Near Invariance for Markov Diffusion Systems

Fritz Colonius and Tobias Gayer
Institut für Mathematik, Universität Augsburg,
86135 Augsburg, Germany,

Wolfgang Kliemann
Department of Mathematics, Iowa State University,
Ames, Iowa 50011, U.S.A.

November 7, 2004

Abstract

A concept of 'almost invariance' is developed starting from sets that are actually invariant under smaller perturbations. This is based on a theory for system dynamics of Markov diffusion processes illuminating the idea of 'large' noise perturbations turning invariant sets for smaller noise ranges into transient sets. This also allows for numerical computation of almost invariant sets, the exit times from these sets, and the exit locations under varying perturbation ranges. The controllability behavior of associated deterministic systems plays a crucial role. Two examples, a perturbed version of the escape equation, and a one degree of freedom system with double well potential and additive perturbation are included.

1 Introduction

Almost invariance is an often used concept for stochastic dynamical systems that intends to describe sets such that the system

- stays with a set in the state space for a 'long' time,
- exits from the set only under 'large' noise perturbations,

- and may return to this set at a later, much 'longer' time.

Hence almost invariance tries to describe a transient phenomenon of stochastic systems, but on 'large' time intervals. The interpretation of 'large' time intervals and 'large' perturbations usually depends on the application one has in mind.

Applications of almost invariance include, e.g., the analysis of molecular dynamics where they can symbolize conformations of a protein that are essential for its chemical properties (see, for instance, Deuflhard, Huisinga, Fischer, and Schütte [8]); the study of set oriented numerical methods for dynamical systems (see, e.g., Dellnitz and Junge [7]); the analysis of dynamic reliability when one tries to estimate rare occurrences of system failure due to large perturbations (see, e.g., Colonius et al. in [2]); and other models in science, such as gender determination of turtles that develops during a maturation time depending on varying temperature.

The goal of this paper is to develop a theory that

- defines a plausible concept of 'almost invariant sets' based on the actual system dynamics of Markov diffusion processes,
- illuminates the idea of 'large' noise perturbations turning invariant sets for smaller noise ranges into transient sets,
- explores the idea of invariance over 'large' time intervals,
- and allows for numerical computation of almost invariant sets, the exit times from these sets, and the exit locations under varying perturbation ranges.

Our approach is, roughly, as follows:

- We consider Markov diffusion models (i.e. the system does not anticipate future behavior of the noise) with perturbations entering as parameter or additive noise into the system dynamics, which are modeled as a set of ordinary differential equations

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} \xi_i(t,\omega) X_i(x) \tag{1}$$

on a finite dimensional C^∞ manifold M, where the C^∞ vector field X_0 describes the unperturbed dynamics and $\xi(t,\omega) = (\xi_i(t,\omega),\ i = 1...m)$ is the vector of random perturbation processes with C^∞ dynamics $X_1...X_m$. We model ξ as a function $\xi = f(\eta)$ of a background noise η, $f : N \to U$, where N is the state space of the background noise and $U \subset \mathbb{R}^m$ is the set of perturbation values. We assume η to be a stationary, ergodic Markov process.

- We treat the noise range as a parameter $\rho \geq 0$ of the system by introducing a family $f^\rho : N \to U^\rho$, $\rho \geq 0$ of functions such that the sets U^ρ of perturbation values increase with ρ. Setting $U^0 = \{0\}$, we recover the unperturbed dynamics of the system 1.
- We identify the invariant sets of the stochastic system 1, depending on the noise range. Under mild conditions, the invariant control sets of an associated control are the supports of the invariant measures of 1 and they form the cores of the invariant sets for the system.
- Analyzing the change of the invariant sets as the noise range $\rho \geq 0$ increases leads to the study of the loss of invariance, specifically to the analysis of bifurcation points ρ_0 where an invariant set loses its invariance and becomes transient or 'almost invariant'.
- Finally, we study the exit time distributions from invariant sets as they become transient under the influence of larger perturbations.

This approach develops a concept for almost invariance starting from sets that are actually invariant under smaller perturbations. In other approaches the term 'almost invariance' is used to describe the behavior in certain regions, usually in relation to an invariant probability measure with support on the whole state space, see e.g. Schütte et al. in [23]. In the approach outlined above, such a reference measure need not exist, and we suggest the term 'near invariance' for the concept developed here.

In Section 2 we describe the setup used in this paper and recall some background material on Markov diffusion systems and their qualitative behavior, based on the analysis of associated control systems with varying control range. Section 3 presents the definition of near invariance together with the main result on the existence of nearly invariant sets. Theorem 3.3 and Corollary 3.4 describe the bifurcation points where an invariant and a variant set merge to generate an almost invariant set. The rest of this Section is devoted to the study of the exit sets from variant sets. Section 4 discusses the numerical computation of exit times for nearly invariants sets and the corresponding exit locations. Section 5 analyzes two examples in some detail: a perturbed version of the escape equation, see e.g., [26], [21], or [9] and the references therein, and a one degree of freedom system with double well potential and additive perturbation. The appendix 6 contains some background information on parameter dependent deterministic control systems that is used throughout the paper.

2 Markov Diffusion Systems and Associated Control Systems

In this section we recall some facts about Markov diffusion systems, their relations to associated control systems, and the support theorem of Stroock and Varadhan. We start from the system

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} f_i(\eta_t) X_i(x) \tag{2}$$

on a finite dimensional, C^∞ manifold M with C^∞ vector fields $X_0, ..., X_m$ as in Section 1. First we specify our assumptions on the background noise η. Let N be a compact connected finite dimensional C^∞-manifold on which the stochastic differential equation

$$d\eta = Y_0(\eta)dt + \sum_{j=1}^{l} Y_j(\eta) \circ dW_j \tag{3}$$

is defined. Here $W = (W_j)$ is an l-dimensional Wiener process, $Y_0, \ldots, Y_l$ are C^∞-vector fields on N, and '$\circ$' denotes the Stratonovich stochastic differential. The compactness of the noise space N rules out excitation processes with Gaussian statistics and thus (3) can be regarded as a realistic model of physical systems with bounded noise. We assume that equation (3) admits at least one stationary Markov solution. Imposing the Lie algebra rank condition

$$\dim \mathcal{LA}\{Y_1, \ldots, Y_l\}(q) = \dim N \text{ for all } q \in N \tag{4}$$

as a nondegeneracy condition on N guarantees that this stationary solution is unique (see Kunita [18]) and can be extended to a stationary Markov solution η_t^*, $t \in \mathbb{R}$.

The noise process $\xi_t := f_i(\eta_t)$ in (2) is defined in the following way: Let $U \subset \mathbb{R}^m$ be a compact, convex set with $0 \in \text{int}U$ and $U^\rho = \text{cl int}U^\rho$. Let

$$f : N \to U$$

be a continuous, surjective function such that there exists a closed, connected subset $L \subset N$ with $f|_L$ is C^1 and $Df(\eta)$ has full rank for all $\eta \in L$ with $f(\eta) \in U$, see [4]. Then $\xi_t := f(\eta_t^*)$ is a stationary process with values in U.

We model variations in the size of the noise by introducing a parameter $\rho \geq 0$ and the noise ranges U^ρ, satisfying the same assumption as U above. We consider the process η_t^* as a background noise, which for every ρ is mapped into the stochastic perturbation space $\mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \textit{measurable}\}$ by a continuous surjective function

$$f^\rho : N \to U^\rho,$$

that satisfies the assumptions on f above. Combining this perturbation model with system (1), we arrive at the Markov diffusion system

$$\begin{array}{rcl} d\eta & = & Y_0(\eta)dt + \sum_{j=1}^{\ell} Y_i(\eta) \circ dW_j, \quad \eta_0 = \eta_0^*, \\ \dot{x} & = & X_0(x) + \sum_{i=1}^{m} f_i^\rho(\eta_t) X_i(x) \end{array} \tag{5}$$

on the state space $M \times N$, for which we assume the existence and uniqueness of a strong solution for all $t \geq 0$. This system is degenerate since the Wiener process acts only on the second component. Note that, in general, the component $x(t)$ by itself is not Markovian. The pair process $(x(t), \eta_t)$ is, however, a Markov diffusion process for all ρ, if the initial random variable x_0 in M is independent of the increments of the Wiener process. Compare especially [16] for results on degenerate diffusions along these lines, and [4] and [5] for more details on our setting in general.

The system (5) can be analyzed using control theory via the support theorem presented by Stroock and Varadhan in [24]. To make this more precise, we set up the control system associated with (5) to be

$$\begin{array}{rcl} \dot{\eta} & = & Y_0(\eta) + \sum_{j=1}^{\ell} w_j(t) Y_i(\eta), \\ \dot{x} & = & X_0(x) + \sum_{i=1}^{m} f_i^\rho(\eta_t) X_i(x) \end{array} \tag{6}$$

where $w \in \mathcal{W} := \{w : [0, \infty) \to \mathbb{R}^l, \text{ piecewise constant}\}$, and assume the Lie algebra rank condition (4) for the $\eta-$component. Furthermore, we want the pair system (5) to be regular, i.e. we want the topological support of its transition probabilities from each point $(x,p) \in M \times N$ to have nonvoid interior in $M \times N$. This is guaranteed by

$$\dim \mathcal{LA} \left\{ \begin{pmatrix} X_0 + \sum \eta_i X_i(x) \\ Y_0 + \sum w_j Y_j \end{pmatrix}, w \in \mathbb{R}^l \right\} \begin{pmatrix} x \\ \eta \end{pmatrix} = \dim M + \dim N \quad (7)$$

for all $(x,\eta) \in M \times N$ (see Meyn and Tweedie [20] for a relaxation of this condition). Instead of (6) it will be sufficient to consider the system

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), \ u \in \mathcal{U}^\rho, \quad (8)$$

see the appendix, Section 6 for definitions and notations of control systems. Note that the condition (7) implies local accessibility for the $x-$ component (8).

We fix $\rho \geq 0$ for the remainder of this section, and drop it in the notation. For all $(x,\eta) \in M \times N$ the orbits $\mathcal{O}^+(x,\eta)$ of system (6) are of the form $\mathrm{cl}\mathcal{O}^+(x,\eta) = \mathrm{cl}\mathcal{O}^+(x) \times N$, where $\mathcal{O}^+(x)$ is the forward orbit of the system (8) from $x \in M$. In particular, the invariant control sets $\hat{C} \subset M \times N$ of (6) correspond one-to-one to the invariant control sets $C \subset M$ of (8) via $\hat{C} = C \times N$. This follows from Lemma 3.17 in [4]. (We remark that in the statement of that lemma one has to add the surjectivity assumption for f which is used in the proof). Therefore the global control structure of the $x-$component (8) determines the control structure of the pair process (6).

