On Topological Equivalence of Linear Flows with Applications to Bilinear Control Systems*

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Abstract

This paper classifies continuous linear flows using concepts and techniques from topological dynamics. Specifically, the concepts of equivalence and conjugacy are adapted to flows in vector bundles and the Lyapunov decomposition is characterized using the induced flows on the Grassmann and the flag bundles. These results are then applied to bilinear control systems for which their behavior in $\mathbb{R}^d$, on the projective space $\mathbb{P}^{d-1}$ and on the Grassmannians is characterized.

1 Introduction

This paper classifies continuous linear flows on vector bundles with compact metric base space, presenting a generalization of the classification of linear autonomous differential equations based on topological conjugacies. We refer to the monograph by Cong [6] which includes an exposition of equivalences and normal forms for nonautonomous linear differential equations (emphasizing results based on ergodic theory). For linear autonomous equations, it is a classical theorem (Robinson [12]), that topological conjugacies of the corresponding flows in $\mathbb{R}^d$ only give a rough classification, since all exponentially stable equations

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are equivalent. In [1] the authors have presented a classification and normal form theory for linear differential equations $\dot{x} = Ax$ related to the exponential growth rates and the corresponding decomposition of $\mathbb{R}^d$ into subspaces of equal exponential growth rates. These are the Lyapunov spaces given by the sums of the generalized eigenspaces corresponding to eigenvalues of A with equal real part. The purpose of the present paper is to develop a similar theory for general linear flows. One of our main motivations comes from bilinear control systems, which are analyzed in the final section of this paper.

Spaces of equal exponential behavior are of interest, since they form the basis of results on invariant manifolds and Grobman-Hartman type theorems. For nonautonomous problems or linear flows, there are several concepts generalizing the real parts of eigenvalues, in particular, the Sacker-Sell spectrum based on exponential dichotomies and the Morse spectrum based on exponential growth behavior of chains and the related subbundle decomposition [14, 13, 3, 5]. Here we follow the latter approach, since it is based on topological dynamics and it is also well suited for control systems. Thus the main goal of this paper is to classify linear flows according to their (exponential) subbundle decompositions that were first studied by Selgrade.

In Section 2 we introduce concepts of equivalence and conjugacy for linear flows. Section 3 studies topological equivalence in vector bundles. It turns out that, just as in the matrix case, this concept characterizes the stable and unstable bundles of hyperbolic linear flows. Section 4 introduces the spectrum, the Lyapunov index and the short Lyapunov index of linear flows. Section 5 characterizes linear flows with the same short Lyapunov index via a graph constructed from the induced flows on the Grassmann bundles assuming that they have a natural finest Morse decomposition. Section 6 derives some sufficient conditions for this property. Finally Section 7 presents an application to the classification of bilinear control systems. Here some more specific information can be obtained due to the specific nature of control flows.

2 Conjugacy and Equivalence

In this section we present dynamical concepts of 'equivalence' and 'conjugacy' that are adequate for linear flows on vector bundles and, more generally, for skew product flows.

Recall that flows (topological dynamical systems) on a metric space X are given by a continuous map $\Phi : \mathbb{R} \times X \to X$ with $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for all $s, t \in \mathbb{R}$ and $x \in X$. One defines topological conjugacy and equivalence in the following way, see, e.g., Hirsch and Smale [8] and Wiggins [16].

Definition 2.1 *Let $\Psi_i : \mathbb{R} \times X_i \to X_i$, be topological dynamical systems defined on metric spaces X_i,$i = 1, 2$. We say that Ψ_1 and Ψ_2 are*

(i) conjugate if there exists a homeomorphism $h : X_1 \to X_2$ such that $h(\Psi_1(t, x)) = \Psi_2(t, h(x))$ for all $x \in X_1$ and $t \in \mathbb{R}$.

(ii) equivalent if there exists a homeomorphism $h : X_1 \to X_2$ and for each $x \in X_1$ a strictly increasing and continuous time parametrization map $\tau_x : \mathbb{R} \to \mathbb{R}$ such that $h(\Psi_1(t, x)) = \Psi_2(\tau_x(t), h(x))$ for all $x \in X_1$.

Here we are interested in flows on vector bundles of product form $\pi : \mathcal{V} = B \times H \to B$ where B is a metric space, H is a finite dimensional Hilbert space and π is the projection onto the first component. Usually, we take $H = \mathbb{R}^d$ with the Euclidean scalar product.

Remark 2.2 *For general vector bundles, one only requires that locally they are the product of an open subset of the metric space B with H; see Karoubi [10] (or [5, Appendix B]). We refrain from writing down the proofs in this general case, since this is not relevant for our intended applications. However, the general case would require only minor modifications that are technically somewhat involved.*

We always assume that the base space B is compact. For $b \in B$ the set $\mathcal{V}_b = \pi^{-1}(b)$ is called the fiber over the base point b. A linear flow Φ on a vector bundle $\pi : \mathcal{V} \to B$ is a flow Φ on $\mathcal{V}$ which has the form

$$\Phi(t, b, x) = (\theta(t, b), \varphi(t, b, x)), \tag{1}$$

where $\theta(t, b)$ is a flow on the base space B (corresponding to transport of the fibers) and $\varphi : \mathbb{R} \times B \times H \to B \times H$ is linear in x, i.e., for all $\alpha \in \mathbb{R}, x_1, x_2 \in H$ and $b \in B$

$$\varphi(t, b, \alpha(x_1 + x_2)) = \alpha\varphi(t, b, x_1) + \alpha\varphi(t, b, x_2).$$

Thus a linear flow respects the fibers and is linear in each fiber. Where notationally convenient, we write instead of $\Phi(t, v)$ either $\Phi_t(v)$ or $\Phi(t)v$ with $v = (b, x) \in \mathcal{V}$.

We define adequate concepts of conjugacy which preserve the fiber structure in the slightly more general setting of skew product flows $\Phi : \mathbb{R} \times X \times Y \to X \times Y$ on metric spaces X and Y, which have the form

$$\Phi(t, x, y) = (\theta(t, x), \varphi(t, x, y)).$$

where θ and φ are as above, but omitting the linearity requirement. For these flows the adequate concepts of conjugacy and equivalence respect the skew product structure.

Definition 2.3 *For $i = 1, 2$ let X_i and Y_i be metric spaces and let $\Phi_i : \mathbb{R} \times X_i \times Y_i \to X_i \times Y_i$, $\Phi_i = (\theta_i, \varphi_i)$ be skew product flows. We say that Φ_1 and Φ_2 are*

(i) skew conjugate if there exists a skew homeomorphism $h = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$ such that $h(\Phi_1(t, x, y)) = \Phi_2(t, h(x, y))$, i.e., $f : X_1 \to X_2$ and $g : X_1 \times Y_1 \to Y_2$ with

$$f(\theta_1(t, x)) = \theta_2(t, f(x)) \text{ for all } (t, x) \in \mathbb{R} \times X_1, \text{ and}$$
$$g(\theta_1(t, x), \varphi_1(t, x, y)) = \varphi_2(t, f(x), g(x, y)) \text{ for all } (t, x, y) \in \mathbb{R} \times X_1 \times Y_1;$$

(ii) skew equivalent if there exists a homeomorphism $h = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$ as above that maps trajectories of Φ_1 onto trajectories of Φ_2, preserving the orientation, but possibly with a time shift. I.e., for each $(x, y) \in X_1 \times Y_1$ there exists a continuous, strictly increasing time parametrization $\tau_{x,y} : \mathbb{R} \to \mathbb{R}$ such that $h(\Phi_1(t, x, y)) = \Phi_2(\tau_{x,y}(t), h(x, y))$;

(iii) base conjugate if the base flows are conjugate, i.e., there exists a homeomorphism $f : B_1 \to B_2$ such that $f(\theta_1(t, b)) = \theta_2(t, f(b))$ for all $(t, b) \in \mathbb{R} \times B_1$, and analogously for base equivalence.

Clearly, base conjugacy (base equivalence) is a prerequisite for skew conjugacy (skew equivalence).

3 Topological Conjugation and Equivalence in Vector Bundles

This section is devoted to the study of topological conjugacy of linear flows in vector bundles. Just as for matrices, i.e. for linear differential equations of the form $\dot{x}(t) = Ax(t)$, $A \in gl(d, \mathbb{R})$, the key point is to show that any two exponentially stable (or unstable) linear flows are topologically conjugate. The proofs in this section are modeled after the matrix case, see, e.g., Robinson [12, proof of Theorem IV.5.1, and page 113].

Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with compact base space B.

Lemma 3.1 *Suppose that for some norm $\|\cdot\|$ on $\mathcal{V}$ there are $a > 0$ and $C > 0$ with*

$$\|\Phi(t, v)\| \leq C\ e^{-at} \|v\| \text{ for all } t \geq 0.$$

Then for all $\alpha < a$ there exists $\tau = \tau(\alpha) > 0$ such that for all $v \in \mathcal{V}$ and all $t \geq \tau$

$$\|\Phi(t, v)\| \leq e^{-\alpha t} \|v\| .$$

Proof. Let $\alpha < a$. Then there is $\tau = \tau(\alpha) > 0$ such that for all $t \geq \tau$

$$C < e^{t(a-\alpha)}$$

and hence for all $v \in \mathcal{V}$ with $\|v\| = 1$

$$\|\Phi(t, v)\| \leq C\ e^{-at} < e^{t(a-\alpha)} e^{-at} = e^{-\alpha t}.$$

This implies for all $v \in \mathcal{V}$ and all $t \geq \tau$

$$\|\Phi(t, v)\| = \left\| \Phi(t, \frac{v}{\|v\|}) \right\| \|v\| \leq e^{-\alpha t} \|v\| .$$

■

We proceed to the existence of an adapted norm.

Proposition 3.2 *Let $a \in \mathbb{R}$ and suppose that for some (and hence for every) norm $\|\cdot\|$ on $\mathcal{V}$ there is $C > 0$ with*

$$\|\Phi_t v\| \leq C\ e^{-at} \|v\| \text{ for all } t \geq 0.$$

Then for every $\alpha < a$ there is a norm $\|\cdot\|_b^*$ depending continuously on $b \in B$ with

$$\|\Phi_t v\|_{b \cdot t}^* \leq e^{-\alpha t} \|v\|_b^* \text{ for all } t \geq 0,$$

where we use the short form $b \cdot t := \theta(t, b)$.

