m} u_i(t)A_i]x(t) \quad \text{in } \mathbb{R}^d, u \in \mathcal{U}, \tag{3}$$

where $A_i = \frac{\partial}{\partial y} f_i(y)|_{y=y^0}$ denote the Jacobians at y^0. We rewrite (1) in the form

$$\dot{y} = A(u(t))y(t) + F(y(t),u(t)), \tag{4}$$

where

$$F(y(t),u(t)) = f_0(y(t)) - A_0 y(t) + \sum_{i=1}^{m} u_i(t)(f_i(y(t)) - A_i y(t))$$

denotes the nonlinearity.

In the following we assume that the linearized system is in block diagonal form and that the nonlinearity is C^k-bounded. More precisely we assume

(A1) $A = \operatorname{diag}(A^{(1)}, \ldots, A^{(n)})$ with n, $1 \le n \le d$, blocks $A^{(i)} : U \to \mathbb{R}^{d_i \times d_i}$, $d_1 + \cdots + d_n = d$.

(A2) $\|D_y^i F(y^0,u)\| \le M$ for $i = 1,\ldots,k$, $u \in \mathcal{U}$, with a constant $M > 0$.

The block diagonalization of the linearized system into the matrices $A^{(i)}$ corresponds to a decomposition of $\mathbb{R}^d$ into d_i-dimensional subspaces. Corresponding to the block diagonal structure of A one can write $y = (y^{(1)}, \ldots, y^{(n)}) \in \mathbb{R}^d$

and $F = (F^{(1)}, \ldots, F^{(n)})$ with the component functions $F^{(i)} : \mathbb{R}^d \times U \to \mathbb{R}^{d_i}$. For a multi index $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}_0^\ell$ let $|\ell| = \ell_1 + \cdots + \ell_n$ denote the *order* and define

$$D_y^\ell F = D_{y^{(1)}}^{\ell_1} \cdots D_{y^{(n)}}^{\ell_n} F \quad \text{and} \quad y^\ell = \underbrace{y^{(1)} \cdots y^{(1)}}_{\ell_1\text{-times}} \cdots \underbrace{y^{(n)} \cdots y^{(n)}}_{\ell_n\text{-times}} .$$

Now we can expand $F(\cdot, u(t))$ into a Taylor-series at y^0

$$F(y, u(t)) = \sum_{\ell \in \mathbb{N}_0^n \,:\, 2 \le |\ell| \le k} \frac{1}{\ell!} D_y^\ell F(y^0, u(t)) \cdot (y - y^0)^\ell + o(\|y - y^0\|^k) ,$$

where $\ell! = \ell_1! \cdots \ell_n!$. For simplicity we assume without loss of generality that $y^0 = 0$. We are looking for a condition which ensures the existence of a C^k conjugacy which eliminates the j-th component $D_y^\ell F^{(j)}(0, u(t)) \cdot y^\ell$ of a summand in the Taylor expansion of F.

Let $\Phi = \operatorname{diag}(\Phi^{(1)}, \ldots, \Phi^{(n)})$ denote the solution of the linearized system (3), i.e., $\Phi^{(i)}(t, u) x^{(i)}$ solves the control system

$$\dot{x}^{(i)}(t) = A^{(i)}(u(t)) x^{(i)}(t) \quad \text{in } \mathbb{R}^{d_i},\ u \in \mathcal{U},$$

with $\Phi^{(i)}(0, u) x^{(i)} = x^{(i)}$. In order to specify the nonresonance condition, we associate to each $\Phi^{(i)}$ an interval $\lambda_i = [a_i, b_i]$ such that for every $\varepsilon > 0$

$$\|\Phi^{(i)}(s, u)^{-1}\| \le K e^{-(a_i - \varepsilon)s} \text{ and } \|\Phi^{(i)}(s, u)\| \le K e^{(b_i + \varepsilon)s} \quad \text{for } s \ge 0,\ u \in \mathcal{U}, \tag{5}$$

with a $K = K(\varepsilon) > 0$.