The natural probability space to work in is $\hat{\Omega} := \mathcal{C}(\mathbb{R}_0^+, M \times N) = \{\omega : R_0^+ \to M \times N, \textit{continuous}\}$ and for fixed initial conditions $(x,q) \in M \times N$ the pair process (5) induces a probability measure $\hat{P}_{(x,q)}$ on $\hat{\Omega}$. By $\hat{P}_{(x,\eta^*)}$ we denote the measure corresponding to the stationary Markov solution $\{\eta_t^*, t \geq 0\}$ in the $\eta-$component. Its marginal distribution on $\Omega := \mathcal{C}(\mathbb{R}_0^+, M)$ will be denoted by $P_x, x \in M$. The trajectories of the pair process are $(\varphi(t,(x,q),\omega), \eta(t,q,\omega))$ for $(x,q) \in M \times N$, and we will write the $x-$component under $\{\eta_t^*, t \geq 0\}$ as $\varphi(t,x,\omega)$, $x \in M$. Then the 'transition probability' from $x \in M$ to a set $A \subset M$ in time $t \geq 0$ is

$$P(t,x,A) = P_x(\varphi(t,x,\omega) \in A). \quad (9)$$

Using the tube method introduced by Arnold and Kliemann in [1], it follows (compare [15]) from the support theorem that

$$\mathrm{supp}\ P(t,x,\cdot) = \mathrm{cl} \left\{ \begin{array}{c} y \in M \mid \text{ there is a piecewise continuous} \\ u \in \mathcal{U} \text{ such that } \varphi(t,x,u) = y \end{array} \right\}. \quad (10)$$

It now follows from [16] and [4] that the invariant Markov probability measures μ of (5) have support given by $\text{supp}\mu = C \times N$, where C is an invariant control set of (8), and these measures are unique on each set of this form. We call *ergodic sets* those invariant control sets C of (8) such that $C \times N$ is the support of some invariant Markov measure, which includes, in particular, all bounded invariant control sets. All other points in $M \times N$ are transient.

To describe the consequences of the support theorem for the relationship between the Markov diffusion process (5) and the control system (8) in more detail, we define the *first entrance time* of (5) to a set $A \subset M$ from a point $x \in M$ as the random variable

$$\tau_x(A) := \inf\{t \geq 0, \ \varphi(t, x, \omega) \in A\},$$

and the *first exit time* of (5) from a set $A \subset M$ starting at a point $x \in M$ as the random variable

$$\sigma_x(A) := \inf\{t \geq 0, \ \varphi(t, x, \omega) \notin A\}.$$

The corresponding *exit location* is given as

$$h_x(A)(\omega) := \begin{cases} y \in M, \ y = \varphi(\sigma_x(A), x, \omega) & \text{for } \sigma_x(A)(\omega) < \infty \\ \emptyset & \text{for } \sigma_x(A)(\omega) = \infty. \end{cases}$$

Due to Theorem 3.19 in [4], for invariant control sets $C \subset M$ of system (8) the equation $P_x(\sigma_x(C) < \infty) = 0$ holds for all $x \in C$. For bounded variant control sets $D \subset M$ on the other hand, it holds that $P_x(\sigma_x(D) < \infty) = 1$ for all $x \in D$. Under the measure P_x we even have that the expectation of the sojourn time $E_x[\sigma_x(D)]$ is finite (see [2], Theorem 11).

3 Near Invariance and Mergers of Control Sets

If a bounded invariant control set C^ρ for $\rho \leq \rho_0$ becomes variant for $\rho > \rho_0$, then the corresponding ergodic set of the Markov process disappears and becomes transient. Nevertheless, although the disappearance of an ergodic set changes the global behavior of a stochastic system considerably, we expect the system to experience large exit times from the resulting variant control set as long as ρ is close to ρ_0 (see [14] for an example that can serve as a prototype of this phenomenon). This behavior is captured more generally in the following definition.

Definition 3.1 *Consider the family of Markov diffusion systems (5)$^\rho$. A closed set $A \subset M$ with* int$A \neq \varnothing$ *is* nearly invariant *in $x_0 \in$* int$A$ *for $\rho > \rho_0$ if*
(i) $\sigma^\rho_{x_0}(A) < \infty$ with positive probability for $\rho > \rho_0$, and
(ii) for all $x \in A$ one has $\sigma^\rho_x(A) \nearrow \infty$ almost surely for $\rho \searrow \rho_0$ and $\sigma^{\rho_0}_x(A) = \infty$ almost surely.
If A is nearly invariant in every $x_0 \in$ int$A$*, the set A is called* nearly invariant.

The following theorem reduces the search for nearly invariant sets to the search for closed sets A which are invariant for the control range U^{ρ_0} and lose their invariance under increased control ranges.

Theorem 3.2 *Suppose the Markov diffusion systems (5)$^\rho$ satisfy the Lie algebra rank conditions (7) and (4) and that U^ρ increases upper semicontinuously with respect to $\rho \in (\rho_*, \rho^*)$. Let $x_0 \in$* int$A$ *for some closed set $A \subset M$,* int$A \neq \varnothing$ *and consider $\rho_0 \in (\rho_*, \rho^*)$. Then the set A is nearly invariant in x_0 if and only if the set A is positively invariant for ρ_0, and for each $\rho > \rho_0$*

$$\mathrm{int}(\mathcal{O}^{\rho,+}(x_0) \setminus A) \neq \emptyset \text{ for all } \rho > \rho_0. \tag{11}$$

Proof. First we show that from positive invariance of A and upper semicontinuity of U^ρ at $\rho = \rho_0$ property (ii) of Definition 3.1 follows. By Lemma 6.1, also int$A$ is positively invariant and hence $\sigma^{\rho_0}_x(A) = \infty$ almost surely. Now assume contrary to the other assertion that there are $x \in A$, a positive time $T > 0$ and $\rho_n \searrow \rho_0$ such that $P_x(\sigma^{\rho_n}_x(A) < T) > 0$. Then from (10) it follows that for all ρ_n there is a control $u_n \in \mathcal{U}^{\rho_n}$ with $\varphi(T, x, u_n) \notin A$, and due to continuity, there are positive times $t_n < T$ such that $\varphi(t_n, x, u_n) \in \partial A$. Since U^ρ is increasing, we can look upon the sequence u_n as a sequence in the compact set $\mathcal{U}^{\rho_1}$ endowed with the weak$^*-$topology. Then there are subsequences, called t_n and u_n again, such that $t_n \to t_*$ and $u_n \to u_*$. By (19) it follows that $\varphi(t_n, x, u_n) \to \varphi(t_*, x, u_*)$. Now observe that on a bounded interval weak$^*-$convergence in L_∞ implies weak convergence in L_2; and here a subsequence of a weakly convergent sequence converges pointwise. Hence upper semicontinuity of the closed sets U^ρ implies that $u_* \in \mathcal{U}^{\rho_0}$, because if $u_*(t)$ was not in U^{ρ_0} for some t, this would contradict $u_n(t) \in U^{\rho_n}$ for all n. Then by continuity it follows that $\varphi(t_*, x, u_*) \in \partial A$, contradicting the positive invariance of int$A$.

Next we prove that assumption (11) implies property (i) of near invariance by showing that $P_{x_0}(\sigma^\rho_{x_0}(A) < \infty) > 0$ for all $\rho > \rho_0$. Pick $\rho > \rho_0$, then there are some open set $V \subset \text{int}(\mathcal{O}^{\rho,+}(x_0) \setminus A)$, a positive time $t_0 < \infty$, and a piecewise constant control $u_0 \in \mathcal{U}^\rho$ such that $\varphi(t_0, x_0, u_0) \in V$. By continuous dependence of the solutions of $(8)^\rho$ on u, there is an open neighborhood $\mathcal{V}(u_0) \subset \mathcal{U}^\rho$ such that $\varphi(t_0, x_0, u) \in V$ for all $u \in \mathcal{V}(u_0)$. The support theorem implies that $P(\eta \in \mathcal{C}(\mathbb{R}^+_0, N), f^\rho(\eta) \in \mathcal{V}(u_0)) > 0$. Since the trajectories of (5) are continuous, we obtain

$$\begin{aligned} P_{x_0}(\sigma^\rho_{x_0}(A) < \infty) &\geq P_{x_0}(\sigma^\rho_{x_0}(A) < t_0) \\ &\geq P(\eta \in \mathcal{C}(\mathbb{R}^+_0, N), f^\rho(\eta) \in \mathcal{V}(u_0)) > 0. \end{aligned} \tag{12}$$

For the converse implication assume that A is nearly invariant in $x_0 \in \text{int}A$ for $\rho > \rho_0$. Then $\sigma^\rho_{x_0}(A) < \infty$ with positive probability for $\rho > \rho_0$. Thus for every $\rho > \rho_0$ there is a realization of η and a time T such that with $u^\rho := f^\rho(\eta) \in \mathcal{U}^\rho$

$$\varphi(T, x_0, u^\rho) \notin A.$$

Thus $\varphi(T, x_0, u^\rho) \in \mathcal{O}^{\rho,+}(x_0) \setminus A$. Local accessibility of (8) implies that

$$\mathcal{O}^{\rho,+}(x_0) \subset \text{cl int}\mathcal{O}^{\rho,+}(x_0).$$

Since A is closed, we see that for every $\rho > \rho_0$ condition (11) holds.

It remains to show that the set A is positively invariant for ρ_0. This follows from $\sigma^{\rho_0}_x(A) = \infty$ almost surely. In fact, if A is not positively invariant, we obtain a contradiction using the same reasoning as above in the proof that (11) implies property (i) of near invariance . ■

This result shows that we have to look for closed sets which are positively invariant for ρ_0 and lose their invariance for $\rho > \rho_0$. Naturally, the sets A that are nearly invariant for all $x_0 \in \text{int}A$, are of particular interest. These sets are specified in the following theorem. Recall from Section 6 that $\mathbf{A}^{inv}(I)$ denotes the largest invariant set in the domain of attraction of a set I.

Theorem 3.3 *(i) Let the assumptions of Theorem 3.2 be satisfied and let C^{ρ_0} be a compact invariant control set for ρ_0. For each $\rho > \rho_0$ denote by*

C^ρ the unique control set of (8)$^\rho$ for which $C^{\rho_0} \subset C^\rho$. Suppose that there is $x \in \mathrm{int} C^{\rho_0}$ with

$$\mathrm{int}(\mathcal{O}^{\rho,+}(x) \setminus C^{\rho_0}) \neq \emptyset \text{ for all } \rho > \rho_0. \tag{13}$$

Then the invariant control set C^{ρ_0} is nearly invariant for $\rho > \rho_0$.

(ii) For every compact set $K \subset M$ the intersection $\mathbf{A}^{inv}(C^{\rho_0}) \cap K$ is nearly invariant for ρ_0, if the intersection is positively invariant for ρ_0.

(iii) If the invariant control set C^{ρ_0} is nearly invariant for $\rho > \rho_0$ and bounded, then $P_{x_o}\{\sigma^\rho_{x_0}(C^{\rho_0}) < \infty\} = 1$ for all $x_0 \in C^{\rho_0}$ and all $\rho > \rho_0$.

(iv) Condition (13) is satisfied, in particular, if C^{ρ_0} merges with a variant control set D^{ρ_0} with nonvoid interior, i.e., $D^{\rho_0} \subset C^\rho$ for all $\rho > \rho_0$, or if all $(u, x) \in \mathcal{U}^{\rho_0} \times C^{\rho_0}$ are inner pairs of system (8)$^\rho$ for every $\rho > \rho_0$, compare the appendix, Section 6.

Proof. (i) We show that C^{ρ_0} is nearly invariant for $\rho > \rho_0$. Since $\mathrm{int}(C^\rho \setminus C^{\rho_0}) \neq \emptyset$ and C^ρ is a control set, there are $y \in \mathrm{int}(C^\rho \setminus C^{\rho_0})$ and $x \in \mathrm{int} C^{\rho_0}$ such that $y \in \mathcal{O}^{\rho,+}(x)$. Due to continuity, it follows that there is an open neighborhood $V(y) \subset \mathrm{int}(C^\rho \setminus C^{\rho_0})$ of y such that $V(y) \subset \mathcal{O}^{\rho,+}(C^{\rho_0})$ and therefore condition (11) holds.