Proof. *Since all norms on $\mathcal{V}$ are equivalent, it does not matter which norm is used in the assumption. Let $\alpha < a$, pick $\tau = \tau(\alpha) > 0$ according to Lemma 3.1 and define a norm on the vector bundle by*

$$\|v\|^* = \int_0^\tau e^{\alpha s} \|\Phi(s, v)\| \, ds.$$

Furthermore, for all $t > 0$ we write

$$t = n\tau + T \text{ with } 0 \leq T < \tau.$$

Then

$$\begin{aligned}
\|\Phi(t, v)\|^* &= \int_0^\tau e^{\alpha s} \|\Phi(s, \Phi(t, v))\| \, ds \\
&= \int_0^\tau e^{\alpha s} \|\Phi(s + t, v)\| \, ds \\
&= \int_0^\tau e^{\alpha s} \|\Phi(n\tau + T + s, v)\| \, ds \\
&= \int_0^{\tau - T} e^{\alpha s} \|\Phi(n\tau, \Phi(T + s, v))\| \, ds + \int_{\tau - T}^\tau e^{\alpha s} \|\Phi((n+1)\tau, \Phi(T - \tau + s, v))\| \, ds \\
&= \int_T^\tau e^{\alpha(s - T)} \|\Phi(n\tau, \Phi(s, v))\| \, ds + \int_0^T e^{\alpha(s - T + \tau)} \|\Phi((n+1)\tau, \Phi(s, v))\| \, ds
\end{aligned}$$

using the time transformations $s := T + s$ and $s := T - \tau + s$, respectively.

Observe that by choice of τ one has for all $w \in \mathcal{V}$ and all $n = 0, 1, ...$

$$\|\Phi(n\tau, w)\| \leq e^{-\alpha n\tau} \|w\|.$$

Hence one finds

$$\begin{aligned}
\|\Phi(t, v)\|^* &\leq \int_T^\tau e^{\alpha(s - T)} e^{-\alpha n\tau} \|\Phi(s, v)\| \, ds + \int_0^T e^{\alpha(s - T + \tau)} e^{-\alpha(n+1)\tau} \|\Phi(s, v)\| \, ds \\
&= \int_T^\tau e^{\alpha(s - T - n\tau)} \|\Phi(s, v)\| \, ds + \int_0^T e^{\alpha(s - T + \tau - (n+1)\tau)} \|\Phi(s, v)\| \, ds \\
&= e^{-\alpha t} \int_0^\tau e^{\alpha s} \|\Phi(s, v)\| \, ds \\
&= e^{-\alpha t} \|v\|^*.
\end{aligned}$$

■

This proposition shows that an exponentially stable linear flow $\Phi : \mathbb{R} \times \mathcal{V} \longrightarrow \mathcal{V}$ admits an adapted norm on the vector bundle relative to which the orbits decrease uniformly. As in the matrix case, this is a key tool in characterizing skew equivalent flows.

Theorem 3.3 *Let Φ and Ψ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$, and $\mathcal{W} = C \times \mathbb{R}^d \to C$ respectively with compact bases. If the flows are base equivalent and both are exponentially stable, then they are skew equivalent.*

Proof. By Proposition 3.2, there exist $\alpha, \beta > 0$ and adapted norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Psi$ such that for all v and all $t \geq 0$

$$\|\Phi(t,v)\|_\Phi \leq e^{-\alpha t} \|v\|_\Phi \text{ and } \|\Phi(t,v)\|_\Psi \leq e^{-\beta t} \|v\|_\Psi .$$

Running times backward, we get for $t \leq 0$

$$\|\Phi(t,v)\|_\Phi^* \geq e^{\alpha |t|} \|v\|_\Phi \text{ and } \|\Phi(T,v)\|_\Psi \geq e^{\beta |t|} \|v\|_\Psi .$$

Using the above estimates, we see that for each $v \neq 0$, i.e., not in the zero section Z, the trajectory $\Phi(t,v)$ crosses the unit sphere bundle $\mathcal{S}_\Phi = \{w \in \mathcal{V},\ \|w\|_\Phi = 1\}$ exactly once and each trajectory $\Psi(t,v)$ crosses the unit sphere bundle $\mathcal{S}_\Psi = \{w \in \mathcal{W},\ \|w\|_\Psi = 1\}$ exactly once.

First define a homeomorphism h_0 from $\mathcal{S}_\Phi$ to $\mathcal{S}_\Psi$. Denote the base equivalence by $g : B \to C$ and define for $v = (b,x) \in \mathcal{S}_\Phi$

$$h_0(b,x) = \left(g(b), \frac{x}{\|x\|_\Psi} \right);$$

here $\|x\|_\Psi$ denotes the adapted norm of $(g(b), x) \in C \times \mathbb{R}^d$. Notice that the inverse of h_0 exists and is given by

$$h_0^{-1}(c,y) = \left(g^{-1}(c), \frac{y}{\|y\|_\Phi} \right),\ w = (c,y) \in \mathcal{S}_\Psi$$

To extend h_0 to all of $\mathcal{V}$ we define $\tau(v)$ for $v \in \mathcal{V}$ to be the time with

$$\|\Phi(\tau(v), v)\|_\Phi = 1.$$

This time depends continuously on $v \in \mathcal{V}$. Because of the definition, it follows that

$$\tau(\Phi(t,v)) = \tau(v) - t. \tag{2}$$

Now define a homeomorphism $h : \mathcal{V} \to \mathcal{W}$ by

$$h(v) = \begin{cases} \Psi(-\tau(v), h_0(\Phi(\tau(v), v))) & \text{for} \quad v \notin Z \\ (g(b), 0) & \text{if} \quad v \in Z \end{cases}.$$

Then h is a conjugacy: First observe that h maps fibers into fibers by base equivalence of Φ and Ψ. Furthermore, the conjugation property follows using

(2) from

$$\begin{aligned} h(\Phi(t,v)) &= \Psi(-\tau(\Phi(t,v)), h_0(\Phi(\tau(\Phi(t,v)), \Phi(t,v)))) \\ &= \Psi(-[\tau(v)-t], h_0(\Phi(\tau(v)-t), \Phi(t,v)) \\ &= \Psi(t, \Psi(-\tau(v), h_0(\Phi(\tau(v), v)) \\ &= \Psi(t, h(v)). \end{aligned}$$

Because τ and the flows Φ and Ψ are continuous, it follows that h is continuous at points $v \notin Z$. To check continuity at the zero section of the vector bundle, notice that if v_j converges to an element in the zero section, then $\tau_j = \tau(v_j)$ goes to negative infinity. Letting

$$w_j = h_0(\Phi(\tau_j, v_j)$$

we have that $|w_j|_\Psi = 1$. Thus, by definition of h and stability of Ψ,

$$\|h(v_j)\|_\Psi = \|\Psi(-\tau_j, w_j)\|_\Psi \leq e^{-\beta|\tau_j|}$$

must go to zero. Therefore $h(v_j)$ converges to $0 = h(0)$. This proves the continuity at zero.

To show that h is injective, take v, w with $h(v) = h(w)$. If v is in the zero section, then $0 = h(v) = h(w)$, so $v = w$, since both are in the same fiber. Now assume that v is not in the zero section. Then $h(w) = h(v)$ and hence w is not in the zero section. Letting $\tau = \tau(v)$ one finds

$$h(\Phi(\tau, v)) = \Psi(\tau, h(v)) = \Psi(\tau, h(w)) = h(\Psi(\tau, w)).$$

This shows that $h(\Phi(\tau, w)) = h(\Phi(\tau, v)) \in \mathcal{S}_\Psi$ (since $\Phi(\tau, v) \in \mathcal{S}_\Phi$), so $\Phi(\tau, w) \in \mathcal{S}_\Phi$ and $\tau(w) = \tau(v) = \tau$.

Since $h_0(\Phi(\tau, v)) = h(\Phi(\tau, v)) = h(\Phi(\tau, w)) = h_0(\Phi(\tau, w))$, and h_0 is injective, we have $\Phi(\tau, v) = \Phi(\tau, w)$ and so $v = w$. Thus h is injective in all cases.

Reversing the roles of Φ and Ψ in the arguments above, we get that h^{-1} exists (and so h is surjective) and is continuous. This completes the proof. ■

Corollary 3.4 *Let Φ and Ψ be linear flows on vector bundles with compact bases.*

(i) The flows are skew equivalent if they are base equivalent and both flows are exponentially unstable.

(ii) Suppose that both flows are hyperbolic, i.e. the vector bundles can be written as the Whitney sums of exponentially stable and unstable subbundles . Then they are skew conjugate iff they are base conjugate and the dimensions of their stable (and unstable) subbundles coincide.

Proof. Part (i) of this corollary is proved via time reversal. Skew conjugacy in part (ii) follows by piecing together the stable and the unstable parts of a flow, just as in the matrix case. Conversely, base conjugacy follows trivially and the dimension condition follows by the invariance of domain theorem, see Warner [**?**]. since a conjugacy maps fibers $\{u\} \times \mathbb{R}^{l^+}$ of the stable bundle onto fibers of the stable bundle. ■

This corollary shows that topological conjugacy in vector bundles gives a very rough classification of linear flows in terms of stable and unstable subbundles. Smooth conjugacies, however, result in a very fine classification: Recall the situation for linear ordinary differential equations $\dot{x} = Ax$ and $\dot{y} = By$ with linear flows φ and ψ in $\mathbb{R}^d$, respectively. The systems φ and ψ in $\mathbb{R}^d$ are C^k-conjugate (for $k \geq 1$) iff they are linearly conjugate iff the matrices A and B are similar. The corresponding result for linear flows is given in the next proposition.