Remark 4 *These intervals can be obtained in the following way. Choose*

$$\lambda_i := [\min \mathcal{S}_{\text{dyn}}(\Phi^{(i)}), \max \mathcal{S}_{\text{dyn}}(\Phi^{(i)})]$$

with the dynamical spectrum *of $\Phi^{(i)}$ (see Sacker and Sell [5])*

$$\mathcal{S}_{\text{dyn}}(\Phi^{(i)}) := \{\gamma \in \mathbb{R} : \Phi_\gamma^{(i)} \textit{ admits no exponential dichotomy over } \mathcal{U}\}.$$

Here $\Phi_\gamma^{(i)}(t, u) := e^{-\gamma t} \Phi^{(i)}(t, u)$ admits an exponential dichotomy over $\mathcal{U}$ if and only if there exist a continuous family $\mathcal{U} \ni u \mapsto P(u) \in \mathbb{R}^{d \times d}$ of projections and positive constants K and α such that

$$\begin{aligned} \|\Phi_\gamma^{(i)}(t, u) P(u) \Phi_\gamma^{(i)}(s, u)^{-1}\| &\le K e^{-\alpha(t-s)} && \textit{for } t \ge s , \\ \|\Phi_\gamma^{(i)}(t, u) [I - P(u)] \Phi_\gamma^{(i)}(s, u)^{-1}\| &\le K e^{\alpha(t-s)} && \textit{for } t \le s . \end{aligned}$$

A related way is to consider the system in projective space induced by the linearized system (3). Then (see [3]) the Morse spectrum associates to each chain recurrent component of the associated control flow (i.e., to each chain control set) an interval λ_i. The unions of overlapping intervals coincide with the dynamical spectrum. In any case one has to check if the associated subbundles yield a block diagonalization of the linearized system.

3 Normal Forms

In this section we state and prove the main result of this paper, a normal form theorem for control systems at a singular point. It shows that nonresonant terms in the Taylor expansion can be eliminated without changing the coefficients up to the same order.

For compact sets K_1, $K_2 \subset \mathbb{R}$ and integers $\ell_1, \ell_2 \in \mathbb{Z}$ we define the compact set $\ell_1 K_1 + \ell_2 K_2 := \{\ell_1 k_1 + \ell_2 k_2 : k_i \in K_i\}$ and we write $K_1 < K_2$ iff $\max K_1 < \min K_2$ and similarly for $K_1 > K_2$.

Theorem 5 *Consider a class of C^k control affine systems (1) satisfying assumptions (A1) and (A2). Suppose that to each block an interval λ_i is associated with property (5). Let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}_0^n$ be a multi index of order $2 \leq |\ell| \leq k$ and assume that the nonresonance condition*

$$\lambda_j < \sum_{i=1}^{n} \ell_i \lambda_i \qquad \text{or} \qquad \lambda_j > \sum_{i=1}^{n} \ell_i \lambda_i$$

holds. Then define $H(x,u) = x + h(x,u)$ with $h = (h^{(1)}, \ldots, h^{(n)}) : \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}^d$ and

$$h^{(i)}(x,u) = \begin{cases} 0 & \text{if } i \neq j\,, \\ \int_0^\infty \Phi^{(j)}(s,u)^{-1} \frac{1}{\ell!} D_x^\ell F^{(j)}(0,u(s)) \cdot [\Phi(s,u)x]^\ell \, ds, & \\ \qquad \text{if } i = j \text{ and } \lambda_j > \sum_{i=1}^{n} \ell_i \lambda_i \text{ holds}\,, & \\ -\int_{-\infty}^0 \Phi^{(j)}(s,u)^{-1} \frac{1}{\ell!} D_x^\ell F^{(j)}(0,u(s)) \cdot [\Phi(s,u)x]^\ell \, ds, & \\ \qquad \text{if } i = j \text{ and } \lambda_j < \sum_{i=1}^{n} \ell_i \lambda_i \text{ holds} & \end{cases} \,. \tag{6}$$

This defines a C^k conjugacy between (4) and a class of local control affine systems

$$\dot{y} = A(u(t))y(t) + G(y(t), u(t)) \tag{7}$$

which eliminates the j-th Taylor component $\frac{1}{\ell!} D_x^\ell F^{(j)}(0,u(t)) \cdot x^\ell$ belonging to the multi index ℓ and leaves fixed all other Taylor coefficients up to order $|\ell|$, i.e., for all $\kappa \in \mathbb{N}_0^n$ with $1 \leq |\kappa| \leq |\ell|$ and all $i \in \{1, \ldots, n\}$ the identity

$$D_x^\kappa G^{(i)}(0,u) \equiv \begin{cases} D_x^\kappa F^{(i)}(0,u), & \text{for } \kappa \neq \ell \text{ or } i \neq j \\ 0, & \text{for } \kappa = \ell \text{ and } i = j \end{cases} \tag{8}$$

holds.