(ii) Condition (13) implies that (11) is satisfied for every $x_0 \in A := \mathbf{A}^{inv}(C^{\rho_0})$, since

$$\mathcal{O}^{\rho,+}(x) \subset \mathcal{O}^{\rho,+}(x_0) \text{ for all } x_0 \in \mathbf{A}^{inv}(C^{\rho_0}).$$

(iii) According to [16], all points $x \in M$ are either recurrent or transient and points in variant control sets are transient. Furthermore, the first exit time from bounded sets of transient points is a.s. finite.

(iv) If C^{ρ_0} merges with a variant control set D^{ρ_0} with nonvoid interior, one has $D^{\rho_0} \cap C^{\rho_0} = \emptyset$ and $D^{\rho_0} \subset C^\rho$ for $\rho > \rho_0$, and therefore condition (13) is satisfied. Finally, from the assumption that all $(u, x) \in \mathcal{U}^{\rho_0} \times C^{\rho_0}$ are inner pairs of system (8)$^\rho$ for $\rho > \rho_0$ it ensues that $C^{\rho_0} \subset \mathrm{int} C^\rho$ according to Theorem 6.4. Therefore there is some open set $V \subset C^\rho \setminus C^{\rho_0}$ and condition (13) holds. ■

This theorem shows that control sets C^{ρ_0} that are invariant for the perturbation range ρ_0, but variant for $\rho > \rho_0$, are the key nearly invariant sets of a stochastic system. They are contained in the variant control sets $D^\rho \supset C^{\rho_0}$ as 'almost invariant' sets. If these nearly invariant sets are also bounded, then Property (i) of 3.1 holds with probability 1. In this situation, we also have the following consequence.

Corollary 3.4 *Let the assumptions of Theorem 3.2 be satisfied and let C^{ρ_0} be a compact invariant control set for ρ_0. For each $\rho > \rho_0$ denote by C^ρ the unique control set of (8)$^\rho$ for which $C^{\rho_0} \subset C^\rho$. Assume that C^{ρ_0} merges with a variant control set D^{ρ_0} with nonvoid interior, i.e., $D^{\rho_0} \subset C^\rho$ for all $\rho > \rho_0$. If C^ρ is bounded, then $P_x\{\sigma_x^\rho(C^\rho) < \infty\} = 1$ for all $x \in C^\rho$, $\rho > \rho_0$, and $\sigma_x(C^\rho)$ has finite expectation. This holds, in particular, for $x \in C^{\rho_0}$.*

The proof of this lemma is a direct consequence of Theorem 11 in [2].

We now analyze how the stochastic system can exit from variant control sets. The following propositions show how the continuity results for exit-boundaries of control sets (see Section 6) can be translated to the stochastic situation.

Proposition 3.5 *Suppose the family of Markov diffusion systems (5)$^\rho$ fulfills the Lie algebra rank conditions (4) and (7) for all $\rho \in [\rho_*, \rho^*]$.*

Let $D^\rho \subset M$ be a bounded variant control set of (8)$^\rho$ with nonvoid interior such that $D^{\rho_*} \subset D^\rho$, and let $x \in D^\rho$. For each ρ we define a probability measure on M via

$$Q_x(D^\rho)(A) := P_x(\omega \in \Omega,\, h_x(D^\rho)(\omega) \in A) \qquad \textit{for all Borel sets } A \subset M,$$

with support $\mathrm{cl}\partial_{ex} D^\rho$. If the mapping $\rho \to \mathrm{cl} D^\rho$ is continuous in the Hausdorff distance at ρ_0 and if the perturbation range U^ρ increases lower semicontinuously at ρ_0, then the support of $Q_x(D^\rho)$ changes continuously.

Proof. Recall that $P_x(\sigma_x(D) < \infty) = 1$ for a bounded variant control set D with $x \in D$ and since all trajectories $\varphi(t, x, \omega)$ are continuous, $Q_x(D^\rho)$ is a probability measure. Equation (10) implies that $\mathrm{supp} Q_x(D^\rho) = \mathrm{cl}\partial_{ex} D^\rho$ by definition of $\partial_{ex} D^\rho$. The desired continuity follows from the deterministic situation in Theorem 6.5. ■

Finally, we study the exit locations when an invariant control set merges with a variant control set. The deterministic situation is described in Theorem 6.5.

Proposition 3.6 *Suppose the family of Markov diffusion systems (5)$^\rho$ fulfills the Lie algebra rank conditions (4) and (7) for all $\rho \in [\rho_*, \rho^*]$. For $\rho_o \in (\rho_*, \rho^*)$ let C^{ρ_0} and D^{ρ_0} be an invariant and a variant control set, respectively, that satisfy the conditions of Theorem 6.6.*

Then for the stochastic system $(5)^{\rho_0}$ *we have for the first entrance time* $\tau_x(C^{\rho_0})$ *to the set* C^{ρ_0} *that the probability* $p_x := P_x(\tau_x(C^{\rho_0}) < \infty) < 1$ *for* $x \in D^{\rho_0}$. *By*

$$Q_{x\not\to C^{\rho_0}}(D^\rho)(A) := \frac{1}{1-p_x} P_x\big(\omega \in \Omega, \quad h_x(D^\rho) \in A \text{ and } \tau_x(C^{\rho_0}) = \infty\big)$$
$$\text{for all Borel sets } A \subset M$$

a probability measure is defined on M *with support* $\mathrm{cl}\partial^{ex\not\to C^{\rho_0}} D^\rho$. *Furthermore, for the variant control set* $F^\rho \supset C^{\rho_0} \cup D^{\rho_0}$ *we have that*

$$\mathit{supp}Q_x(F^\rho) \to \mathit{supp}Q_{x\not\to C^{\rho_0}}(D^{\rho_0}) \text{ for } \rho \searrow \rho_0$$

in the Hausdorff metric.

Proof. We first show that $p_x < 1$ for $x \in D^{\rho_0}$. Since it is assumed that the exit boundary of D^{ρ_0} can be non-trivially decomposed into $\partial^{ex\to C^{\rho_0}} D^{\rho_0}$ and $\partial^{ex\not\to C^{\rho_0}} D^{\rho_0}$, it follows that $\mathrm{cl}\partial^{ex\not\to C^{\rho_0}} D^{\rho_0} \neq \emptyset$. Then (10) implies $p_x < 1$.

Thus $Q_{x\not\to C^{\rho_0}}(D^{\rho_0})$ is well defined and $Q_{x\not\to C^{\rho_0}}(D^{\rho_0})(M) = 1$. As before due to (10) and the continuity of the trajectories, $\mathrm{supp}Q_{x\not\to C^{\rho_0}}(D^{\rho_0}) = \mathrm{cl}\partial^{ex\not\to C^{\rho_0}} D^{\rho_0}$. Now the asserted right continuity follows from Theorem 6.6. ■

4 Computation of Exit Times and Exit Locations for Nearly Invariant Sets

In this section we present an algorithm to compute exit times of stochastic systems from sets, based on set oriented methods as they were developed for dynamical systems by Dellnitz, Hohmann, and Junge (see [6], [7]), and for control systems by Szolnoki (cf. [25]). We start from the setup in Theorem 3.3 and Corollary 3.4: For the parameter interval $[\rho_*, \rho^*]$ we assume that there is a 'bifurcation point' ρ_0 such that C^{ρ_0} is an invariant control set that is contained in a variant control set C^ρ for $\rho > \rho_0$. According to Theorem 3.3, points x in the set C^{ρ_0} and in $\mathbf{A}^{inv}(C^{\rho_0}) \cap K$ of the stochastic system $(5)^{\rho_0}$ can be expected to be identified in the analysis of system $(5)^\rho$ for $\rho > \rho_0$, with $\rho - \rho_0$ small, by significantly large first exit times. However, it is impossible to analytically compute $\sigma_x(C^\rho)$ in general. We know, however, that for bounded, variant C^ρ we have $P_x(\sigma_x(C^\rho) < \infty) = 1$ for all

$x \in C^\rho$. For more detailed information on exit time distributions, one has to use numerical methods.

The following algorithm produces a numerical approximation to the distribution of exit times from sets in the state space. We will concentrate here on the distribution $P_x\{\sigma_x(C^\rho) \le t\}$, $t \ge 0$, for bounded, variant control sets C^ρ of the system $(5)^\rho$.

Algorithm

Step 1 Compute the bounded variant control set $C^\rho \subset M$ of the control system $(8)^\rho$.

Step 2 Choose a compact set $K \subset M$ with $\mathrm{cl}C^\rho \subset \mathrm{int}K$ and define a partition $\mathcal{P}$ of K into finitely many boxes B_i. Define the collection $\mathcal{C} = \{B_1, B_2, \ldots, B_N\}$ of all boxes in $\mathcal{P}$ that have nonvoid intersection with C^ρ, and denote by B_{N+1} the 'sink box' which symbolizes the area outside of $\bigcup_{i=1}^N B_i$. Since $C^\rho \subset \bigcup_{i=1}^N B_i$, and we are interested in the first exit time, one box suffices to cover the area of 'no return'.

Step 3 Choose a discretization time $T > 0$, and compute the 'transition probabilities' $p_{ij} := \frac{1}{m(B_i)} \int_{B_i} P(T, y, B_j)\, dy$ for the ensuing discretized system, with $P(T, y, B_j)$ as defined in (9) for $i = 1...N$. Here $m(\cdot)$ denotes the Lebesgue measure. We set $p_{N+1,j} = 1$ for $j = 1...N+1$. The resulting matrix $P := (p_{ij}) \in \mathbb{R}^{(N+1)\times(N+1)}$ is row stochastic and hence the transition matrix of a certain Markov chain on the box space.

Step 4 Compute the cumulative distribution function (cdf) of the first exit time $\sigma_x(C^\rho)$ for $x \in B_i$: $P\{\sigma_x(C^\rho) \le nT\}$ is approximated by the $i-th$ entry in the last column $(p_{i,N+1}^{(n)})$ of P^n. Specifically, for a given time T_{exit} we find n_e with $(n_e - 1)T \le T_{exit} \le n_e T$, and the last column of P^{n_e} approximates the probability to exit C^ρ from B_i until time T_{exit}.

For the approximation of the control sets, numerical methods have been developed by Szolnoki (cf. [25]). They rely on subdivision techniques for the numerical analysis of dynamical systems developed by Dellnitz, Hohmann, and Junge (see [6], [7]). These references also describe the generation of a partition $\mathcal{P}$ and of the boxes.

For the approximation of the dynamics of $(5)^\rho$ we have created a Markov chain on a finite box partition. After choosing a discretization time T in Step 3, the transition probabilities between the states are computed by Monte Carlo simulation. This idea is rather old and goes back to Ulam, Metropolis, and von Neumann (see [19]). Although in the meantime many sophisticated variants for different disciplines have been developed, there are no general error estimates available, hence one can never be sure that the Monte Carlo simulation recognizes all relevant behavior of the stochastic system. This is especially problematic if one wants to compute stationary measures or long time simulations of stochastic processes that visit certain areas of the state space only infrequently. There have been some developments to overcome these problems for specific systems. For instance, for systems with purely additive noise, the deterministic part and the noise influence can be decoupled as has been done by Fischer in [9] and [10] following some work by Froyland [11]. Subsequent application of the so-called exhaustion algorithm produces some error bounds for such systems. In the algorithm described above we start from a given partition $\mathcal{P}$, a fixed discretization time T, and several starting points within each box B_i. Hence this algorithm does not follow a simulated trajectory of one initial point over a long time period, and it has proven to be quite reliable.