Definition 3.5 *Let Φ and Ψ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$, and $\mathcal{W} = C \times \mathbb{R}^d \to C$ respectively, with compact bases. We say that $h = (f, g) : B \times \mathbb{R}^d \to C \times \mathbb{R}^d$ is a C^k-conjugacy ($k \geq 1$) between Φ and Ψ if h is a skew conjugacy and for all $b \in B$ the maps $g(b, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ are C^k-diffeomorphisms.*

Proposition 3.6 *Let $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$ be linear flows on vector bundles $\mathcal{V} = B \times \mathbb{R}^d \to B$, and $\mathcal{W} = C \times \mathbb{R}^d \to C$ respectively, with compact bases. If Φ and Ψ are C^1-conjugate via $h = (f, g)$ then they are linearly conjugate in the following sense*

$$\varphi(t, \cdot, b) = [D_x g(\theta_t b, 0)]^{-1} \circ \psi(t, \cdot, f(b)) \circ D_x g(b, 0). \tag{3}$$

Proof. We will use the following notation: $\Phi_t(b, x) = (\theta_t b, \varphi(t, x, b))$, $\Psi_t(c, y) = (\vartheta_t c, \psi(t, y, c))$ and $h(b, x) = (f(b), g(b, x))$. The conjugation property yields

$$\begin{aligned} h \circ \Phi_t(b, x) = h(\theta_t b, \varphi(t, x, b)) &= (f(\theta_t b), g(\theta_t b, \varphi(t, x, b)) \\ &= (\vartheta_t f(b), \psi(t, g(b, x), f(b))), \end{aligned}$$

and hence

$$\varphi(t, x, b) = g^{-1}(\theta_t b, \psi(t, g(b, x), f(b))). \tag{4}$$

Differentiation of (4) with respect to x yields

$$\varphi(t, \cdot, b) = D_x g^{-1}(\theta_t b, \psi(t, g(b, x), f(b))) \circ D_x \psi(t, g(b, x), f(b)) \circ D_x g(b, x). \tag{5}$$

The zero section $B \times \{0\}$ is invariant under the flow, hence evaluating (5) at the zero section yields with $\psi(t, g(b, 0), f(b)) = g(\theta_t b, \varphi(t, 0, b)) = g(\theta_t b, 0)$ and $D_x g^{-1}(\theta_t b, g(\theta_t b, 0)) = [D_x g(\theta_t b, g(\theta_t b, 0))]^{-1} = [D_x g(\theta_t b, 0)]^{-1}$ for all $t \in \mathbb{R}$ the result

$$\varphi(t, \cdot, b) = [D_x g(\theta_t b, 0)]^{-1} \circ \psi(t, \cdot, f(b)) \circ D_x g(b, 0). \tag{6}$$

■

Remark 3.7 *In the case of linear differential equations, linear conjugations preserve the eigenvalues and the Jordan structure of the matrices. If the conjugating skew homeomorphism $h = (f, g)$ of two linear flows is linear, i.e.,*

$g : B \times \mathbb{R}^d \to \mathbb{R}^d$ is linear in the second argument, the flows are called cohomologous. According to Proposition 3.6 this holds if two linear flows are C^1-conjugate. Cohomologous flows preserve the Morse spectrum (see below for the definition) and the associated subbundle decomposition (see [5], Proposition 5.4.4 for the case of identical base flows, but the proof is easily extended to shift conjugate base flows using uniform continuity of f).

Corollary 3.4 and Proposition 3.6 show which type of classification can be achieved using topological and C^k-conjugacies of linear flows. As in the matrix case, neither of them results in a dynamic characterization of flows whose subbundles have the same exponential behavior. In the next two sections, we make this precise and provide a characterization using conjugacies of the induced flows on the Grassmann bundles.

4 Spectrum and Lyapunov Index of Linear Flows

This section recalls the (Morse) spectrum of a linear flow and introduces its Lyapunov index.

Recall the following notions from topological dynamics (see e.g. [7] or [5, Appendix B]). For a flow Φ on a compact metric space Y a compact subset $K \subset Y$ is called *isolated invariant*, if it is invariant and there exists a neighborhood N of K, i.e., a set N with $K \subset \operatorname{int} N$, such that $\Phi(t, x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$. A *Morse decomposition* is a finite collection $\{\mathcal{M}_i,\ i = 1, ..., n\}$ of nonvoid, pairwise disjoint, and isolated compact invariant sets such that

(i) for all $x \in X$ one has $\omega(x)$, $\alpha(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$; and

(ii) suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l}$ and $x_1, ..., x_l \in X \setminus \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, ..., l$; then it follows that $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_l}$, where $\alpha(x)$ and $\omega(x)$ denote the α- and ω-limit set, respectively, from a point $x \in Y$.

The elements of a Morse decomposition are called *Morse sets*. Observe that $\mathcal{M}_i \preceq \mathcal{M}_j$, if $\alpha(x) \subset \mathcal{M}_i$ and $\omega(x) \subset \mathcal{M}_j$ for some x, defines an order on the Morse sets. A Morse decomposition is finer than another one, if all elements of the second one are contained in element of the first. If a finest Morse decomposition exists, its elements are the maximal chain transitive sets, i.e. maximal sets that have the property that for all elements x, y and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)-$chain from x to y given by $n \in \mathbb{N}$, $T_0, ..., T_{n-1} \geq T$, and $x_0 = x, ..., x_n = y$ with $d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, ..., n-1$.

The following theorem goes back to Selgrade [14] and provides a decomposition via chain transitivity properties in the projective bundle.

Theorem 4.1 (Selgrade) *Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with chain transitive flow on the base space B. Then the chain recurrent set of the induced flow $\mathbb{P}\Phi$ on the projective bundle $\mathbb{P}\mathcal{V}$ has finitely many, linearly ordered, components $\{\mathcal{M}_1, ..., \mathcal{M}_l\}$, and $1 \leq l \leq d := \dim \mathcal{V}_b$, $b \in B$. The $\mathcal{M}_i$*

provide the finest Morse decomposition. Every maximal chain transitive set $\mathcal{M}_i$ *defines an invariant subbundle of* $\mathcal{V}$ *via*

$$\mathcal{V}_i = \mathbb{P}^{-1}\left(\mathcal{M}_i\right) = \{v \in \mathcal{V},\ v \notin Z \textit{ implies } \mathbb{P}v \in \mathcal{M}_i\}$$

and the following decomposition into a Whitney sum holds:

$$\mathcal{V} = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_l.$$

In analogy to the matrix case, we call this subbundle decomposition the *Lyapunov* decomposition of $\mathcal{V}$ relative to Φ.

With an appropriate concept of exponential growth rates, this decomposition also yields a notion of spectrum. Denote by $Z := \{(b,0),\, b \in B\}$ the zero section of the bundle $\mathcal{V}$. For points $v \in \mathcal{V}$ not in the zero section $Z \subset \mathcal{V}$ the Lyapunov exponent (or exponential growth rate of the corresponding trajectory) is given by

$$\lambda(v) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_t v\| \tag{7}$$

and the Lyapunov spectrum Σ_{Ly} of the linear flow Φ is the set of all Lyapunov exponents

$$\mathbf{\Sigma}_{Ly} = \{\lambda(v),\ v = (b,x) \in \mathcal{V} \text{ with } x \neq 0\}\,. \tag{8}$$

While the Lyapunov spectrum may be very complicated, the concept of Morse spectrum [5] yields a simple structure. It is defined via (ε, T)-chains in the projective bundle. Recall that for ε, $T > 0$ an (ε, T)-chain ζ in $\mathbb{P}\mathcal{V}$ of Φ is given by $n \in \mathbb{N}$, $T_0, ..., T_{n-1} \geq T$, and $p_0, ..., p_n$ in $\mathbb{P}\mathcal{V}$ with $d(\Phi(T_i, p_i), p_{i+1}) < \varepsilon$ for $i = 0, ..., n-1$. Define the finite time exponential growth rate of such a chain ζ (or "chain exponent") by

$$\lambda(\zeta) = \left(\sum_{i=0}^{n-1} T_i\right)^{-1} \sum_{i=0}^{n-1} \left(\log \|\Phi(T_i, v_i)\| - \log \|v_i\|\right),$$

where $v_i \in \mathbb{P}^{-1}(p_i)$. For a Lyapunov subbundle $\mathcal{V}_i$ projecting to a maximal chain transitive set $\mathcal{M}_i$ in the projective bundle, the Morse spectrum of $\mathcal{V}_i$ is

$$\mathbf{\Sigma}_{Mo}(\mathcal{V}_i) = \left\{ \begin{array}{c} \lambda \in \mathbb{R},\ \text{there are } \varepsilon^k \to 0,\, T^k \to \infty \text{ and } (\varepsilon^k,\, T^k)\text{-chains} \\ \zeta^k \text{ in } \mathcal{M}_i \text{ with } \lambda(\zeta^k) \to \lambda \text{ as } k \to \infty \end{array} \right\}.$$

The main results on the Morse spectrum are collected in the following theorem.

Theorem 4.2 *Let* Φ *be a linear flow on a vector bundle* $\pi : \mathcal{V} \to B$ *with chain transitive flow on the base space* B*. Then the Morse spectrum*

$$\mathbf{\Sigma}_{Mo}(\Phi) := \bigcup_{i=1}^{l} \mathbf{\Sigma}_{Mo}(\mathcal{V}_i)$$

contains the Lyapunov spectrum and for every i

$$\mathbf{\Sigma}_{Mo}(\mathcal{M}_i) = \left[\kappa^*(\mathcal{V}_i), \kappa(\mathcal{V}_i)\right],$$

where $\kappa^*(\mathcal{V}_i) = \inf \Sigma_{Mo}(\mathcal{V}_i)$, $\kappa(\mathcal{V}_i) = \sup \Sigma_{Mo}(\mathcal{V}_i)$, *with* $\kappa^*(\mathcal{V}_i) < \kappa^*(\mathcal{V}_j)$ *and* $\kappa(\mathcal{V}_i) < \kappa(\mathcal{V}_j)$ *for* $i < j$*; the boundary points* $\kappa^*(\mathcal{V}_i), \kappa(\mathcal{V}_i)$ *are Lyapunov exponents of* Φ*.*

For each Lyapunov subbundle $\mathcal{V}_i$, the Morse spectrum $\Sigma_{Mo}(\mathcal{V}_i)$ describes the exponential behavior of the solutions $\varphi(\cdot, b, x)$ with $(b, x) \in \mathcal{V}_i$. Hence our interest is in finding dynamical characterizations of the Lyapunov decomposition and the dimensions of the subbundles. For matrices, the Lyapunov forms summarize such a characterization, see [1]. This idea is generalized to linear flows in the following way.