Proof. We follow the proof of the normal form theorem in Siegmund [6]. Let ϕ, ψ and Φ denote the control systems which are generated by (4), (7) and the linearization (3), respectively. Assume that $\lambda_j > \sum_{i=1}^n \ell_i \lambda_i$, the other case is similar. It is easy to see that H is well-defined. Therefore choose ε with

$0 < \varepsilon < (a_j - \sum_{i=1}^n \ell_i b_i)/(|\ell|+1)$ and $K>0$ such that (5) holds. Then

$$\|h^{(j)}(x,u)\| \le \int_0^\infty K^{|\ell|+1}\frac{1}{\ell!}Me^{-(a_j-\sum_{i=1}^n \ell_i b_i-(|\ell|+1)\varepsilon)s}\|x^{(1)}\|^{\ell_1}\cdots\|x^{(n)}\|^{\ell_n}\,ds \tag{9}$$

$$= \frac{K^{|\ell|+1}M}{\ell!(a_j-\sum_{i=1}^n \ell_i b_i-(|\ell|+1)\varepsilon)}\|x^{(1)}\|^{\ell_1}\cdots\|x^{(n)}\|^{\ell_n}\,. \tag{10}$$

We prove that the conditions of Definition 1 are satisfied for H. Obviously

$$\mathbb{R}^d\times\mathcal{U}\ni(x,u)\mapsto(H(x,u),u)\in\mathbb{R}^d\times\mathcal{U}$$

preserves the zero section $\{0\}\times\mathcal{U}$.

(i) Lebesgues theorem implies the existence of the derivative

$$D_xh^{(j)}(x,u) = \sum_{i=1,\dots,n\,:\,\ell_i\ge1}\frac{\ell_i}{\ell!}\int_0^\infty\Phi^{(j)}(s,u)^{-1}D_x^\ell F^{(j)}(0,u(s))\cdot$$
$$\cdot\,[\Phi^{(1)}(s,u)x^{(1)}]^{\ell_1}\cdots[\Phi^{(i)}(s,u)\cdot]\cdot[\Phi^{(i)}(s,u)x^{(i)}]^{\ell_i-1}\cdots[\Phi^{(n)}(s,u)x^{(n)}]^{\ell_n}\,ds$$

and for every $0<L<1$ we get a constant $\delta=\delta(L,M,K,\ell,\lambda_1,\dots,\lambda_n,\varepsilon)>0$ such that for $u\in\mathcal{U}$ and $x\in B_\delta(0)$

$$\|D_xh(x,u)\|\le\sum_{i=1,\dots,n\,:\,\ell_i\ge1}\frac{\ell_iK^{|\ell|+1}M}{\ell!(a_j-\sum_{i=1}^n\ell_ib_i-(|\ell|+1)\varepsilon)}\|x\|^{|\ell|-1}\le L$$

and therefore $\|h(x,u)-h(\bar{x},u)\|\le L\|x-\bar{x}\|$. Then with the Neumann series the inverse for $x\in B_\delta(0)$ and each fixed $u\in\mathcal{U}$ is given explicitly by

$$H(x,u)^{-1}=x-h(x,u)+r(x,u)\qquad\text{with } r(x,u)=\sum_{i=2}^\infty(-h)^i(x,u),$$

where $(-h)^{i+1}(x,u)=(-h)^i(h(x,u),u)$. Hence $B_\delta(0)\ni x\mapsto H(x,u)$ is a homeomorphism. Now Proposition 2.5.6 in Abraham, Marsden and Ratiu [1, pp. 119-121] implies that this map is a C^k diffeomorphism for a smaller $\delta>0$ with inverse $B_{\delta'}(0)\ni x\mapsto H(x,u)^{-1}\in B_\delta(0)$ for a $0<\delta'<\delta$ independent of $u\in\mathcal{U}$.