To approximate the dynamics of $(5)^\rho$, in Step 3 we first simulate a large number of trajectories $\hat{\eta}^l$, $l = 1, \ldots, s_1$, of the background noise process η. For this we choose initial values in the compact space N according to the stationary solution η^* (provided this is known) and approximate solutions of the stochastic differential equation (3) until time T. Strong schemes are the methods of choice for the approximation because information about the whole solution path of $(3)^\rho$ is needed for solving the $x-$component of $(5)^\rho$ (see Kloeden and Platen [17] for an introduction to numerical methods for stochastic differential equations).

Subsequently, s_2 starting points x^k are picked in each box B_i. From each starting point, the solution of the $x-$component of $(5)^\rho$ is approximated for all samples $\hat{\eta}^l$ generating $s_1 s_2$ target points, denoted by $\hat{\varphi}(T, x^k, \hat{\eta}^l)$. The transition probability from box B_i to B_j is then approximated by

$$p_{ij} = \frac{1}{m(B_i)} \int_{B_i} P(T, x, B_j) dx \approx \frac{1}{s_1 s_2} \sum_{k=1}^{s_2} \sum_{l=1}^{s_1} \chi_{B_j} \left(\hat{\varphi}(T, x^k, \hat{\eta}^l) \right),$$

where χ_{B_j} denotes the characteristic function of the set B_j. The question as to how many boxes, starting points, and sample paths of the background process should be used, depends on the properties of the system, the time length T, and the box size—and, of course, on availability of computing resources. While the number of boxes $N + 1$ is mainly limited by available memory (note that it is necessary to multiply full matrices with $(N + 1)^2$ entries in Step 4), we have observed that the algorithm is more sensitive to a change of the noise realization than to a change of the initial values within a box. It seems that the solution trajectories $\eta^*_\cdot(\omega)$ of (3) are less smooth then the solutions of the system (2). Therefore it is reasonable to increase the number of realizations of the background noise at the expense of initial values in each box when computing resources become an issue.

Repeated multiplication of the matrix P with itself in Step 4 may pose a problem for fine partitions, particularly in higher dimensions. When computing the cdf of the first exit time, this problem cannot be avoided. If one is interested mainly in the probability of exit until some large time T_{exit}, one can save certain iterations: Instead of performing $n_e = \frac{T_{exit}}{T}$ multiplications with P, we find $\widehat{n} = \max\{n \in N, 2^{\widehat{n}} \leq n_e\}$ and compute $P^{2^{\widehat{n}}}$ in $\widehat{n}$ steps. If $2^{\widehat{n}} < n_e$, we continue the same process with $n_e - 2^{\widehat{n}}$, etc., until P^{n_e} is computed. (Of course, bases other than 2 can be used and sometimes lead to less factors in the decomposition of n_e.) For $T_{exit} = 10^4$ and $T = 10^{-2}$, this process results in 25 matrix multiplications instead of 10^6. If the cdf of the first exit time is not required in a resolution corresponding to n_e time intervals, one can proceed similarly by expressing the size of the desired resolution in powers of a prime, e.g., of 2. In our example, choosing a resolution of $10^3 T$, we compute P^{1000} with 14 multiplications, and then P^{1000k}, $k = 2...1000$, resulting in 1013 steps.

Recall that for bounded, variant control sets C^ρ the expected exit time from a point $x \in C^\rho$ is finite and given by

$$E[\sigma_x(C^\rho)] = \int_0^\infty t \, dP_\sigma,$$

where P_σ is the distribution of σ_x. This expected value can be approximated by

$$\hat{E}[\sigma_x(C^\rho)] = T \sum_{n=1}^{\infty} n \, (p_{i,N+1}^{(n)} - p_{i,N+1}^{(n-1)}) \qquad \text{for } x \in B_i.$$

For the actual computation, naturally an upper limit n_{max} on n has to be chosen, which results in an approximation of the expected exit time before $n_{max} T$.

To compute the exit locations for the system (2), we again approximate its dynamics by the Markov chain defined in Step 3. For an initial value $x \in C^\rho$ we identify the box B_i with $x \in B_i$. As before, $p_{i,j}^{(n)}$ is the probability to reach the state B_j from B_i in n steps. If $B_j \neq B_{N+1}$, and if $p_{j,N+1}^{(n+1)} > 0$, then the Markov chain exits from $\mathcal{C}$ in step $n+1$. In this case the state B_j is an exit state for the chain, starting from B_i. Let h_i denote the corresponding random exit location. We then have

$$P\{h_i = B_j\} = \sum_{n=0}^{\infty} p_{ij}^{(n)} p(j, N+1),$$

and this distribution approximates the one of $h_x(C^\rho)$ as defined in Section 2. In practice, one will have to choose again a maximal time $T_{exit} \in \mathbb{N}$, and the finite sum with $T_{exit} + 1$ terms is computed

5 Examples

5.1 A Perturbed Escape Equation

As first example we will present some results for the perturbed escape equation. It describes the movement of a particle with unit mass in the potential $V(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$ with inertia and linear viscous damping under the influence of some perturbation. This equation has attracted great interest and has been analyzed thoroughly (see e.g., [26], [21], or [9] and the references therein). We consider the perturbed escape equation

$$\ddot{x} + \gamma\dot{x} + x - x^2 = \rho \sin \eta_t$$

with a background noise process η_t on the one-dimensional sphere $\mathbb{S}^1$. The Wiener process on this sphere is considered as the one dimensional Wiener process on $\mathbb{R}$ modulo 2π. For $t \geq 0$ and $\bar{x}, \bar{y} \in \mathbb{S}^1$ and $x, y \in \mathbb{R}$ such that $\bar{x} \equiv x \bmod 2\pi$ and $\bar{y} \equiv y \bmod 2\pi$, the transition densities of this process, resulting from the corresponding normally distributed process on $\mathbb{R}$, are given by

$$p(t, \bar{x}, \bar{y}) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} exp(-\frac{(y - x + 2n\pi)^2}{2t}).$$

The sum on the right hand side converges uniformly and absolutely. Then for an integrable nonnegative function $f : \mathbb{S}^1 \to \mathbb{R}$ it holds that

$$\begin{aligned} U_t f(\bar{x}) &:= \int_{S^1} p(t, \bar{x}, \bar{y})\, f(\bar{y})\, d\bar{y} \\ &= \tfrac{1}{\sqrt{2\pi t}} \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} exp(-\tfrac{(y-x+2n\pi)^2}{2t}) \right) f(y)\, dy \\ &= \tfrac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} exp(-\tfrac{(y-x)^2}{2t})\, f(y \bmod 2\pi)\, dy. \end{aligned}$$

The function $f(\bar{x}) \equiv \frac{1}{2\pi}$ fulfills $U_t f(\bar{x}) = f(\bar{x})$. Thus $f(\bar{x})$ is the unique stationary density of the noise process because (4) obviously holds.

The perturbed escape equation driven by this background process is given by

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -\gamma\, y(t) - x(t) + x(t)^2 + \rho \sin(\eta_t) \\ d\eta_t &= dW_t \mathrm{mod} 2\pi, \end{aligned} \tag{14}$$

As we saw, the stationary process η_t^* has the uniform distribution on $\mathbb{S}^1$ as its one-dimensional distribution.

The associated controlled version of this equation on $\mathbb{R}^2$ reads

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -\gamma\, y(t) - x(t) + x(t)^2 \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix} \tag{15}$$

where $u(t) \in U^\rho := [-\rho, \rho]$. For our computations we set the damping coefficient γ to 0.1. Computation of the control sets using the method described in 4, yields for $\rho = 0.04$ the existence of one invariant control set $C^{0.04}$ that contains the stable fixed point $(0,0)$ of the uncontrolled equation and one variant control set $D^{0.04}$ containing the hyperbolic fixed point $(1,0)$ of the uncontrolled equation (cf. Figure 1). Increasing the control range one finds that the two control sets merge for some ρ_0 close to 0.0411 (see [12]) to form one variant control set. The assumptions of Theorem 6.6 are satisfied for this example.

For the computation of the exit times from the merged control set we set $\rho = 0.15$ and distinguish two different scenarios. The first one explores the exit time distribution for very short time, i.e. $T_{exit} \leq 1.0$. In this case we choose a fine partition of the compact set K containing $D^{0.15}$. The second one aims at long times, and we choose a coarser partition to accelerate the computation time. In both cases we pick only the center of each box as initial value because the system (14) proves to be more sensitive to a variation in the noise sample than to a small change of the initial value.

In order to approximate the background noise process in the short time case ($T_{exit} \leq 1.0$), we choose $\hat{\eta}_0^l = l \cdot 2\pi/100$ for $l = \{0, 1, \ldots, 99\}$ as initial values to represent the uniform distribution of η_t^*. Then the background noise part of (14) is solved for each of these initial values with step size 0.1 until time 1.0, generating 100 sample paths $\hat{\eta}^l$ of the Wiener process on $\mathbb{S}^1$.

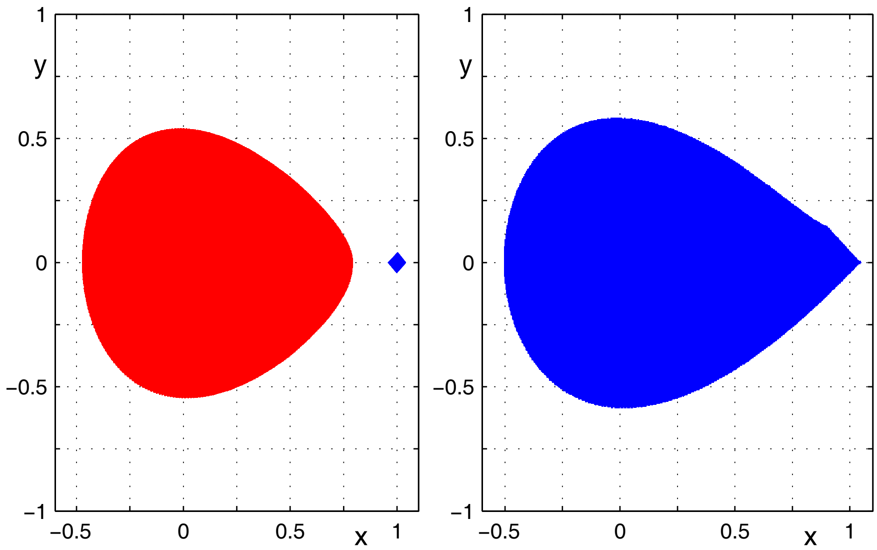

Figure 1: Control sets for the controlled escape equation for $\rho = 0.04$ (left) and $\rho = 0.045$ (right)

For this integration, a simple Euler scheme can be used efficiently because drift and diffusion coefficients are both constant. The exit probability from a box B_i is then approximated directly by solving the (x, y)-component for each sample $\hat{\eta}^l$ starting at the center of B_i. That way, the upper left sketch in Figure 2, was produced where different colors represent different exit probabilities until time $T_{exit} = 1.0$. The other three graphs in Figure 2 follow the same procedure for $T_{exit} = 5$, 30, and 220.