Definition 4.3 *Consider a linear flow* Φ *on a vector bundle* $\pi : \mathcal{V} \to B$ *with Lyapunov decomposition* $\mathcal{V} = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_l$*. The Lyapunov index* $L(\Phi)$ *of* Φ *is the matrix*

$$\begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_l \end{bmatrix} \text{ with } \Lambda_i = \begin{bmatrix} \kappa^*(\mathcal{V}_i),\ \kappa(\mathcal{V}_i) & & 0 \\ & \ddots & \\ 0 & & \kappa^*(\mathcal{V}_i),\ \kappa(\mathcal{V}_i) \end{bmatrix},$$

where the block size of Λ_i *is the dimension* $\dim \mathcal{V}_i$ *of the corresponding subbundle. The blocks are arranged according to the order of the Lyapunov bundles. Two linear flows* Φ_i *are called Lyapunov equivalent if* $L(\Phi_1) = L(\Phi_2)$*.*

Remark 4.4 *Lyapunov equivalence is an equivalence relation on the set of linear flows with fixed dimension* d *(on compact chain transitive base spaces). Each class has a unique Lyapunov index given by* l *pairs of real numbers* $\kappa^*(\mathcal{V}_1) < ... < \kappa^*(\mathcal{V}_l)$, $\kappa(\mathcal{V}_1) < ... < \kappa(\mathcal{V}_l)$ *and* l *natural numbers* $d_i = m(\mathcal{V}_i)$*.*

Remark 4.5 *For matrices, one can find in every Lyapunov equivalence class a unique (diagonal) flow of the form* $e^{\Lambda t}$ *hence representing a normal form, called the Lyapunov normal form. For general linear flows we use the matrix above only as a symbol for the corresponding equivalence class of linear flows. In particular, for a given base flow one should not expect that a linear flow of such a form exists.*

Following the matrix case in [1] we also define:

Definition 4.6 *The short Lyapunov index* $SL(\Phi)$ *of a linear flow* Φ *is given by the vector of the dimensions* d_i *of the* l *Lyapunov subbundles (in their natural order):* $SL(\Phi) = (l, d_1, ..., d_l)$*.*

Two linear flows Φ_1 and Φ_2 have the same short Lyapunov index if and only if the (ordered) blocks of $L(\Phi_1)$ and $L(\Phi_2)$ have the same dimensions. This form does not contain stability information, since it does not include the actual size of the Lyapunov exponents, only their order. To separate the stable, center and unstable bundles, one may also introduce the following definitions.

Definition 4.7 *(i) The short zero-Lyapunov index $SL_0(\Phi)$ is given by the vector of the dimensions d_i of the Lyapunov subbundles (in their natural order), and the number of subbundles for which the Morse spectrum is, negative, includes zero, and is positive: $SL_0(\Phi) = (l^-, l^0, l^+, d_1, ..., d_k)$ where $l = l^- + l^0 + l^+ \leq d$ is the number of Lyapunov subbundles.*
(ii) The stability Lyapunov index of Φ is given by the dimensions of the stable, center and unstable subbundles, i.e., $SL^s(\Phi) = (l^-, l^0, l^+)$.

Clearly, a system is hyperbolic if $l^0 = 0$.

5 Grassmann Graphs and Finest Morse Decompositions

In this section we provide a characterization of Lyapunov equivalent linear flows using a graph (the Grassmann graph) constructed from the induced flows on the Grassmann bundles. This characterization generalizes the matrix case.

First we recall some facts on Morse decompositions in Grassmann bundles. We denote by $\mathbb{G}_i$ the Grassmannian of $i-$dimensional subspaces of $\mathbb{R}^d$ (which may be identified with a subset of the projective space of the exterior product $\Lambda^i\mathbb{R}^d$). The $k-$th flag of $\mathbb{R}^d$ is given by the following $k-$sequences of subspace inclusions,

$$\mathbb{F}_k = \{F_k = (V_1, ..., V_k),\ V_i \subset V_{i+1} \text{ and } \dim V_i = i \text{ for } i = 1, ..., k\}.$$

For $k = d$ we obtain the complete flag $\mathbb{F} = \mathbb{F}_d$. For a vector bundle $\mathcal{V} = B \times \mathbb{R}^d$ one obtains bundles of Grassmannians $\mathbb{G}_k\mathcal{V} = B \times \mathbb{G}_k$ and flags $\mathbb{F}_k\mathcal{V} = B \times \mathbb{F}_k$. For simplicity we denote the flows induced by a linear flow Φ on these bundles also by Φ. It is known that there are finest Morse decompositions on the Grassmann and flag bundles. However, we will use the following more specific information (see [4], Theorem 6).

Theorem 5.1 *Let Φ be a linear flow on a vector bundle $\pi : \mathcal{V} \to B$ with chain transitive compact base space B and dimension d. Let $\mathcal{V}_i$ with dimension d_i, $i = 1, ..., l$, be the Lyapunov subbundles. Define for $1 \leq k \leq d$ the index set*

$$I(k) := \{(k_1, k_2, ..., k_l),\ k_1 + k_2 + ... + k_l = k \text{ and } 0 \leq k_i \leq d_i\}.$$

Then a Morse decomposition in the Grassmann bundle $\mathbb{G}_k\mathcal{V} \to B$ is given by the sets

$$\mathcal{N}^k_{k_1,...,k_l} = \mathbb{G}_{k_1}\mathcal{V}_1 \oplus ... \oplus \mathbb{G}_{k_l}\mathcal{V}_l,\ (k_1, ..., k_l) \in I(k), \quad (9)$$

with the interpretation that on the right hand side we have in every fiber $\mathcal{V}_b$ over $b \in B$ the sum of arbitrary k_i-dimensional subspaces of $\mathcal{V}_{i,b}$. In particular, every chain recurrent component of $\mathbb{G}_k\mathcal{V}$ is contained in one of these Morse sets.

The Morse decomposition (9) is the *natural Morse decomposition* on $\mathbb{G}_k\mathcal{V}$. We work under the assumption that for each k this is also the finest Morse decomposition of the flow on $\mathbb{G}_k\mathcal{V}$. This will allow us to characterize short Lyapunov equivalence using the flows on the Grassmann bundles.

Remark 5.2 *The linear flow of an autonomous linear differential equation has the natural finest Morse decompositions. By Floquet theory, one sees that this also follows for periodic linear differential equations. Furthermore, Braga Barros and San Martin [2] have shown that a large class of linear flows has this property. In Section 6 we discuss some situations in which one can show directly that (9) is, indeed, the finest Morse decomposition.*

Next we define an order (with associated graph) via the finest Morse decompositions on the Grassmann bundles. On each $\mathbb{G}_k\mathcal{V}$, $k = 1, ..., d$ we use the order $\preceq_k$ related to the finest Morse decomposition. And for Morse sets $\mathcal{N}^k$, $\mathcal{N}^{k-1}$ in $\mathbb{G}_k\mathcal{V}$ and in $\mathbb{G}_{k-1}\mathcal{V}$, respectively, we set $\mathcal{N}^{k-1} \sqsubseteq_{k-1} \mathcal{N}^k$ if $\mathcal{N}^k$ projects down to $\mathcal{N}^{k-1}$. Combined, $\preceq_k$ and $\sqsubseteq_{k-1}$ define the graph of an order relation.

Finite graphs that represent orders are directed graphs without loops. For these graphs one can define "elementary graphs" that only consider "nearest neighbors", i.e., without edges that result from transitivity. Here the situation is slightly more complicated, since these graphs represent the d different orders $\preceq_k$ and the $(d-1)$ different orders $\sqsubseteq_{k-1}$. Since the order $\sqsubseteq_{k-1}$ only involves the Grassmann bundles $\mathbb{G}_k\mathcal{V}$ and $\mathbb{G}_{k-1}\mathcal{V}$, all these edges of these graphs are "nearest neighbors". Hence edges of $\sqsubseteq_{k-1}$ cannot be used in a transitive way without destroying the order. On the other hand, the orders $\preceq_k$ on each Grassmann bundle $\mathbb{G}_k\mathcal{V}$ involve all Morse sets on $\mathbb{G}_k\mathcal{V}$, hence elementary versions on each level k make sense. More precisely: Let G be an order graph in $\mathbb{F}\mathcal{V}$ and G_k its subgraphs corresponding to level k. An edge (e_1, e_2) in G_k is called a transitivity edge, if there exist nodes $n_1, ..., n_l$, $l \geq 3$ such that $(n_1 = e_2, ..., n_l = e_2)$ is a path in G_k. The elementary graph $E(G_k)$ has the same nodes as G_k, but with all transitivity edges removed. We arrive at the following definition.

Definition 5.3 *For a linear flow Φ consider the graph corresponding to the order relations $\preceq_k$ and $\sqsubseteq_{k-1}$. The Grassmann graph of Φ is the graph obtained from this graph by replacing on each level k (i.e., in each Grassmann bundle) the corresponding subgraph by its elementary version.*

One easily checks that the Grassmann graph is unique.

Remark 5.4 *Theorem 5.1 describes an indexing system for the finest Morse decomposition on each Grassmann bundle $\mathbb{G}_k\mathcal{V}$ that corresponds to the parametrization of the short Lyapunov index.*

We proceed to discuss how one can regain information about the Lyapunov bundles from the Grassmann graph.

Definition 5.5 *Let G be the Grassmann graph of a linear flow. An* increasing path *p in G is a path from level $\mathbb{G}_1\mathcal{V}$ to level $\mathbb{G}_d\mathcal{V}$ that follows the order, $\sqsubseteq_1$, ..., $\sqsubseteq_{d-1}$. The* in-order *of a node $n \in G$ is the number of edges that end in n and the* out-order *is the number of edges that begin in n. For an increasing path $p = (n_1, ..., n_d)$ in G we define its* simple length

$$\text{sl}(p) = \max\{k, \text{ in-order}(n_k) \leq 1\}.$$

For a node n on the level $\mathbb{G}_1\mathcal{V} = \mathbb{P}^{d-1}\mathcal{V}$ we define its multiplicity *as*

$$\text{mult}(n) = \max\{\text{sl}(p), p \textit{ is an increasing path with initial node } n\}.$$

Lemma 5.6 *Let a linear flow Φ on $\mathcal{V}$ with the natural finest Morse decompositions on the Grassmann bundles be given. For a Lyapunov bundle $\mathcal{V}_i$ denote the corresponding Morse set of the flow $\mathbb{P}\Phi$ by $\mathcal{M}_i = \mathbb{P}\mathcal{V}_i \subset \mathbb{P}\mathcal{V}$. Then the multiplicity* $\text{mult}(\mathcal{M}_i)$ *of $\mathcal{M}_i$ in the Grassmann graph of Φ is equal to the (linear) dimension* $\dim \mathcal{V}_i$.

Proof. Follows directly from Theorem 5.1 and the assumption. ■

The lemma above says that one can recover the dimensions of the Lyapunov bundles from the orders $\sqsubseteq$ on the Grassmann graph. Furthermore, the order of the Lyapunov bundles can be recovered from the order $\preceq$ on level $\mathbb{G}_1\mathcal{V}$ of the graph. Hence we can hope to use Grassmann graphs for the characterization of the short Lyapunov index of a linear flow.

Definition 5.7 *Let G and G' be finite directed graphs. A map $h : G \to G'$ is called a* graph homomorphism *if for all edges (n_1, n_2) in G, $(h(n_1), h(n_2))$ is an edge in G'. Furthermore, h is a graph isomorphism if h is bijective and h and h^{-1} are graph homomorphisms.*

Theorem 5.8 *Let Φ and Ψ be linear flows on vector bundles with equal dimensions, compact chain transitive base spaces, and natural finest Morse decompositions on the Grassmann bundles. Then the short Lyapunov indices $SL(\Phi)$ and $SL(\Psi)$ coincide iff the Grassmann graphs $G(\Phi)$ and $G(\Psi)$ are isomorphic.*

Proof. Let the Grassmann graphs $G(\Phi)$ and $G(\Psi)$ be isomorphic.