(ii) We first show that $(x,u)\mapsto H(x,u)$ is continuous. Arguing for each component separately the proof of this claim reduces to the verification that $h^{(j)}$ is continuous and this follows with Lebesgues theorem from (9), since the integrand is continuous for almost all $s\in(0,\infty)$.

To prove the continuity of $(x,u)\mapsto H(x,u)^{-1}$ we use the estimate $\|h(x,u)-h(\bar{x},u)\|\le L\|x-\bar{x}\|$ for $x,\bar{x}\in B_\delta(0)$ and $u\in\mathcal{U}$ to get

$$\|H(x,u)-H(\bar{x},u)\|\le(1+L)\|x-\bar{x}\|\,.$$

Moreover $\|y-\bar{y}\|-L\|y-\bar{y}\|\le\|y-\bar{y}\|-\|h(y,u)-h(\bar{y},u)\|\le\|H(y,u)-H(\bar{y},u)\|$ implies with $y=H(x,u)^{-1}$, $\bar{y}=H(\bar{x},u)^{-1}$ for $x,\bar{x}\in B_{\delta'}(0)$ and $u\in\mathcal{U}$

$$\|H(x,u)^{-1}-H(\bar{x},u)^{-1}\|\le\frac{1}{1-L}\|x-\bar{x}\|\,.$$

Now choose $(x,u),(x^0,u^0)\in B_{\delta'}(0)\times\mathcal{U}$. Then

$$\|H(x,u)^{-1}-H(x^0,u^0)^{-1}\|\leq\frac{1}{1-L}\|x-x^0\|+\|H(x^0,u)^{-1}-H(x^0,u^0)^{-1}\|\ .$$

To estimate the second summand consider for all $u\in\mathcal{U}$ the identity

$$x_0=H(H(x^0,u)^{-1},u)=H(x^0,u)^{-1}+h(H(x^0,u)^{-1},u)\ .$$

We get

$$H(x^0,u)^{-1}-H(x^0,u^0)^{-1}=-h(H(x^0,u)^{-1},u)+h(H(x^0,u^0)^{-1},u^0)$$

and

$$\begin{aligned}\|H(x^0,u)^{-1}-H(x^0,u^0)^{-1}\|&\leq L\|H(x^0,u)^{-1}-H(x^0,u^0)^{-1}\|\\&\quad+\|h(H(x^0,u^0)^{-1},u)-h(H(x^0,u^0)^{-1},u^0)\|\ .\end{aligned}$$

Solving for $\|H(x^0,u)^{-1}-H(x^0,u^0)^{-1}\|$ the claim follows from the continuity of h.

(iii) We define $\psi(t,x,u):=H(\phi(t,H(x,u)^{-1},u),\theta_tu)$ for $t\in\mathbb{R}$, $x\in B_{\delta'}(0)$, and $u\in\mathcal{U}$. Then ϕ and ψ are C^k conjugate via H and moreover

$$\psi(t,x,u)=\phi(t,H(x,u)^{-1},u)+o(\|x\|^{|\ell|-1})\ . \tag{11}$$

We have to show that ψ solves a control affine system (7) which satisfies (8). Since $H^{(i)}(x,u)=x$ for $i\neq j$, we get $\psi^{(i)}(t,x,u)=\phi^{(i)}(t,H(x,u)^{-1},u)$ and therefore

$$\frac{d}{dt}\psi^{(i)}(t,x,u)=A^{(i)}(u(t))\psi^{(i)}(t,x,u)+F^{(i)}(\phi(t,H(x,u)^{-1},u),u(t))$$

for almost all $t\in\mathbb{R}$. Now $F^{(i)}$ is nonlinear in x and with (11) $\psi^{(i)}$ solves

$$\dot{x}^{(i)}(t)=A^{(i)}(u(t))x^{(i)}(t)+F^{(i)}(x(t),u(t))+R^{(i)}(x(t),u(t))$$