To compute the distribution of the exit times $\sigma_x(D^{0.15})$, which requires large time intervals, we follow the same scheme to integrate the Wiener process but compute more samples by starting from $\hat{\eta}_0^l = l \cdot 2\pi/10000$ for $l = \{0, 1, \ldots, 9999\}$ to compensate for the increased box sizes. Once again, the approximation of the (x, y)-component for each sample $\hat{\eta}^l$ starts at the center of B_i. Here the limiting factor for the number of boxes is the multiples of the transition matrix P that are to be computed. Multiples P^n of P are computed for $n = 2, \ldots, 1500$. The minimum over all boxes of the exit probabilities $\min_i p_{i,N+1}^{(1500)}$ until $T_{exit} = 1500$ is then 0.98, and the computation was terminated. The left hand graph in Figure 3 shows the distribution of the exit probability until time $n = 1500$ for the initial value $(0, 0)$, and Fig-

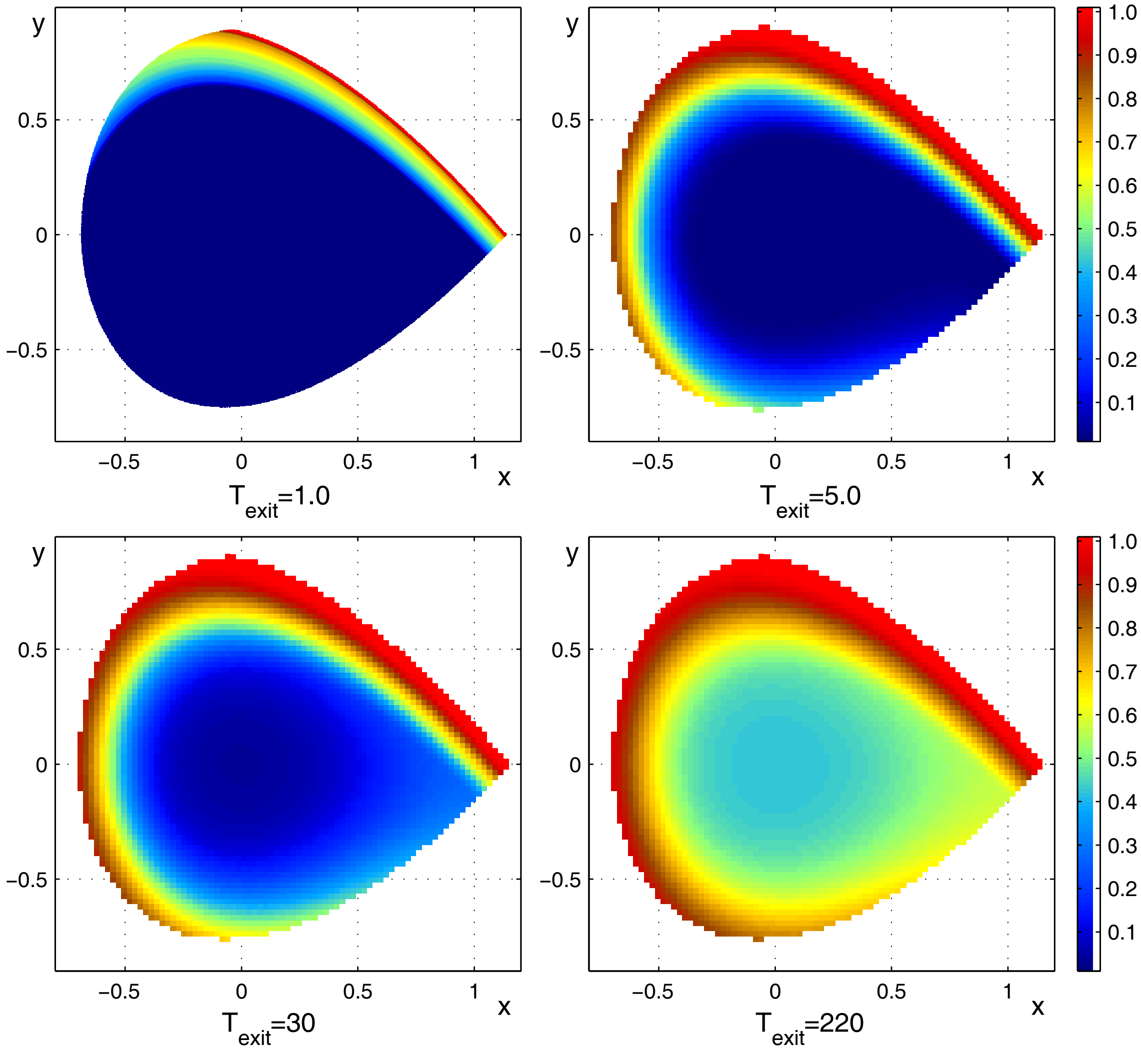

Figure 2: Exit probabilities from $D^{0.15}$ for $\rho = 0.15$ until T_{exit}

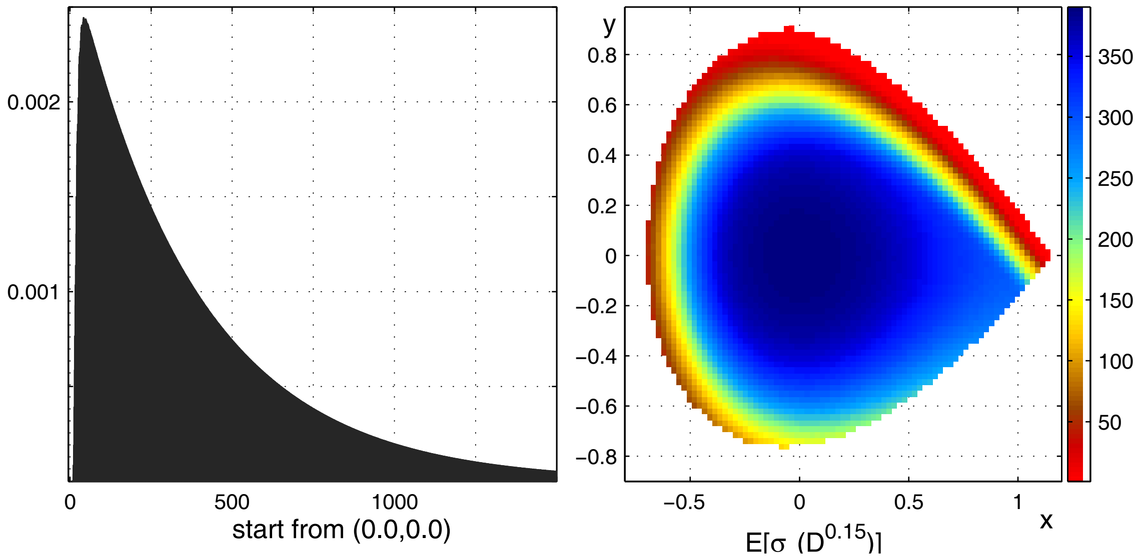

Figure 3: Exit time distribution amd expected exit times until time 1500

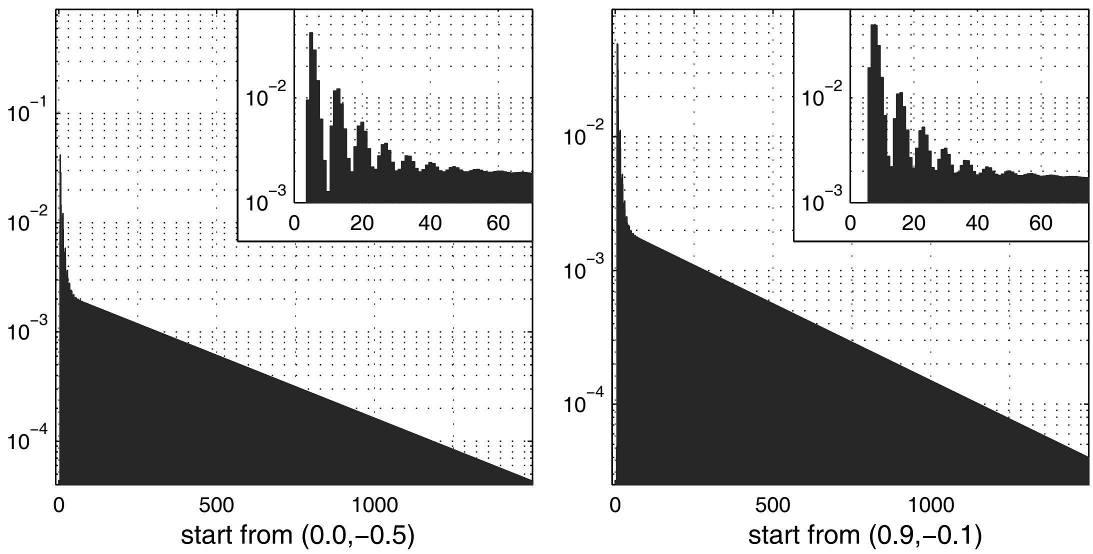

Figure 4: Exit time distribution starting from $(0.0, -0.5)$ and $(0.9, -0.1)$

ure 4 shows the distribution for the initial values $(0.0, -0.5)$ and $(0.9, -0.1)$, now on a logarithmic scale. Both graphs show an exponential tail for the exit time distribution. Indeed, these numerically computed distributions (after some oscillations during the initial settling-in period) closely resemble a 3-parameter Weibull distribution, which is the standard model for lifetime distributions in reliability theory. The oscillations stem from the deterministic dynamics of system (14). Computing an unperturbed solution that starts not too far away from $(0, 0)$ on the positive x-axis, one obtains roughly a time of 6.5 before the trajectory intersects the positive x-axis again. This is exactly the average distance between two maxima in the histograms of the distributions. The right hand graph in Figure 3 shows the expected value of the exit time from all boxes in $D^{0.15}$. These expected times reflect the separation between long sojourn times in the formerly invariant region and short ones outside this area, compare Figure 1.

5.2 A System with Perturbed Double Well Potential

Next we investigate a particle in a two-well potential and consider the following equation:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -\gamma\, y(t) - x^2(t)\,(x^2(t)/2 + 2\,x(t)/3 - 2) + \rho \sin(\eta_t) \\ d\eta_t &= dW_t \mathrm{mod} 2\pi, \end{aligned} \tag{16}$$

with associated control system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -\gamma\, y(t) - x^2(t)\,(x^2(t)/2 + 2\,x(t)/3 - 2) \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix} \tag{17}$$

where again $u(t) \in U^\rho := [-\rho, \rho]$ and the damping coefficient γ is set to 0.1. For $\rho = 0.07$ there are two invariant control sets $C_1^{0.07}$ and $C_2^{0.07}$ that contain the stable fixed points $(1, 0)$ and $(-2, 0)$, respectively, of the uncontrolled equation and one variant control set $D^{0.07}$ containing the hyperbolic fixed point $(0, 0)$ of the uncontrolled equation. Increasing the control range, one finds that the control sets $C_1^{\rho_0}$ and D^{ρ_0} merge for some ρ_0 close to 0.085 and form one variant control set (see Figure 5). Note that before the merger of the control sets, the variant control set increases discontinuously and forms a ring around the invariant control set.