(i) We construct the orders $\preceq$ and $\sqsubseteq$ as follows. The only node with out-order 0 is the unique node n_l on the highest level l. All nodes n for which there is an edge (n, n_l) are on the level $l-1$. All nodes n' that are not on level $l-1$ and for which there is an edge (n', n) with n on level $l-1$, are on level $l-2$, etc. This algorithm stops after l' steps, i.e., after determining the nodes on level l', and all nodes are associated with some level. Then $l - l' + 1 = d$, the dimension of the underlying vector bundles. We re-index the levels such that the smallest level is 1. Then, the edges between nodes on the same level k determines the order $\preceq_k$. And edges between nodes on levels $k-1$ and k determine the order $\sqsubseteq_{k-1}$. Note that the node corresponding to the Morse set $\mathcal{M}_1$ on $\mathbb{G}_1\mathcal{V}$ is the unique node with in-order 0.

(ii) The length of any increasing path $(n^1, n^2, ..., n^d)$ determines the dimension d.

(iii) For each node in level $\mathbb{G}_1\mathcal{V}$, its multiplicity defines the dimension of the corresponding Lyapunov space.

(i)-(iii) mean that for linear flow its short Lyapunov index can be uniquely reconstructed from the Grassmann graph, hence isomorphic Grassmann graphs belong to linear flows with identical short Lyapunov indices. Vice versa, short Lyapunov indices determine Grassmann graphs by their construction. ■

Theorem 5.8 characterizes for a linear flow Φ the Lyapunov subbundles and their dimensions, i.e. the short Lyapunov form $SL(\Phi)$. Together with the topological characterization in Corollary 3.4 one also obtains results on the short zero-Lyapunov index $SL_0(\Phi)$, generalizing the situation for linear differential equations $\dot{x} = Ax$. In Section 7 we will analyze bilinear control systems in more detail.

6 Natural Finest Morse Decompositions on Grassmann Bundles

In this section, we derive a condition which ensures that a linear flow Φ on a vector bundle $\pi : \mathcal{V} \to B$ has the natural finest Morse decompositions on each Grassmann bundle $\mathbb{G}_k\mathcal{V}$. For this purpose we need more detailed information on the relation between Morse sets in the flags and in the Grassmannians. We cite the following results (Theorem 5 and Proposition 2 of [4]).

Theorem 6.1 *Let $\mathcal{M}_i$, $i \in \{1, ..., d\}^{k-1}$ be a chain recurrent component in the flag bundle $\mathbb{F}_{k-1}\mathcal{V}$ and consider the $d - k + 1$-dimensional vector bundle $\pi : \mathcal{W}(\mathcal{M}_i) \to \mathcal{M}_i$ with fibers $\mathcal{W}(\mathcal{M}_i)_{F_{k-1}} = \mathcal{V}_b/V_{k-1}$ for $F_{k-1} = (b, V_1, ..., V_{k-1}) \in \mathcal{M}_i$. Then every chain recurrent component ${}_{\mathbb{P}}\mathcal{M}_{i_j}$, $j = 1, ..., k_i \leq d - k + 1$, of the projective bundle $\mathbb{P}\mathcal{W}(\mathcal{M}_i)$ determines a chain recurrent component ${}_k\mathcal{M}_{i_j}$ of $\mathbb{F}_k\mathcal{V}$ via*

$$ {}_k\mathcal{M}_{i_j} = \{F_k = (F_{k-1}, V_k) \in \mathbb{F}_k\mathcal{V},\ F_{k-1} \in \mathcal{M}_i \text{ and } \mathbb{P}(V_k/V_{k-1}) \subset {}_{\mathbb{P}}\mathcal{M}_{i_j}\}, $$

and every chain recurrent component in $\mathbb{F}_k\mathcal{V}$ is of this form.
For every k, the finest Morse decomposition in the Grassmannian $\mathbb{G}_k\mathcal{V}$ is given by the projection of the chain recurrent components from the complete flag $\mathbb{F}$.

Using Theorem 6.1 we obtain the following characterization of the finest Morse decomposition in the full flag.

Proposition 6.2 *Consider a linear flow Φ that has the natural finest Morse decompositions in the Grassmann bundles. Then the Morse sets in the finest Morse decomposition in the full flag bundle are given by the following sets: For each $k = 1, ..., d$ consider index sets*

$$ (i_1^k, ..., i_l^k) \in I(k) $$

such that

$$ k < k' \text{ implies } i_j^k \leq i_j^{k'} \text{ for all } j = 1, ..., l \tag{10} $$

Then for every such $d-$tuple of index sets a Morse set in the finest Morse decomposition of the full flag bundle $\mathbb{F}\mathcal{V}$ is given by

$$ \left\{(V_1, ..., V_d) \in \mathbb{F}\mathcal{V},\ V_k \in \mathcal{N}^k_{i_1^k, ..., i_l^k} \text{ for all } k\right\} \tag{11} $$

and every Morse set is of this form.

Proof. For each k the set $\mathcal{N}^k_{i_1^k,...,i_l^k}$ is a set in the finest Morse decomposition in the Grassmann bundle. Condition (10) guarantees that for $k < k'$ the corresponding Morse set in $\mathbb{G}_{k'}\mathcal{V}$ projects down to the Morse set in $\mathbb{G}_k\mathcal{V}$. Hence every set in (11) is nonvoid. These sets provide a Morse decomposition, since they are compact invariant and isolated; furthermore, they contain all $\alpha-$ and $\omega-$limit sets and satisfy the no-cycle condition. Now the assertion follows, since this certainly is the finest Morse decomposition which projects down to the finest Morse decomposition in the Grassmannians. ■

The following simple lemma shows that an extension of the base space can only decrease the number of Lyapunov bundles.

Lemma 6.3 *Let Φ and $\hat{\Phi}$ be linear flows with chain transitive base flows on vector bundles $\pi : \mathcal{V} \to B$ and $\hat{\pi} : \hat{\mathcal{V}} \to \hat{B}$, respectively, and suppose that $B \subset \hat{B}$ with $\dim \mathcal{V} = \dim \hat{\mathcal{V}}$ and $\hat{\Phi} \mid_{\mathcal{V}} = \Phi$. Then in the fibers over each $b \in B$ the corresponding Lyapunov bundles of $\hat{\Phi}$ are sums of Lyapunov bundles of Φ. In particular, the numbers l and $\hat{l}$ of Lyapunov bundles for Φ and $\hat{\Phi}$, respectively, satisfy $\hat{l} \leq l$.*

Proof. The Lyapunov decomposition for $\hat{\Phi}$ corresponds to the finest Morse decomposition of the projective flow on $\mathbb{P}\hat{\mathcal{V}}$. Its restriction to $\mathbb{P}\mathcal{V}$ coincides with the projective flow induced by Φ. Thus every chain transitive set of $\mathbb{P}\Phi$ is contained in a chain transitive set of $\mathbb{P}\hat{\Phi}$. This entails that for every $b \in B \subset \hat{B}$ the fibers of the corresponding Lyapunov bundles satisfy

$$\mathcal{V}_i(b) \ \subset \hat{\mathcal{V}}_{j(i)}(b), \ i = 1, ..., l.$$

On the other hand, the dimension condition implies that for every $b \in B$

$$\mathcal{V}(b) = \mathcal{V}_1(b) \oplus ... \oplus \mathcal{V}_l(b) = \hat{\mathcal{V}}(b) = \hat{\mathcal{V}}_1(b) \oplus ... \oplus \hat{\mathcal{V}}_{\hat{l}}(b).$$

This implies $\mathcal{V}_i(b) \subset \hat{\mathcal{V}}_{j(i)}(b)$ for some $j(i)$ and every $\hat{\mathcal{V}}_j(b)$ contains some $\mathcal{V}_i(b)$. Hence the assertion follows. ■

The next proposition applies this result to linear flows used in the construction of the finest Morse decomposition on flag bundles.

Proposition 6.4 *Consider the situation of Lemma 6.3. Then the numbers l_k and $\hat{l}_k$ of elements in the finest Morse decompositions of Φ and $\hat{\Phi}$ in each $k-$flag bundle satisfies $\hat{l}_k \leq l_k$.*

Proof. By Theorem 6.1 the chain recurrent components of Φ and $\hat{\Phi}$ in the $k-$flag bundle are in $1-1-$correspondence to the chain recurrent components ${}_{\mathbb{P}}\mathcal{M}_{i_j}$, $j = 1, ..., k_i \leq d-k+1$, of the projective bundle $\mathbb{P}\mathcal{W}(\mathcal{M}_i)$ where $\mathcal{M}_i$, $i \in \{1, ..., d\}^{k-1}$ is a chain recurrent component in the flag bundle $\mathbb{F}_{k-1}\mathcal{V}$. For $k = 1$, Lemma 6.3 shows that for $b \in B$ the fibers of the corresponding subbundles in $\mathbb{P}\mathcal{V} = \mathbb{F}_1\mathcal{V}$ and $\mathbb{P}\hat{\mathcal{V}} = \mathbb{F}_1\hat{\mathcal{V}}$ satisfy an inclusion. For $k = 2$ the chain recurrent components in $\mathbb{F}_2\mathcal{V}$ are determined by the chain recurrent components in the vector bundles

$$\pi : \mathcal{W}(\mathcal{M}_i) \to \mathcal{M}_i \text{ and } \widehat{\pi} : \mathcal{W}(\hat{\mathcal{M}}_i) \to \hat{\mathcal{M}}_i$$

where $\mathcal{M}_i$ and $\hat{\mathcal{M}}_i$ are the chain recurrent components in $\mathbb{F}_1\mathcal{V}$ and $\mathbb{F}_1\hat{\mathcal{V}}$. Since the base spaces satisfy $\mathcal{M}_i \subset \hat{\mathcal{M}}_{j(i)}$, Lemma 6.3 implies that the fibers for the corresponding Whitney sum decompositions (and hence for the chain recurrent components in the projective bundles $\mathbb{P}\mathcal{W}(\mathcal{M}_i)$ and $\mathbb{P}\mathcal{W}(\hat{\mathcal{M}}_i)$) satisfy an inclusion. Thus for all $m \in \mathcal{M}_i$ and $i_j = (i, i_2)$

$$_2\mathcal{M}_{i_j}(m) \subset \ _2\hat{\mathcal{M}}_{i_j}(m).$$

Thus also the fibers over $b \in B$ satisfy such an inclusion. This entails that the fibers of the corresponding subbundles of $\mathbb{F}_2\mathcal{V} \to B$ and $\mathbb{F}_2\hat{\mathcal{V}} \to \hat{B}$ satisfy such an inclusion and hence the result for $k = 2$. Proceeding in the same way for $k = 3, ...$ one sees that the assertion follows. ■

Now the following consequence is obvious.