with $R^{(i)}(x,u)=o(\|x\|^{|\ell|})$. It is a little bit more complicated to construct the equation for $\psi^{(j)}(t,x,u)=\phi^{(j)}(t,H(x,u)^{-1},u)+h^{(j)}(\phi^{(j)}(t,H(x,u)^{-1},u),\theta_tu)$. Treating the two summands separately we get with (11)

$$\begin{aligned}\frac{d}{dt}\phi^{(j)}(t,H(x,u)^{-1},u)&=A^{(j)}(u(t))[\psi^{(j)}(t,x,u)-h^{(j)}(\phi(t,H(x,u)^{-1},u),\theta_tu)]\\&\quad+F^{(j)}(\psi^{(j)}(t,x,u),u(t))+o(\|x\|^{|\ell|})\end{aligned}$$

for almost all $t\in\mathbb{R}$. To compute the derivative of the second summand we use a simple transformation of the integral in $h^{(j)}$, Lebesgues theorem to differentiate and the formulas $\frac{d}{dt}[\Phi(s-t,\theta_tu)]=-\Phi(s-t,\theta_tu)A(u(t))$ and

$\frac{d}{dt}[\Phi(s-t,\theta_t u)^{-1}] = A(u(t))\Phi(s-t,\theta_t u)$. We get

$$
\begin{aligned}
&\frac{d}{dt} h^{(j)}(\phi(t,H(x,u)^{-1},u),\theta_t u) \\
&= \frac{d}{dt}\int_t^\infty \Phi^{(j)}(s-t,\theta_t u)^{-1}\frac{1}{\ell!}D_x^\ell F^{(j)}(0,u(s))\cdot[\Phi(s-t,\theta_t u)\phi(t,H(x,u)^{-1},u)]^\ell\,ds \\
&= -\frac{1}{\ell!}D_x^\ell F^{(j)}(0,u(s))\cdot\phi(t,H(x,u)^{-1},u)^\ell + A^{(j)}(u(t))h^{(j)}\big(\phi(t,H(x,u)^{-1},u),\theta_t u\big) \\
&- D_x h^{(j)}\big(A(u(t))\cdot\phi(t,H(x,u)^{-1},u),\theta_t u\big) \\
&+ D_x h^{(j)}\big(A(u(t))\cdot\phi(t,H(x,u)^{-1},u) + F(\phi(t,H(x,u)^{-1},u),u(t)),\theta_t u\big)
\end{aligned}
$$

for almost all $t\in\mathbb{R}$. Adding the derivatives of the two summands and using (11) $\psi^{(j)}$ solves

$$
\begin{aligned}
\dot{x}^{(j)}(t) &= A^{(j)}(u(t))x^{(j)}(t) + F^{(j)}(x(t),u(t)) \\
&\quad - \frac{1}{\ell!}D_x^\ell F^{(j)}(0,u(s))\cdot x(t)^\ell + R^{(j)}(x(t),u(t))
\end{aligned}
$$

with $R^{(j)}(x,u) = o(\|x\|^\ell)$ and the theorem is proved. ■

4 Examples

We consider examples to demonstrate our normal form theorem, Theorem 5. The first example illustrates the elimination of a nonresonant monomial, the second example has a resonant term and the sign determines the qualitative behavior of the control system. The control functions u may vary in $[-\rho,\rho]$ for some $\rho > 0$.

(a) Let $p\in\mathbb{N}$ and consider the system

$$
\begin{aligned}
\dot{x} &= x^3 + ux \\
\dot{y} &= xy^p - y.
\end{aligned}
$$

The linearization at 0 is decoupled and the spectral intervals of the first and second equation are $\lambda_1 = [-\rho,\rho]$ and $\lambda_2 = \{-1\}$, respectively. The nonresonance condition for the elimination of the monomial xy^p in the y-equation is

$$
\lambda_2 < \lambda_1 + p\lambda_2 \qquad \text{or} \qquad \lambda_2 > \lambda_1 + p\lambda_2
$$

and this is equivalent to $p < 1-\rho$ (which is impossible) or $p > 1+\rho$. If the latter holds, Theorem 5 yields the transformation $H(x,y,u) = (x,y) + h(x,y,u)$ with $h = (h^{(1)},h^{(2)})$, $h^{(1)}(x,y,u) = 0$ and

$$
h^{(2)}(x,y,u) = \int_0^\infty \Phi^{(2)}(s,u)^{-1}\cdot\Phi^{(1)}(s,u)x\cdot[\Phi^{(2)}(s,u)y]^p\,ds
$$

which transforms the system into

$$\begin{aligned}\dot{x} &= x^3 + ux \\ \dot{y} &= -y + o(\|(x, y)\|^{p+1}).\end{aligned}$$