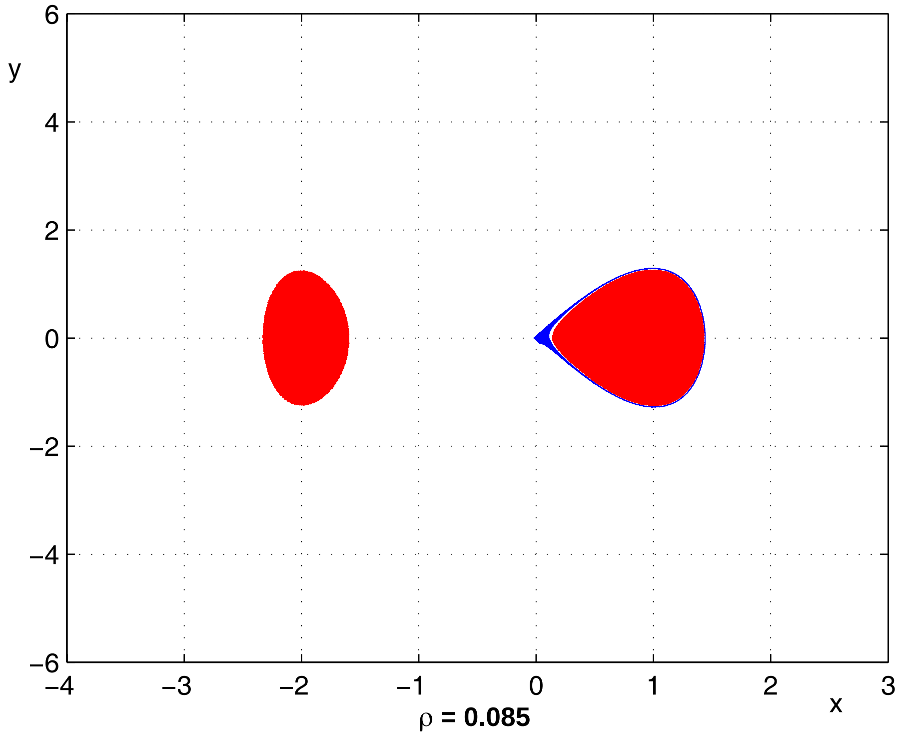

Figure 5: Control sets for the double well potential at $\rho = 0.085$

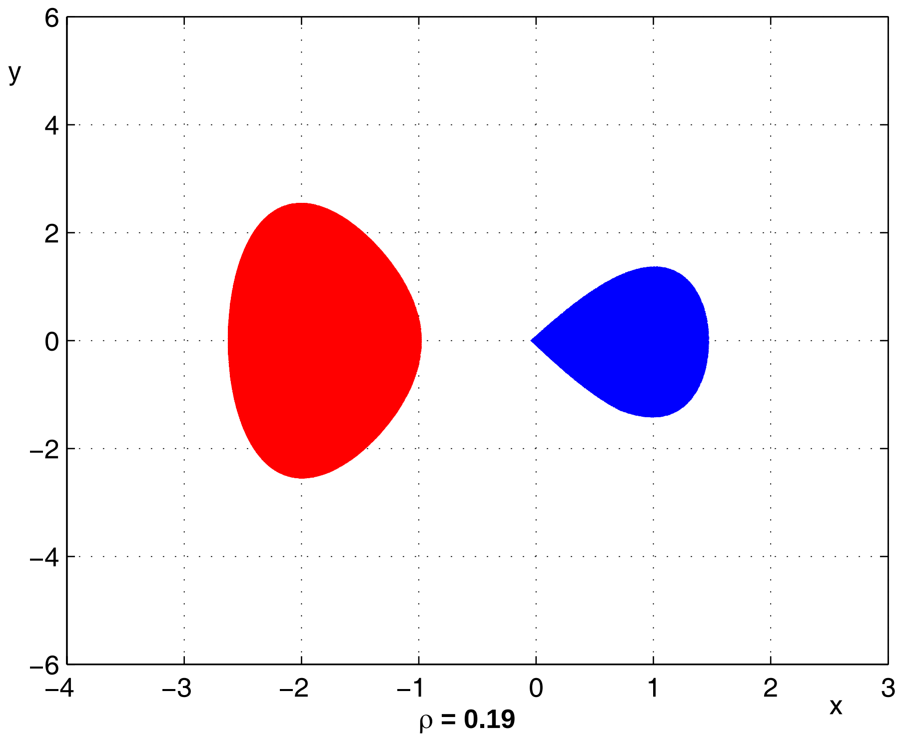

Figure 6: Control sets for the double well potential at $\rho = 0.19$

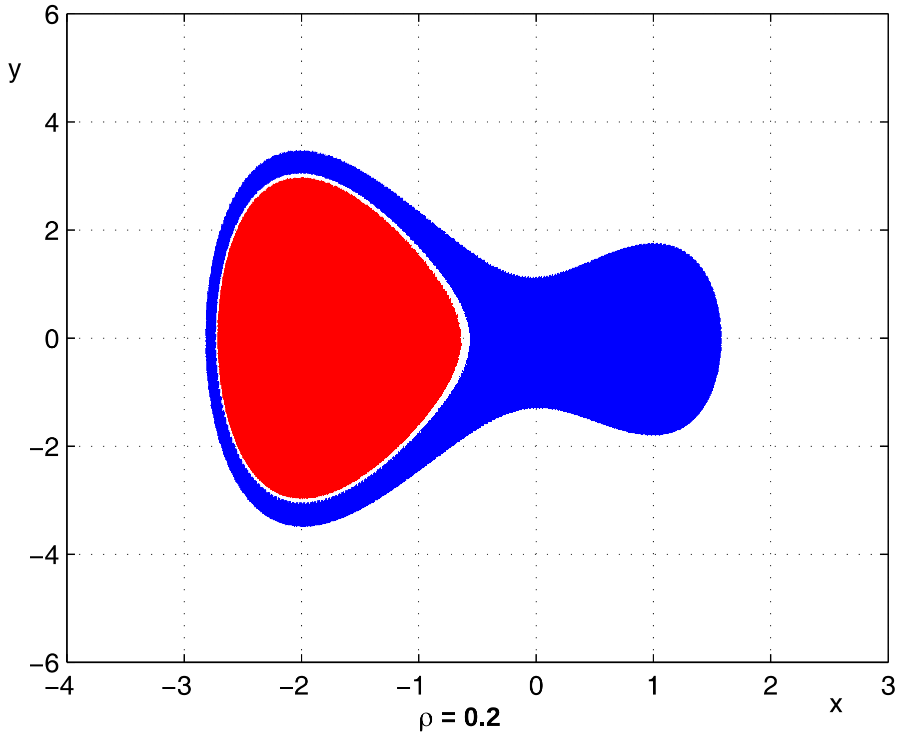

Figure 7: Control sets for the double well potential at $\rho = 0.2$

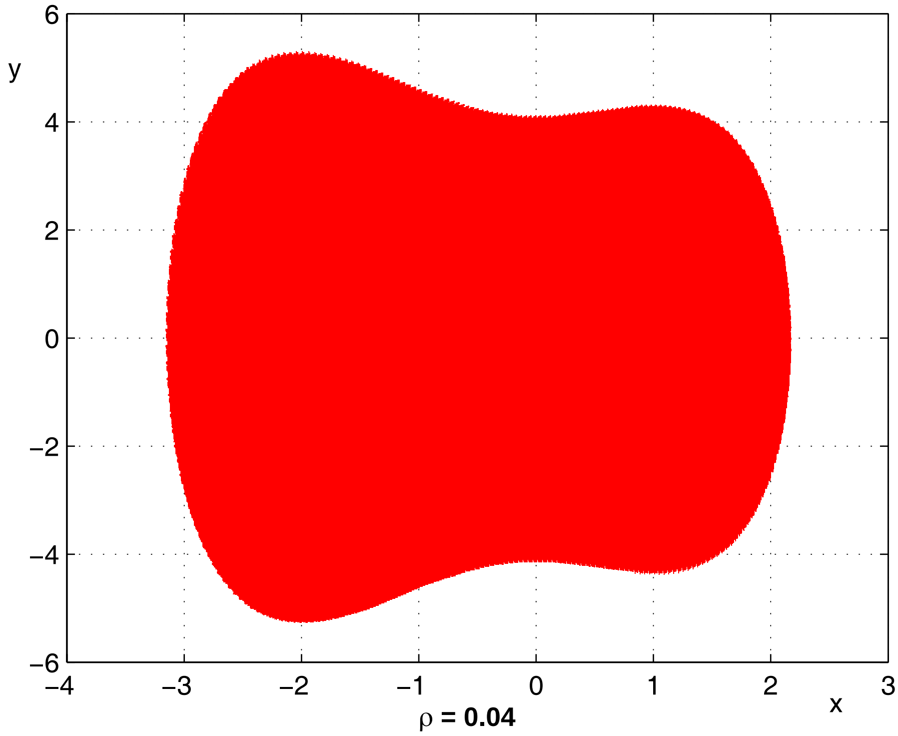

Figure 8: Control sets for the double well potential at $\rho = 0.4$

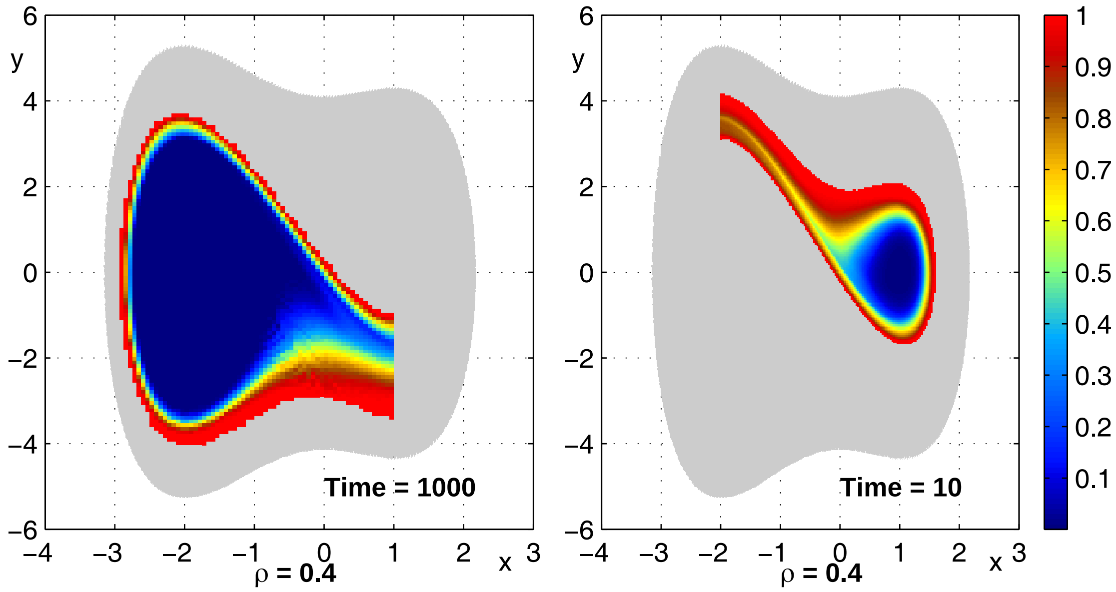

Figure 9: Exit probabilities from the colored region around $C_1^{\rho_0}$ until time $T = 10$ (right) and from the colored region around $C_2^{\rho_0}$ until time $T = 1000$ (left) for $\rho = 0.4$. Parts of the the invariant domains of attraction $A^{inv}(C_1^{\rho_0})$ and $A^{inv}(C_2^{\rho_0})$ become visible.

At some ρ_1 close to $\rho = 0.2$ the remaining control sets $C_2^{\rho_1}$ and D^{ρ_1} merge in a similar way (see Figures 6, 7, and 8).