Corollary 6.5 *If a linear flow Φ contains a subflow, which has the natural finest Morse decompositions in the Grassmann bundles, then Φ has the same property..*

In the next section we will apply this result to bilinear control systems. Another example concerns the linearization of ordinary differential equations:

Example 6.6 *Consider a differential equation in $\mathbb{R}^d$ given by a C^1 vector field $f : \mathbb{R}^d \to \mathbb{R}^d$*

$$\dot{x} = f(x). \tag{12}$$

Assume that (12) has a compact chain transitive set $E \subset \mathbb{R}^d$. Over E we linearize the system to obtain

$$\dot{x} = f(x), \ \dot{y} = Df(x)y, \tag{13}$$

with associated linear flow Φ on $E \times \mathbb{R}^d$. Assume that E contains an equilibrium e, i.e. $f(e) = 0$ (e.g., E may consist of an equilibrium together with a homoclinic orbit). The subflow of Φ on $\{e\} \times \mathbb{R}^d \subset E \times \mathbb{R}^d$ given by

$$x \equiv e, \ \dot{y} = Df(e)y,$$

has the natural finest Morse decompositions on the Grassmann bundles, hence by Corollary 6.5 Φ has this property as well. Using Remark 5.2 one finds that this also holds if E contains a periodic solution.

7 Applications to Bilinear Control Systems

In the last twenty years, the problem to classify control systems allowing state and feedback transformations has been extensively studied. In particular, we mention the approach due to Kang and Krener [9] based on Taylor expansions and more geometric approaches to equivalence for (nonlinear) control systems that are based on equivalent distributions defined by a system on the tangent bundle. This point of view allows for the redefinition of controls (via feedback) and requires that the control range is a linear, unbounded space (see e.g. the recent survey by Respondek and Tall [11]). This section approaches the classification of bilinear control systems from a topological point of view, as is common in the theory of dynamical systems, see, e.g., [8] and [16]. Most of the proofs are based on results from the previous sections and in some cases more specific information can be obtained due to the specific nature of bilinear control flows.

We denote the set of $d \times d$ matrices with real entries by $gl(d, \mathbb{R})$.

Definition 7.1 *A bilinear control system in* $\mathbb{R}^d$ *is given by a set of matrices* $\{A_0, ..., A_m\} \subset gl(d, \mathbb{R})$ *and a control range* $U \subset \mathbb{R}^m$*, which we assume to be a compact and convex set with* $0 \in \text{int}U$*:*

$$\dot{x}(t) = A(u(t))x(t) = [A_0 + \sum_{i=1}^{m} u_i(t)A_i]x(t), \tag{14}$$

$$u \in \mathcal{U} := \{u : \mathbb{R} \to U \quad \textit{for all} \quad t \in \mathbb{R}, \textit{locally integrable}\}.$$

For all $(u, x) \in \mathcal{U} \times \mathbb{R}^d$, the system has a unique solution $\varphi(t, x, u),\ t \in \mathbb{R}$, with $\varphi(0, x, u) = x$. We denote by $\mathbf{B}(d, m, U)$ the set of bilinear control systems $\Sigma = (A_0, ..., A_m, U)$ in $\mathbb{R}^d$ with m controls and control range U.

Associated with a control system is a linear dynamical system (the control flow) in the following way, compare [5].

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d,\ \Phi(t, u, x) = (\theta(t, u), \varphi(t, x, u)), \tag{15}$$

where we denote the shift in the base by $\theta(t, u(\cdot)) = u(t + \cdot)$. The dynamical system (15) is a linear skew-product flow on the vector bundle $\mathcal{U} \times \mathbb{R}^d$. Continuity of Φ follows if $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ is endowed with the weak* topology, i.e., the weakest topology such that for all $\psi \in L_1(\mathbb{R}, \mathbb{R}^m)$ the maps

$$L_\infty(\mathbb{R}, \mathbb{R}^m) \to \mathbb{R},\ u \mapsto \int_{\mathbb{R}} u(t)^T \psi(t)\ dt$$

are continuous. We also note that $\mathcal{U}$ becomes a compact metrizable space; a metric is obtained by choosing a countable dense subset $\{\psi_n\}$ in $L_1(\mathbb{R}, \mathbb{R}^m)$ and defining

$$\mathrm{d}(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\left|\int_{\mathbb{R}} [u(t) - v(t)]^T \psi_n(t)\ dt\right|}{1 + \left|\int_{\mathbb{R}} [u(t) - v(t)]^T \psi_n(t)\ dt\right|}.$$

Note that the shift on $\mathcal{U}$ is chain transitive and chain recurrent.

7.1 Base Conjugation for Bilinear Control Systems

In this section we analyze when two bilinear control systems are base conjugate, i.e., the corresponding shifts on the control functions are conjugate. Note that the base $\mathcal{U}$ is considered in the weak* topology of L_∞, hence continuity of the conjugation map enforces that the conjugation respects, in an appropriate way, the duality relation between L_∞ and L_1.

Two subsets U_1, U_2 of $\mathbb{R}^m$ are called affinely isomorphic if there exists an invertible affine map H on $\mathbb{R}^m$ with $H[U_1] = U_2$. This means that there are an invertible matrix $M \in \mathbb{R}^{m\times m}$ and a vector $b \in \mathbb{R}^m$ with

$$H(x) = Mx + b. \tag{16}$$

Then the inverse is

$$H^{-1}(y) = M^{-1}y - M^{-1}b. \tag{17}$$

Proposition 7.2 *Let $U_1, U_2 \subset \mathbb{R}^m$ be compact and convex and consider for $i = 1, 2$*

$$\mathcal{U}_i := \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m), u(t) \in U_i \text{ for a.a. } t \in \mathbb{R}\},$$

with shifts $\theta_i : \mathbb{R} \times \mathcal{U}_i \longrightarrow \mathcal{U}_i$. If the sets U_1, U_2 are affinely isomorphic, then there there exists a homeomorphism $f : \mathcal{U}_1 \to \mathcal{U}_2$ in the weak topology with*

$$f(\theta_1(t, u)) = \theta_2(t, f(u)) \text{ for all } t \in \mathbb{R}.$$

Proof. Using the affine isomorphism H as in (16) between U_1 and U_2 define a map

$$f : \mathcal{U}_1 \to \mathcal{U}_2,\ u \mapsto (f(u))(s) = (H(u(s)),\ s \in \mathbb{R}.$$

Then the conjugation property

$$f(u(t + \cdot)) = f(u)(t + \cdot)$$

holds. This map is continuous, since for $u_n \to u$ in $\mathcal{U}_1$ and every $\psi \in L_1(\mathbb{R}, \mathbb{R}^m)$

$$\begin{aligned} \int_{\mathbb{R}} [f(u_n)(t) - f(u)(t)]^T \psi(t)\ dt &= \int_{\mathbb{R}} [H(u_n(t)) - H(u(t))]^T \psi(t)\ dt \\ &= \int_{\mathbb{R}} [M(u_n(t) - u(t))]^T \psi(t)\ dt \\ &= \int_{\mathbb{R}} [u_n(t) - u(t)]\, M^T \psi(t)\ dt. \end{aligned}$$

Since $M^T\psi(\cdot) \in L_1$, this converges to zero for $n \to \infty$. The inverse of f is constructed using the inverse (17) of H. ■

As a consequence of this proposition we obtain that any two shift flows with scalar control are conjugate.

Corollary 7.3 *Each of the following conditions implies that U_1 and U_2 are affinely isomorphic and hence the corresponding shifts are conjugate:*

(i) The sets U_1, U_2 are compact intervals in $\mathbb{R}$ with nonvoid interior.

(ii) The sets U_1 and U_2 are the convex hull of $2m$ points in $\mathbb{R}^m$ in the form

$$U_i = co(v_i^1, ...v_i^m, -v_i^1, ... - v_i^m\}.$$

Proof. (i) The affine isomorphism is obtained by shifting each interval such that the origin becomes the middle point and then mapping the boundary points to each other.

(ii) Define a linear isomorphism H on the linear basis by $H(v_1^j) = v_2^j$ for $j = 1, ..., m$. ■

Corollary 7.4 *Let $\rho > 0$ and consider for $U \subset \mathbb{R}^m$ the control range $\rho \cdot U$. Then the shifts on $\mathcal{U}$ and on $\mathcal{U}^\rho := \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in \rho U \text{ for all } t \in \mathbb{R}\}$ are conjugate.*

Proof. Clear by Proposition 7.2, since $H : u \mapsto \rho u$ is linear. ■

7.2 Topological Conjugation and Equivalence in $\mathbb{R}^d$ and $\mathbb{P}^{d-1}$

The results in Section 3 are immediately applicable to bilinear control systems. Let $\Sigma_1 = (A_0, ..., A_m, U_1)$ and $\Sigma_2 = (B_0, ..., B_m, U_2)$ be two bilinear control systems in $\mathbf{B}(d, m, U_i)$ with linear flows $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$, respectively.

Corollary 7.5 *Consider two bilinear control systems with conjugate base flows.*
(i) If both flows are exponentially (un)stable, then they are skew conjugate.
(ii) If both flows are hyperbolic, i.e. the vector bundles $\mathcal{U}_i \times \mathbb{R}^d$ can be written as the Whitney sums of exponentially stable and unstable subbundles and if the dimensions of their stable (and unstable) subbundles coincide, then they are skew conjugate.

The proof follows directly from Corollary 3.4. Corollary 7.5 generalizes the well-known result for hyperbolic matrices to bilinear control systems.