(b) Let $\alpha \in \mathbb{R}$ and consider the two equations

$$\dot{x} = x^3 + (\alpha + u)x \qquad \text{and} \qquad \dot{x} = -x^3 + (\alpha + u)x.$$

The spectrum of the linearization $\dot{x} = (\alpha + u)x$ is the interval $\lambda = [\alpha - \rho, \alpha + \rho]$. Thus, for both equations, one has for all control functions u that for $\alpha < -\rho$ the origin is stable and for $\alpha > \rho$ it is unstable.

If $0 \notin \lambda$ and $\rho < \frac{|\alpha|}{2}$ then the nonresonance condition $\lambda < 3\lambda$ or $\lambda > 3\lambda$ is satisfied and the monomials $\pm x^3$ can be eliminated with Theorem 5.

If $0 \in \lambda$, the nonresonance condition is not satisfied and Theorem 5 yields no C^3 transformation which eliminates x^3 and $-x^3$, respectively.

We remark that these two families of control systems have qualitatively different behaviors. Recall from [3] that a control set is a maximal subset of approximate controllability. The origin is a singular point and hence it is a trivial control set. The control sets with nonvoid interiors correspond to the maximal ω-limit sets of the control flow (2). One obtains easily (compare also [3, pp. 337-338]) that for the first equation one has

- for $\alpha < -\rho$ there are the two control sets $D_1^\alpha = (-\sqrt{\alpha+\rho}, -\sqrt{\alpha-\rho})$, $D_2^\alpha = (-\sqrt{\alpha+\rho}, \sqrt{\alpha+\rho})$;
- for $-\rho \leq \alpha \leq \rho$ there are two control sets $D_1^\alpha = (-\sqrt{\alpha+\rho}, 0)$ and $D_2^\alpha = (0, \sqrt{\alpha+\rho})$.

Observe that these control sets are variant, i.e., there are trajectories leaving them in finite time.

On the other hand, for the second equation one has

- for $-\rho \leq \alpha \leq \rho$ there are the two control sets $D_1^\alpha = [-\sqrt{\alpha+\rho}, 0)$ and $D_2^\alpha = (0, \sqrt{\alpha+\rho})$;
- for $\rho < \alpha$ there are the two control sets $D_1^\alpha = [-\sqrt{\alpha+\rho}, -\sqrt{\alpha-\rho}]$, $D_2^\alpha = [-\sqrt{\alpha+\rho}, \sqrt{\alpha+\rho}]$.

Observe that these control sets are invariant, i.e., no trajectory leaves them in finite time.

Thus these equations present singular pitchfork bifurcations of control sets; they are subcritical and supercritical, respectively, .

References

[1] R. Abraham, J. E. Marsden, T. Ratiu, *Manifolds, Tensor Analysis, and Applications (Applied Mathematical Sciences 75). Springer-Verlag, New York (1983).*

[2] L. Arnold, *Random Dynamical Systems, Springer-Verlag, New York (1998).*

[3] F. Colonius, W. Kliemann, *The Dynamics of Control. Birkhäuser (2000).*

[4] S. Grünvogel, *Lyapunov Spectrum and Control Sets, Dissertation, Universität Augsburg (2000)*, *Augsburger Mathematisch-Naturwissenschaftliche Schriften no. 34, Wißner Verlag, Augsburg, 2000.*

[5] R.J. Sacker and G.R. Sell, *A Spectral Theory for Linear Differential Systems, J. Diff. Equations, 27 (1978), 320-358.*

[6] S. Siegmund, *Normal Forms for Nonautonomous Differential Equations. To appear in Journal of Differential Equations.*