Thus the corresponding stochastic system (16) possesses one nearly invariant region $C_1^{\rho_0}$ and one nearly invariant region $C_2^{\rho_1}$. Figure 9 shows the exit probabilities until the given exit times from the colored subsets for $\rho = 0.4$. Again, a comparison of the regions of large exit time in Figure 9 with the invariant control sets $C_1^{\rho_0}$ in Figure 5 and $C_2^{\rho_1}$ in Figure 7 show remarkable agreement. Also the invariant domains of attraction of the control sets become visible in Figure 9 as regions, whose exit times are rather large.

6 Appendix: Some Background on Nonlinear Control Systems

In this appendix, we recall some facts on nonlinear control systems. See for example [3] for more information.

6.1 Accessibility and Control Sets

Consider the control-affine system (8) given by

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, X_i(x(t)) \tag{18}$$

with C^∞-vector fields $X_0, \ldots, X_m$ on a C^∞-manifold M of dimension $d < \infty$. We obtain a family of systems by specifying an increasing family of compact, convex control ranges $0 \in intU^\rho \subset \mathbb{R}^m$ with $U^\rho = \mathrm{cl}\,\mathrm{int}U^\rho$ for all $\rho \in [\rho_*, \rho^*]$ and define corresponding sets of control functions $\mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho,$ measurable$\}$. Setting $u \equiv 0$ models the uncontrolled system. We assume that there exists a unique solution $\varphi(t, x, u)$ of (18) for each ρ, for every $u \in \mathcal{U}^\rho$, for every initial state $x \in M$, and for all $t \in [-\infty, \infty]$. If the dependence on ρ is not important, we will simply omit the notation of ρ in the following.

The positive and negative orbits at time $t > 0$ are

$$\mathcal{O}_t^+(x) = \{\varphi(t, x, u),\ u \in \mathcal{U}\},\ \mathcal{O}_t^-(x) = \{\varphi(-t, x, u),\ u \in \mathcal{U}\},$$

and we set

$$\mathcal{O}_{\leq T}^+(x) = \bigcup_{t\in[0,T]} \mathcal{O}_t^+(x),\ \mathcal{O}^-(x) = \bigcup_{t\in[0,T]} \mathcal{O}_t^-(x),$$

$$\mathcal{O}^+(x) = \bigcup_{t\in[0,\infty)} \mathcal{O}_t^+(x),\ \mathcal{O}^-(x) = \bigcup_{t\in[0,\infty)} \mathcal{O}_t^-(x),$$

respectively. A set $D \subset M$ with nonvoid interior is a control set if it is a maximal set with the property $D \subset \mathrm{cl}\mathcal{O}^+(x)$ for every $x \in D$. A control set C with $C = \mathrm{cl}\mathcal{O}^+(x)$ for every $x \in C$ is an invariant control set, the others are called variant. Throughout we assume that system (8) is locally accessible, i.e.,

$$\mathrm{int}\mathcal{O}_{\leq T}^+(x) \neq \varnothing \text{ and } \mathrm{int}\mathcal{O}_{\leq T}^-(x) \neq \varnothing \text{ for all } T > 0.$$

This is guaranteed by the Lie algebra rank condition $\dim \mathcal{LA}\{X_0, ..., X_m\}(x) = d$ for all $x \in M$. We endow the set of control functions $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ with the weak* (or L_1-) topology, which makes $\mathcal{U}$ a compact metric space. Then for $t_n \to t, x_n \to x$, and $u_n \to u$ in $\mathcal{U}$ it follows that

$$\varphi(t_n, x_n, u_n) \to \varphi(t, x, u). \tag{19}$$

We note the following lemma which states that the interior of a positively invariant set is positively invariant.

Lemma 6.1 *Suppose that $I \subset M$ is closed and satisfies $\varphi(t, x, u) \in I$ for all $t \geq 0, x \in I, u \in \mathcal{U}$. Then $\varphi(t, x, u) \in \mathrm{int} I$ for all $x \in \mathrm{int} I, u \in \mathcal{U}$ and $t \geq 0$.*

Proof. Suppose that there are $x \in \mathrm{int} I, t > 0$ and $u \in \mathcal{U}$ with $\varphi(t, x, u) \notin \mathrm{int} I$. Then $\tau := \sup\{s \in (0, t],\ \varphi(t, x, u) \in \mathrm{int} I\}$ satisfies $\varphi(\tau, x, u) \in \partial I$. Hence there is a neighborhood V of $\varphi(\tau, x, u)$ with $V \cap (M \setminus I) \neq \varnothing$. Continuous dependence on initial conditions implies that there are $y \in \mathrm{int} I$ with $\varphi(\tau, y, u) \notin I$ contradicting positive invariance of I. ■

Invariant control sets and hence their interiors are positively invariant. For a set $I \subset M$ with nonvoid interior the domain of attraction is

$$\mathbf{A}(I) = \left\{x \in M,\ \mathrm{cl}\mathcal{O}^+(x) \cap \mathrm{int} I \neq \varnothing\right\}.$$

Domains of attraction are open, since by local accessibility $\mathrm{cl}\mathcal{O}^+(x) = \mathrm{cl}\ \mathrm{int}\mathcal{O}^+(x)$. We define the invariant domain of attraction as the largest invariant set contained in $\mathbf{A}(I)$ (sometime called its invariance kernel).

Definition 6.2 *For $I \subset M$ the invariant domain of attraction is*

$$\mathbf{A}^{inv}(I) = \{x \in \mathbf{A}(I),\ \varphi(t, x, u) \in \mathbf{A}(I) \text{ for all } u \in \mathcal{U} \text{ and } t \in \mathbb{R}_+\}.$$

Here we require implicitly, that all trajectories exist on $\mathbb{R}_+$. This set is related to invariant control sets by the following observation.

Proposition 6.3 *Assume that $\mathbf{A}(I) \cap K$ is positively invariant for a compact set K. Then*

$$\mathbf{A}^{inv}(I) \cap K = \left\{ x \in \mathbf{A}(I) \cap K, \begin{array}{c} \text{if } C \subset \mathrm{cl}\mathcal{O}^+(x) \text{ is an invariant} \\ \text{control set, then } C \cap \mathrm{int} I \neq \varnothing \end{array} \right\} \tag{20}$$

and this set is compact. Furthermore, $\mathrm{int}\left[\mathbf{A}^{inv}(I) \cap K\right]$ *is positively invariant.*

Proof. Let $x \in \mathbf{A}^{inv}(I) \cap K$ and suppose that $C \subset \mathrm{cl}\mathcal{O}^+(x)$ is an invariant control set. Then $\mathrm{int}C \subset \mathcal{O}^+(x)$. If $C \cap \mathrm{int}I = \varnothing$, invariance of $\mathrm{int}C$ implies that we can find $y \in C \cap \mathcal{O}^+(x)$, which is not in $\mathbf{A}(I)$ contradicting $x \in \mathbf{A}^{inv}(I)$. For the converse, let $x \in \mathbf{A}(I) \cap K$ be in the set on the right hand side of (20). Consider $\varphi(t, x, u)$ with $u \in \mathcal{U}$ and $t \in \mathbb{R}_+$. Then by [3, Theorem 3.2.8] there is an invariant control set $C \subset \mathrm{cl}\mathcal{O}^+(x) \cap K$. Then $C \cap \mathrm{int}I \neq \varnothing$ and it follows that $\varphi(t, x, u) \in \mathbf{A}(I)$ and hence $x \in \mathbf{A}^{inv}(I) \cap K$. This proves the other inclusion. In order to see closedness, let $x_n \in \mathbf{A}^{inv}(I) \cap K$ with $x_n \to x$. Then $x \in K$ and, again by [3, Theorem 3.2.8], there is an invariant control set $C \subset \mathrm{cl}\mathcal{O}^+(x) \cap K$. We find $T > 0$ and $u \in \mathcal{U}$ with $\varphi(T, x, u) \in \mathrm{int}C$. Then for n large enough, also $\varphi(T, x_n, u) \in \mathrm{int}C$ and hence $C \subset \mathrm{cl}\mathcal{O}^+(x_n)$. Now (20) implies $C \cap \mathrm{int}I \neq \varnothing$ and $x \in \mathbf{A}^{inv}(I) \cap K$ follows. Invariance of the interior follows by Lemma 6.1. ■

Note also that every invariant control set C satisfies $C \subset \mathbf{A}^{inv}(C)$, but not necessarily $C \subset \mathrm{int}\mathbf{A}^{inv}(C)$.

6.2 Parameter Dependent Control Systems

In this section we describe the behavior of control sets under perturbations of the control range. Here, in addition to control sets, also chain control sets are needed. A nonvoid set $E \subset M$ is a chain control set for (18) if it is a maximal set such that for all $x \in E$ there is a control $u \in \mathcal{U}$ with $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$, and for every $\varepsilon > 0, T > 0$ any two points $x, y \in E$ can be connected by controlled $(\varepsilon, T)-$chains, i.e., there are

$$n \in \mathbb{N}, x_0 = x, ...x_n = y,\ u_0, ...u_{n-1} \in \mathcal{U}, \text{ and } T_0, ..., T_{n-1} > T$$

with

$$\mathrm{d}(\varphi(T_i, x_i, u_i), x_{i+1}) < \varepsilon \text{ for all } i = 0...n-1.$$

For a given interval $[\rho_*, \rho^*]$ of parameters, we denote by $(18)^\rho$ the corresponding control system with control range U^ρ, $\rho \in [\rho_*, \rho^*]$. For every control set D^{ρ_*} and every chain control set E^{ρ_*} of the system $(18)^{\rho_*}$ there are unique control sets D^ρ and unique chain control sets E^ρ for each $\rho \in [\rho_*, \rho^*]$ such that $D^{\rho_*} \subset D^\rho$ and $E^{\rho_*} \subset E^\rho$. If all involved sets are bounded, it is well known that the increasing, compact-valued mappings $\rho \mapsto \mathrm{cl}D^\rho$ and $\rho \mapsto \mathrm{cl}E^\rho$ are continuous with respect to the Hausdorff-metric at all but countably many $\rho-$values (Scherbina's Lemma, [22]).

In order to obtain stronger results on the behavior of control sets and chain control sets, the following inner-pair condition is needed. A pair $(x, u) \in M \times \mathcal{U}$ is called an *inner pair* of the control system (18) if there exists $T > 0$ such that $\phi(T, x, u) \in \mathrm{int}\mathcal{O}^+(x)$. The family of systems $(18)^\rho$ is said to satisfy the *inner-pair condition* if for all $\rho_1 < \rho_2$ each pair $(x, u) \in M \times \mathcal{U}^{\rho_1}$ is an inner pair of the ρ_2-system $(18)^{\rho_2}$. We say that a set $K \subset M$ fulfills the *no-return-condition* if $x \in \mathcal{O}^+(K) \cap K^c$ implies that $\mathcal{O}^+(x) \cap K = \emptyset$, where K^c denotes the complement of K in M.

The following theorem (see [3, Lemma 4.7.3, Lemma 4.7.4, and Theorem 4.7.5]) describes the close relation between control sets and chain control sets if the inner-pair condition holds.