Corollary 7.6 *Let $\Phi = (\theta, \varphi)$ and $\Psi = (\vartheta, \psi)$ be the control flows in $\mathbb{R}^d$ of two bilinear control systems $\Sigma_i \in \mathbf{B}(d, m, U_i)$. If Φ and Ψ are C^1-conjugate via $h = (f, g)$ then all matrices $A(u) = A_0 + \sum_{i=1}^m u_i A_i$ and $B(u) = B_0 + \sum_{i=1}^m u_i B_i$ are linearly conjugate in the sense that there exists for each constant control $u \in U_1$ an invertible matrix $T(u) \in Gl(d, \mathbb{R})$ with*

$$A(u) = T^{-1}(u) B(f(u)) T(u).$$

Proof. According to Proposition 3.6 we have for all $t \in \mathbb{R}$ and $u \in \mathcal{U}_1$

$$\varphi(t, \cdot, u) = [D_x g(\theta_t u, 0)]^{-1} \circ \psi(t, \cdot, f(u)) \circ D_x g(u, 0). \tag{18}$$

Note that the constant controls $u(t) \equiv u \in \mathcal{U}_1$ for all $t \in \mathbb{R}$ are fixed points of the shift θ. Hence we obtain for all $u \in U_1$

$$\varphi(t, \cdot, u) = [D_x g(u, 0)]^{-1} \circ \psi(t, \cdot, f(u)) \circ D_x g(u, 0). \tag{19}$$

Differentiation of (19) with respect to t yields at $t = 0$ the result

$$A(u) = T^{-1}(u) B(f(u)) T(u) \text{ for all } u \in U_1, \tag{20}$$

where we have set $T(u) := D_x g(u, 0)$. ■

Remark 7.7 *The proof also shows that for* $U_1 = U_2 = U$ *with* $f = id$*, the relation* $A(u) = T^{-1}(u)B(u)T(u)$ *holds for all* $u \in U$*. One obtains, e.g., for* $u = 0$ *that* A_0 *and* B_0 *are similar matrices.*

Remark 7.8 *Note that linear conjugacy of the matrices in the sense of equation (20) does not automatically imply that the flows* Φ *and* Ψ *are linearly (and hence* C^k*) conjugate. This would follow from simultaneous equivalence of the matrices, i.e. there exists a basis transformation* $T \in Gl(d, \mathbb{R})$ *such that* $A_i = T^{-1}B_iT$ *for* $i = 0, ..., m$*. The result in Corollary 7.6 contains as a special case the situation for linear differential equations by considering* $u = 0$*.*

Next we show that the flow induced on the projective bundle $\mathcal{U} \times \mathbb{P}^{d-1}$ allows us to recover the subbundle decompositions associated with the (Morse) spectrum of a bilinear control system, compare Section 4.

A system $\Sigma \in \mathbf{B}(d, m, U)$ induces a (nonlinear) control system $\mathbb{P}\Sigma$ on the projective space $\mathbb{P}^{d-1}$ in the following way:

$$\dot{s}(t) = \mathbb{P}A(u(t), s(t)) = \mathbb{P}A_0(s) + \sum_{i=1}^{m} u_i \mathbb{P}A_i(s) \tag{21}$$
$$\mathbb{P}A_i(u, s) = (A_i - s^T A_i s \cdot I)s \text{ for all } i = 0, ..., m.$$

Here ^{T} denotes transposition and I is the $d \times d$ identity matrix. For all $(u, s) \in \mathcal{U} \times \mathbb{P}^{d-1}$ the system has a unique solution, denoted by $\mathbb{P}\varphi(t, s, u)$ for all $t \in \mathbb{R}$ with $\mathbb{P}\varphi(0, s, u) = s$. The associated dynamical system reads

$$\mathbb{P}\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{P}^{d-1} \to \mathcal{U} \times \mathbb{P}^{d-1}, \ \mathbb{P}\Phi(t, u, s) = (\theta(t, u), \mathbb{P}\varphi(t, s, u)). \tag{22}$$

The *Morse spectrum* of the system Σ is $\Sigma_{Mo} = \bigcup_{j=1}^{l} \Sigma_{Mo}(\mathcal{E}_j)$, where the $\mathcal{E}_j$ are the maximal chain transitive sets (or Morse sets) of $\mathbb{P}\Phi$. As before, we denote by $\mathcal{V}_j$ the Lyapunov subbundle of Φ associated with the Morse set $\mathcal{E}_j$ for $j = 1, ..., l$. Recall that the projections of the maximal chain transitive sets of $\mathbb{P}\Phi$ onto the projective space, i.e. $E_j = \{y \in \mathbb{P}^{d-1}$, there exists $u \in \mathcal{U}$ with $(u, y) \in \mathcal{E}_j\}$, are the chain control sets of the system (21), see [5], Chapter 4.

Theorem 7.9 *For* $i = 1, 2$*, let* $\Sigma_i \in \mathbf{B}(d, m, U_i)$ *be two bilinear control systems with associated flows* Φ_i *in* $\mathcal{U}_i \times \mathbb{R}^d$ *and projected flows* $\mathbb{P}\Phi_i$ *in* $\mathcal{U}_i \times \mathbb{P}^{d-1}$*. Denote the associated bundle decompositions by* $\bigoplus_{j=1}^{l} \mathcal{V}_i^j = \mathcal{U}_i \times \mathbb{R}^d$*. Let* $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ *be a skew equivalence between* $\mathbb{P}\Phi_1$ *and* $\mathbb{P}\Phi_2$*. Then*
(i) h maps chain recurrent components of $\mathbb{P}\Phi_1$ *onto chain recurrent components of* $\mathbb{P}\Phi_2$*, and hence the chain control sets in* $\mathbb{P}^{d-1}$ *onto chain control sets,*
(ii) h respects the order of the chain recurrent components, and hence of the chain control sets,
(iii) Σ_1 *and* Σ_2 *have the same number of spectral intervals and h respects the order between these intervals,*
(iv) h maps the associated bundle decompositions into each other, and the dimensions of corresponding fibers agree.

Proof. (i) and (ii): Lemma 4.3 and Proposition 5.2 in [1] prove these facts for flows over the same base space. The same proofs, with minor adjustments, go through for skew equivalences of projected flows.

(iii) follows directly from (ii) and the properties of the Morse spectrum, see Section 4.

(iv): Note that for a chain control set $E_i^j \subset \mathbb{P}^{d-1}$ we have that its lift $\mathcal{E}_i^j$ to $\mathcal{U}_i \times \mathbb{P}^{d-1}$ satisfies the relation $\mathcal{V}_i^j = \mathbb{P}^{-1}\mathcal{E}_i^j$. Hence it follows from (i) that $h[\mathbb{P}\mathcal{V}_1^j] = \mathbb{P}\mathcal{V}_2^j$ for all $j = 1, ..., l$. In order to see that the dimensions of $\mathcal{V}_1^j$ and $\mathcal{V}_2^j$ coincide, observe that each (projective) fiber $\mathbb{P}\mathcal{V}_1^j(u) \subset \{u\} \times \mathbb{P}^{d-1} \cong \mathbb{P}^{d-1}$ is mapped homeomorphically onto the fiber $\mathbb{P}\mathcal{V}_2^j(f(u)) \subset \{f(u)\} \times \mathbb{P}^{d-1} \cong \mathbb{P}^{d-1}$. The canonical projection $\pi : \mathbb{R}^d \to \mathbb{P}^{d-1}$ is a submersion, hence the fibers are submanifolds of $\mathbb{P}^{d-1}$, compare, e.g., Warner [15]. Since h is a homeomorphism, the fibers have the same dimension by the invariance of domain theorem, see Warner [15]). Hence the linear dimensions of $\mathcal{V}_1^j(u)$ and $\mathcal{V}_2^j(f(u))$ coincide. ■

Remark 7.10 *We cannot give a complete characterization of bilinear control systems, for which the projected flows on* $\mathcal{U} \times \mathbb{P}^{d-1}$ *are topologically skew conjugate. Indeed, this question is open even for single matrices, compare Remark 5.4 in [1]. It is shown there that topological conjugacy of projected flows also preserves certain detail characteristics within the eigenspace decomposition. Theorem 7.9 shows that the existence of a topological skew conjugacy of the projected flows is a much stronger requirement than the existence of a topological skew conjugacy for the linear flows, compare Corollary 7.5.*

7.3 The Lyapunov Index of Bilinear Control Systems

This section characterizes the (short) Lyapunov index of bilinear control systems as introduced in Section 5 for general linear flows. The specific structure of bilinear systems allows a more complete description of the (exponential) subbundles and their dimension. Note that it follows from Remark 7.10 that conjugacies of the projected flows do not characterize the Lyapunov index of a bilinear control system, because the requirement of $h(\mathbb{P}\Phi_1(t, u, x)) = \mathbb{P}\Phi_2(t, h(u, x))$, i.e., of mapping trajectories into trajectories, is too strong. Hence we employ a concept that relates to mappings of the Morse decompositions of the projected flow, compare Theorem 5.5 in [1].

Theorem 7.11 *Consider two bilinear control systems* $\Sigma_i \in \mathbf{B}(d, m, U_i)$ *which are base conjugate via* $f : \mathcal{U}_1 \to \mathcal{U}_2$*. Then* Σ_1 *and* Σ_2 *have the same short Lyapunov index iff there is a skew homeomorphism* $h = (f, g) : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ *with* $g : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ *that maps the finest Morse decomposition of* $\mathbb{P}\Phi_1$ *into the finest Morse decomposition of* $\mathbb{P}\Phi_2$*, i.e.* h *maps Morse sets into Morse sets and preserves their order.*

Proof. Let $h : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ be a homeomorphism that maps the finest Morse decomposition of $\mathbb{P}\Phi_1$ into the finest Morse decomposition of $\mathbb{P}\Phi_2$. This means, in particular, that both systems have the same number of spectral intervals, and these are ordered according to their minimal (or maximal) elements. It remains to show that the associated bundle decompositions have the same dimension. This follows exactly as assertion (iv) in Theorem 7.9. Hence the short Lyapunov indices of the systems coincide.