Theorem 6.4 *Consider the family of control affine systems $(18)^\rho$ for $\rho \in [\rho_*, \rho^*]$ where $\rho \mapsto U^\rho$ is continuous with respect to the Hausdorff-metric. Let D^{ρ_*} be a control set and E^{ρ_*} be a chain control set of $(18)^{\rho_*}$ such that $D^{\rho_*} \subset E^{\rho_*}$. Then for all ρ it holds that $D^\rho \subset E^\rho$ where the sets D^ρ and E^ρ are defined as above. Suppose $E^{\rho^*} \subset K$ for a compact set $K \subset M$ that fulfills the no-return condition for the ρ^*-system, and assume that the family $(18)^\rho$ satisfies the inner-pair condition in $[\rho_*, \rho^*]$.*

Then for $\rho_1 < \rho_2$ in $(\rho_*, \rho^*]$ it holds that $\mathrm{cl}D^{\rho_1} \subset E^{\rho_1} \subset \mathrm{int}D^{\rho_2}$ and for all up to at most countably many $\rho-$values, the equation $\mathrm{cl}D^\rho = E^\rho$ is satisfied. The map $(\rho_*, \rho^*) \to \mathcal{C}(K) : \rho \mapsto \mathrm{cl}D^\rho$ is continuous at ρ iff $\mathrm{cl}D^\rho = E^\rho$; the same is true for the map $\rho \mapsto E^\rho$. Here $\mathcal{C}(K)$ denotes the space of compact subsets of K.

In [13] it is shown that the inner-pair condition holds for an important class of systems that includes, in particular, the escape equation (15) and the double well equation (17).

We also need some results on the boundaries of control sets D. Define the entrance and exit boundaries by

$$\begin{aligned} \partial^{ex} D &:= \{x \in \partial D \mid \text{there is } y \in \mathrm{int}D \text{ such that } x \in \mathcal{O}^+(y)\}, \\ \partial^{en} D &:= \{x \in \partial D \mid \text{there is } y \in \mathrm{int}D \text{ such that } y \in \mathcal{O}^+(x)\}, \end{aligned} \tag{21}$$

and the tangential boundary $\partial^{tg} D := \partial D \backslash (\partial^{ex} D \cup \partial^{en} D)$. The sets $\partial^{ex} D$ and $\partial^{en} D$ are disjoint and open in ∂D, and $\partial^{tg} D$ is closed in ∂D. Furthermore, $\partial^{tg} D = \mathrm{cl}\, \partial^{ex} D \cap \mathrm{cl}\, \partial^{en} D$ and $\mathrm{int}_{\partial D} \partial^{tg} D = \emptyset$. The following theorem from [13] shows that exit and entrance boundaries change continuously if the control range U^ρ increases lower semicontinuously and if the control sets themselves change continuously.

Theorem 6.5 *Consider the set-valued mapping* $[\rho_*, \rho^*] \to \mathcal{C}(M)$, $\rho \mapsto \mathrm{cl} D^\rho$, *as in the previous theorem, where now* D^{ρ_*} *is a control set of (18)*$^{\rho_*}$ *and* D^ρ *denotes the unique control set of (18)*$^\rho$ *with* $D^{\rho_*} \subset D^\rho$. *If this map is continuous in the Hausdorff distance at* $\rho_0 \in (\rho_*, \rho^*)$, D^{ρ^*} *is bounded and if the control range* U^ρ *increases lower semicontinuously at* ρ_0, *then the mappings* $\rho \mapsto \partial D^\rho$, $\rho \mapsto \mathrm{cl}\, \partial^{ex} D^\rho$, *and* $\rho \mapsto \mathrm{cl}\, \partial^{en} D^\rho$ *are continuous in the Hausdorff distance at* ρ_0.

Next we will examine more closely how an invariant control set C loses its invariance when merging with a variant control set D while the control range U^ρ is increased. For this we introduce two further specifications of exit boundaries: the part from where under all admissible controls exactly one invariant control set C can be reached, and the part from where C can not be reached at all. We denote the first set by

$$\partial^{ex \to C} D := \left\{ \begin{array}{c} x \in \partial^{ex} D \mid \mathcal{O}^+(x) \text{ bounded and if for some invariant} \\ \text{control set } C' \subset M \text{ we have } C' \cap \mathcal{O}^+(x) \neq \emptyset \text{ then } C = C' \end{array} \right\}$$

and the second one by

$$\partial^{ex \not\to C} D := \{ x \in \partial^{ex} D \mid \mathcal{O}^+(x) \cap C = \emptyset \}.$$

Note that from [3, Theorem 3.2.8] it follows that $\mathcal{O}^+(x) \subset \mathcal{O}^-(C) \cap \mathcal{O}^+(D)$ for all $x \in \partial^{ex \to C} D$.

If the exit boundary of D^{ρ_0} can be decomposed into $\partial^{ex \to C^{\rho_0}} D^{\rho_0}$ and $\partial^{ex \not\to C^{\rho_0}} D^{\rho_0}$, then the exit boundary of the merged set is continuous in the following sense [13].

Theorem 6.6 *Let* $K \subset M$ *be a compact set such that all control sets of the control systems (18)*$^\rho$ *have void intersection with the boundary of* K. *Assume that system (18)*$^{\rho_0}$ *has precisely one invariant control set* $C^{\rho_0} \subset K$ *and one variant control set* $D^{\rho_0} \subset K$ *such that* $C^{\rho_0} \cap \mathrm{cl} D^{\rho_0} \neq \emptyset$. *For each* $\rho > \rho_0$ *let there be precisely one variant control set* $F^\rho \subset K$ *of (18)*$^\rho$ *and* $C^{\rho_0} \cup D^{\rho_0} \subset F^\rho$. *Suppose that* $\mathrm{cl} F^\rho$ *are chain control sets of (18)*$^\rho$ *for each* $\rho > \rho_0$ *and* $\mathrm{cl}(\mathcal{O}^{\rho_0,-}(C^{\rho_0}) \cap \mathcal{O}^{\rho,+}(D^{\rho_0}))$ *is a chain control set of (18)*$^{\rho_0}$. *Finally, assume that* U^ρ *depends continuously on* ρ *with respect to the Hausdorff metric at* ρ_0 *and let* $\delta^{ex \to C^{\rho_0}} D^{\rho_0}$ *and* $\delta^{ex \not\to C^{\rho_0}} D^{\rho_0}$ *be a non-trivial decomposition of* $\delta^{ex} D^{\rho_0}$.

Then $\mathrm{cl}\, \partial^{ex} F^\rho \to \mathrm{cl}\, \partial^{ex \not\to C^{\rho_0}} D^{\rho_0}$ *in the Hausdorff metric for* $\rho \searrow \rho_0$.

Acknowledgement. The algorithms used have been implemented into the MATLAB version of the program package GAIO from Junge. Thus the box handling algorithms from Junge could be used. The control sets are found by methods based on Szolnoki [25]. The necessary solvers for stochastic differential equations and the routines for the computation of the transition matrix were added into the GAIO structure.

References

[1] L. Arnold, W. Kliemann, *Qualitative theory of stochastic systems*, in A.T. Barucha-Reid, *Probabilistic Analysis and Related Topics. Volume 3*, (Academic Press, 1983), 1–79.

[2] F. Colonius, G. Häckl, W. Kliemann, *Dynamic reliability of nonlinear systems under random excitation*, in L.A. Bergman, B.F. Spencer (eds.), *Vibration and Control of Stochastic Dynamical Systems*, *Amer. Soc. Mech. Engrg. ASME DE* **84** (1) (1995), 1007–1024.

[3] F. Colonius, W. Kliemann, *The Dynamics of Control*, (Birkhäuser, 2000).

[4] F. Colonius, W. Kliemann, *Topological, smooth, and control techniques for perturbed systems*, in H. Crauel, M. Gundlach (eds.), *Stochastic Dynamics*, (Springer, 1999), 181–208.

[5] F. Colonius, F.J. de la Rubia, W. Kliemann, *Stochastic models with multistability and extinction levels*, *SIAM J. Appl. Math.* **56** (1996), 919–945.

[6] M. Dellnitz, A. Hohmann, *A subdivision algorithm for the computation of unstable manifolds and global attractors*, *Numer. Math.* **75** (1997), 293–317.

[7] M. Dellnitz, O. Junge, *Set oriented numerical methods for dynamical systems*, in B. Fiedler, G. Iooss, N. Kopell (eds.), *Handbook of Dynamical Systems III: Towards Applications*, (World Scientific, 2002) 221–264.

[8] P. Deuflhard, W. Huisinga, A. Fischer, and Ch. Schütte, *Identification of almost invariant aggregates in reversible nearly uncoupled Markov chains*, *Lin. Alg. Appl.* **315** (2000), 39–59.

[9] J. Fischer, *Cell Mapping for Randomly Perturbed Systems. Robustness Analysis of Dynamical Systems*, Dissertation, (VDI, 2002).

[10] J. Fischer, E. Kreuzer, *Generalized cell mapping for randomly perturbed dynamical systems*, *ZAMM Z. Angew. Math. Mech.*, **81** (11) (2001), 769–777.

[11] G. Froyland, *Ulam's method for random interval maps*, *Nonlinearity*, **12** (1999), 1029–1052.

[12] T. Gayer, *Controlled And Perturbed Systems Under Parameter Variation*, Dissertation, (Shaker, 2003).

[13] T. Gayer, *Control sets and their boundaries under parameter variation*, *J. Diff. Equ.*, **201** (1) (2004), 177–200.

[14] T. Gayer, *On Markov chains and the spectra of the corresponding Frobenius–Perron operator*, *Stochastics and Dynamics* **1** (4) (2001), 477–491.

[15] W. Kliemann, *Analysis of nonlinear stochastic systems*, in W. Schiehlen and W. Wedig (eds.), *Analysis and Estimation of Stochastic Mechanical Systems*, (Springer, 1988), 43–102.

[16] W. Kliemann, *Recurrence and invariant measures for degenerate diffusions*, *Ann. Prob.* **15** (1987), 690–707.

[17] P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, (Springer, 1999) 3rd ed.

[18] H. Kunita, *Supports of diffusion processes*, in K. Ito (ed.), *Proc. Int. Symp. Stoch. Diff. Equ.* (Wiley, 1978), 163–185.

[19] N. Metropolis, S. Ulam, *The Monte Carlo method*, *J. Amer. Statist. Assoc.* **44** (247) (1949), 335–341.

[20] S.P. Meyn, R.L. Tweedie, *Stability of Markovian processes II: Continuous-time processes and sampled chains*, *Adv. Appl. Probab.* **25** (1993), 487–517.

[21] H.E. Nusse, E. Ott, J.A. Yorke, *Saddle-node bifurcations on fractal boundaries*, *Phys. Rev. Lett.*, **75** (13) (1995), 2482–2485.

[22] S.Y. Pilyugin, *The Space of Dynamical Systems with C^0-Topology*, (Springer, 1994).

[23] Ch. Schütte, W. Huisinga and S. Meyn, Metastability of Diffusion Processes, accepted for Proceeding of the IUTAM Symposium on Nonlinear Stochastic Dynamics (August 26 - August 30, 2002, Illinois/USA), Ann. Appl. Probab. 14, pp. 419-458 (2004).

[24] D. Stroock, S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, *Proc. 6th Berkeley Symp. Math. Stat. Prob. Vol. 3* (1972), 333–359.

[25] D. Szolnoki, *Set oriented methods for computing reachable sets and control sets*, *Discrete Contin. Dynam. Systems–B* **3** (2003), 361–382.

[26] J.M.T. Thompson, *Chaotic phenomena triggering the escape from a potential well*, *Proc. R. Soc. Lond. A*, **421** (1989), 195–225.