For the converse, we order the Morse sets of $\mathbb{P}\Phi_1$ and $\mathbb{P}\Phi_2$ in their natural order and concentrate on one corresponding pair, say $\mathcal{M}^1$ for $\mathbb{P}\Phi_1$ and $\mathcal{M}^2$ for $\mathbb{P}\Phi_2$. By Theorem 4.1, the lifts $\mathbb{P}^{-1}\mathcal{M}^i$ of the $\mathcal{M}^i$ to $\mathcal{U}_i \times \mathbb{R}^d$ are subbundles $\mathcal{V}_j^i$ with $\mathcal{U}_i \times \mathbb{R}^d = \bigoplus_{j=1}^l \mathcal{V}_j^i$ and one can choose, for every $u_1 \in \mathcal{U}_1$ and $u_2 = f(u_1) \in \mathcal{U}_2$, a basis $x_1^i(u_i), \ldots, x_{k_j}^i(u_i) \in \mathbb{R}^d$ such that $\mathcal{V}_j^i(u_i) = \text{span}$ $\{x_1(u_i), \ldots, x_{k_j}(u_i)\}$ Since the subbundles are continuous decompositions of $\mathcal{U}_i \times \mathbb{R}^d$, these choices can be made continuous. We define a family of linear, invertible maps on $\mathbb{R}^d$ via $T_u(x_k^1(u_1)) = x_k^2(u_2)$, $k = 1, ..., d$. The projection $\mathbb{P}T : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ is the desired skew homeomorphism. ∎

Corollary 7.12 *Consider two bilinear control systems $\Sigma_i \in \mathbf{B}(d, m, U_i)$ such that the corresponding flows Φ_i are base conjugate and hyperbolic. Then Σ_1 and Σ_2 have the same short zero-Lyapunov index iff their linear flows Φ_i in $\mathcal{U} \times \mathbb{R}^d$ are skew conjugate, and there is a skew homeomorphism $h : \mathcal{U}_1 \times \mathbb{P}^{d-1} \to \mathcal{U}_2 \times \mathbb{P}^{d-1}$ respecting the finest Morse decompositions of the projected flows.*

Proof. By Theorem 7.11 the flows have the same short Lyapunov index iff a homeomorphism h as above exists. Additionally, Corollary 7.5 shows that the dimension of the stable subbundles is fixed (and hence the short zero-Lyapunov index is determined) iff the linear flows are conjugate. ∎

Finally we mention the application of Theorem 5.8 on Grassmann graphs to bilinear control systems.

Corollary 7.13 *Let $\Sigma_1 \in \mathbf{B}(d, m, U_1)$ and $\Sigma_2 \in \mathbf{B}(d, m, U_2)$ be bilinear control systems. Then the short Lyapunov indices $SL(\Sigma_1)$ and $SL(\Sigma_2)$ coincide iff the Grassmann graphs of Φ_1 and Φ_2 are isomorphic.*

Proof. Note that a bilinear control system for $u = 0$ is an autonomous differential equation and hence it has the natural finest Morse decompositions on the Grassmann bundles. Thus Corollary 6.5 implies that a bilinear control system has the natural finest Morse decomposition on the Grassmann bundles. The assertion now follows from Theorem 5.8. ∎

7.4 Families of Bilinear Control Systems

For bilinear control systems with compact control range it is of great interest to study the change in system behavior under varying control range, specifically controllability, stability and stabilization, and bifurcation phenomena. The theory developed in this paper further illuminates the properties of these families of control systems.

Consider the family of bilinear control systems

$$\Sigma^\rho \in \mathbf{B}(d, m, U^\rho) \text{ with } U^\rho = \rho \cdot\ U,\ \rho \geq 0, \tag{23}$$

where $U \subset \mathbb{R}^m$ is convex, compact with $0 \in \operatorname{int} U$. Thus the sets of admissible controls are

$$\mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{ locally integrable}\}.$$

For $\rho \in [0,\infty)$ the objects related to $(23)^\rho$ may be denoted by a superscript ρ. The systems Σ^ρ induce (nonlinear) control systems $\mathbb{P}\Sigma^\rho$ on the projective space $\mathbb{P}^{d-1}$ as in (21) with control range U^ρ. For the corresponding chain control sets E_j^ρ we define the maps

$$E_j : [0,\infty] \to \mathcal{C}(\mathbb{P}^{d-1}),\ \rho \mapsto E_j^\rho,\ j = 1,\dots,l, \tag{24}$$

where l is the number of different real parts of the eigenvalues of A_0, and $\mathcal{C}(\mathbb{P}^{d-1})$ is the set of compact subsets of $\mathbb{P}^{d-1}$ with the Hausdorff metric, compare [5]. Note that the maps $E_j(\cdot)$ are increasing in ρ for all $j = 1,...,l$. The maps of the subbundle decompositions corresponding to (24) and their dimensions are given by

$$\begin{aligned} \mathcal{V}_j &: [0,\infty) \to \mathcal{L}(\mathcal{U} \times \mathbb{R}^d),\ \rho \mapsto \mathcal{V}_j^\rho,\ j = 1,\dots,l, \\ m_j &: [0,\infty) \to \{0,...,d\},\ \rho \mapsto \dim(\mathcal{V}_j^\rho) =: m_j^\rho,\ j = 1,\dots,l, \end{aligned} \tag{25}$$

where $\mathcal{L}(\mathcal{U} \times \mathbb{R}^d)$ is the space of linear subbundles of $\mathcal{U} \times \mathbb{R}^d$. Note that the maps $m_j(\cdot)$ are piecewise constant, increasing, with at most $d-1$ points of discontinuity. The exact number of discontinuities depends on the number l of different real parts of the eigenvalues of A_0 and on the successive mergers of the chain control sets $E_j(\rho)$ as ρ increases. Theorem 7.11 implies the following characterization of the family $(23)^\rho$ of bilinear control systems in terms of the short Lyapunov indices..

Theorem 7.14 *Let $\Sigma^\rho \in \mathbf{B}(d,m,U^\rho)$ be a family (23) of bilinear control systems depending on the parameter $\rho \geq 0$. Then the following statements are equivalent for two systems Σ^{ρ_1} and Σ^{ρ_2}:*

(i) Σ^{ρ_1} and Σ^{ρ_2} have the same short Lyapunov indices $SL(\Sigma^{\rho_1}) = SL(\Sigma^{\rho_2})$;

(ii) ρ_1 and ρ_2 are in same (constant) interval of $m_j(\cdot)$ for all $j = 1,\dots,l$;

(iii) there exists a skew homeomorphism $h : \mathcal{U}^{\rho_1} \times \mathbb{P}^{d-1} \to \mathcal{U}^{\rho_2} \times \mathbb{P}^{d-1}$ mapping the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_1}$ into the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_2}$.

(iv) The Grassmann graphs of Σ^{ρ_1} and Σ^{ρ_2} are isomorphic.

Proof. (i) $\Rightarrow$ (ii): If for one $j \in \{1,...,l\}$ we have $m_j(\rho_1) \neq m_j(\rho_2)$, then the subbundle decompositions $\bigoplus \mathcal{V}_j^{\rho_1}$ and $\bigoplus \mathcal{V}_j^{\rho_1}$ do not have the same dimensions and hence the short Lyapunov indices differ.

(ii) $\Rightarrow$ (i) follows directly from the definition of the short Lyapunov index.

(i) $\Leftrightarrow$ (iii): Note first that U^{ρ_1} and U^{ρ_2} are linearly isomorphic via $H : U^{\rho_1} \to U^{\rho_2}$, $H(u) = \frac{\rho_2}{\rho_1} u$, and hence by Proposition 7.2 the shifts on the sets $\mathcal{U}^{\rho_1}$ and $\mathcal{U}^{\rho_2}$ of admissible control functions are conjugate. The result now follows from Theorem 7.11.

(i) ⇔ (iv): follows from Corollary 7.13. ■

If we add hyperbolicity, a characterization in terms of the short zero-Lyapunov indices and conjugacies is obtained.

Corollary 7.15 *Let $\Sigma^\rho \in \mathbf{B}(d, m, U^\rho)$ be a family (23) of bilinear control systems depending on the parameter $\rho \geq 0$. Then for two systems Σ^{ρ_1} and Σ^{ρ_2} with hyperbolic linear flows Φ^{ρ_1} and Φ^{ρ_2}, respectively, the following statements are equivalent*

(i) Σ^{ρ_1} and Σ^{ρ_2} have the same short zero-Lyapunov indices $SL_0(\Sigma^{\rho_1}) = SL_0(\Sigma^{\rho_2})$.

(ii) ρ_1 and ρ_2 are in same (constant) interval of $m_j(\cdot)$ for all $j = 1, \ldots, l$, and the dimensions of the stable subbundles coincide.

(iii) The linear flows Φ^{ρ_1} and Φ^{ρ_2} are skew conjugate and there exists a skew homeomorphism $h : \mathcal{U}^{\rho_1} \times \mathbb{P}^{d-1} \to \mathcal{U}^{\rho_2} \times \mathbb{P}^{d-1}$ mapping the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_1}$ into the finest Morse decomposition of $\mathbb{P}\Phi^{\rho_2}$.

Proof. (i) ⇔ (iii): This follows from Corollary 7.12.

(i) ⇔ (ii): By Theorem 7.14 the short Lyapunov indices coincide, if ρ_1 and ρ_2 are in the same interval. Since the short zero-Lyapunov index only contains the additional information on the dimension of the stable subbundle, the assertion follows. ■

Much more can be said about the discontinuity points of the maps $m_j(\cdot)$, compare [5], Chapter 7.

References

[1] V. Ayala, F. Colonius, and W. Kliemann, *Dynamical characterization of the Lyapunov form of matrices,* Linear Algebra and Its Applications 420(2005), 272 - 290.

[2] C.J. Braga Barros and L.A.B. San Martin, *Chain transitive sets for flows on flag bundles.* Forum Math., to appear.

[3] I. U. Bronstein and A. Y. Kopanskii, *Smooth Invariant Manifolds and Normal Forms,* World Scientific, 1994.

[4] F. Colonius, and W. Kliemann, *Morse decompositions and spectra on flag bundles,* J. Dyn. Diff. Equ.14(2002), 719-741.

[5] F. Colonius and W. Kliemann, *The Dynamics of Control,* Birkhäuser 2000.

[6] Nguyen Dinh Cong, *Topological Dynamics of Random Dynamical Systems*, Oxford Mathematical Monographs, Clarendon Press 1997.

[7] C. Conley, *Isolated Invariant Sets and the Morse Index,* CBMS Regional Conference Series, no. 38, Amer. Math. Soc., 1978.

[8] M.W. Hirsch, S. Smale, and R.L. Devaney, *Differential Equations, Dynamical Systems and an Introduction to Chaos*, Elsevier, 2004.

[9] W. Kang and A.J. Krener, *Extended quadratic controller normal form and dynamic feedback linearization of nonlinear systems*, SIAM J. Control Optim. 30(1992), 1319-1337.

[10] M. Karoubi, *K-Theory, An Introduction,* Springer-Verlag 1978.

[11] W. Respondek and I. Tall, *Feedback equivalence of nonlinear control systems: a survey on formal approach*, to appear in: *Chaos in Automatic Control,* W. Perruquetti and J.-P. Barbot, eds., Taylor&Francis, 2005.

[12] C. Robinson, *Dynamical Systems,* CRC Press, 1995.

[13] D. Salamon and E. Zehnder, *Flows on vector bundles and hyperbolic sets*, Trans. Amer. Math. Soc., 306(1988), 623–649.

[14] J. Selgrade, *Isolated invariant sets for flows on vector bundles*, Trans. Amer. Math. Soc.203(1975), 259-290.

[15] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups,* Springer-Verlag, 1983.

[16] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Applications,* Springer-Verlag 1996.