Growth rates for persistently excited linear systems

Yacine Chitour *   Fritz Colonius †   Mario Sigalotti ‡

Abstract

We consider a family of linear control systems $\dot{x} = Ax + \alpha Bu$ on $\mathbb{R}^d$, where α belongs to a given class of persistently exciting signals. We seek maximal α-uniform stabilisation and destabilisation by means of linear feedbacks $u = Kx$. We extend previous results obtained for bidimensional single-input linear control systems to the general case as follows: if there exists at least one K such that the Lie algebra generated by A and BK is equal to the set of all $d \times d$ matrices, then the maximal rate of convergence of (A, B) is equal to the maximal rate of divergence of $(-A, -B)$. We also provide more precise results in the general single-input case, where the above result is obtained under the simpler assumption of controllability of the pair (A, B).

1 Introduction

In the present paper we address stabilization issues relative to linear systems subject to scalar persistently exciting signals (PE-signals). Such a linear time-dependent system is written as

$$\dot{x} = Ax + \alpha(t)Bu\,, \tag{1}$$

where $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, the matrices A, B have appropriate sizes and the function α is a *scalar* PE-signal, i.e., α takes values in $[0, 1]$ and there exist two positive constants μ, T such that, for every $t \geq 0$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \tag{2}$$

Given two positive real numbers $\mu \leq T$, we use $\mathcal{G}(T, \mu)$ to denote the class of all PE signals verifying (2).

In (1), the PE-signal α can be seen as an input perturbation modelling the fact that the instants where the control u acts on the system are not exactly known. If α only takes the values 0 and 1, then (1) actually switches between the uncontrolled system $\dot{x} = Ax$ and the controlled one $\dot{x} = Ax + Bu$. In that context, the persistence of excitation condition (2) is designed to guarantee some action on the system. Persistent of excitation conditions have appeared both in the identification and in the control literature [2, 3, 5, 6, 16, 17, 19].

Here, we are mainly concerned with the global asymptotic stabilization of system (1) with a constant linear feedback $u = Kx$ *uniformly* with respect to all PE-signals $\alpha \in \mathcal{G}(T, \mu)$. The dual problem consists in exponentially destabilizing system (1) by a constant linear feedback. In order to quantitatively measure these stabilization and destabilization features, we first define, for every K, the exponential rate of convergence for the family of time varying-systems $\dot{x} = (A + \alpha BK)x$ and

*Laboratoire des Signaux et Systèmes, Supélec, Gif s/Yvette, France and Université Paris Sud, Orsay and Team GECO, INRIA Saclay-Île-de-France, chitour@lss.supelec.fr

†Institut für Mathematik, Universität Augsburg, Augsburg, Germany, fritz.colonius@math.uni-augsburg.de

‡INRIA Saclay-Île-de-France, Team GECO, and CMAP, UMR 7641, École Polytechnique, Palaiseau, France, mario.sigalotti@inria.fr

use $\mathrm{rc}(A,B,K)$ to denote it. Similarly, for every K, let $\mathrm{rd}(A,B,K)$ be the rate of divergence for the family of time varying-systems. (For the precise definitions of $\mathrm{rc}(A,B,K)$ and $\mathrm{rd}(A,B,K)$ in terms of Lyapunov exponents, see Section 2.2.) The sign convention on $\mathrm{rc}(A,B,K)$ (respectively, $\mathrm{rd}(A,B,K)$) is such that exponential stabilizability (respectively, destabilizability) of system (1) is equivalent to the existence of some feedback K with $\mathrm{rc}(A,B,K) > 0$ (respectively, $\mathrm{rd}(A,B,K) > 0$). If K is such that $\mathrm{rc}(A,B,K) > 0$ then we say that K is a (T,μ)-stabilizer. Let $\mathrm{RC}(A,B)$ and $\mathrm{RD}(A,B)$ be defined as the supremum over K of $\mathrm{rc}(A,B,K)$ and $\mathrm{rd}(A,B,K)$ respectively.

Recall that if $T=\mu$ then $\alpha\equiv 1$ is the unique choice of PE-signal and in that case the above issues correspond to the classical stabilizability questions associated with time-invariant finite-dimensional linear control systems $\dot{x} = Ax + Bu$. In particular, it follows from the pole-shifting theorem that $\mathrm{RC}(A,B) = +\infty$ if and only if $\mathrm{RD}(A,B) = +\infty$, and this happens if and only if the pair (A,B) is controllable.

The present paper belongs to a line of research initiated in [8] which consists in generalizing the pole-shifting theorem to linear control systems subject to persistence of excitation on the input (for a survey on recent results on persistence of excitation, see [9]). The pole-assignment part of that theorem seems difficult to transpose in the context of persistence of excitation. Therefore, we are more interested in a qualitative feature that we call *generalized pole-shifting property*, namely whether $\mathrm{RC}(A,B)$ and $\mathrm{RD}(A,B)$ are both infinite and to characterize such a property in terms of the data of the problem A,B,T,μ.

When $\mu < T$, the generalized pole-shifting property is not guaranteed. More precisely, it has been proved in [10] that for bidimensional single-input controllable systems of the form (1), there exists $\rho \in (0,1)$ (independent of A,B) such that if $\mu/T < \rho$ then $\mathrm{RC}(A,B)$ is finite. As a consequence, one easily deduces that for λ large enough $\mathrm{rc}(A+\lambda\mathrm{Id},B,K)$ is negative for every K, hence there does not even exist a (T,μ)-stabilizer in that situation. Let us mention that if one restricts $\mathcal{G}(T,\mu)$ to the subclass $\mathcal{D}(T,\mu,M)$ of its elements which are M-Lipschitz for a given $M>0$, then one recovers that, for every $0<\mu<T$, system (1) can be stabilized and destabilized with arbitrarily large exponential rates uniformly with respect to $\alpha \in \mathcal{D}(T,\mu,M)$ (cf. [18]).

Our main goal in this paper is to relate the maximal rates of convergence and divergence associated with the pairs (A,B) and $(-A,-B)$. Recall that in the case $T=\mu$, one trivially has that $\mathrm{RD}(A,B) = \mathrm{RC}(-A,-B)$. On the other hand, it was proved in [10] that $\mathrm{RC}(A,B) = +\infty$ if and only if $\mathrm{RD}(A,B) = +\infty$ for bidimensional single-input controllable systems of the form (1). The main result we obtain in this paper is Theorem 5.4. It shows that the maximal rate of convergence for a persistently excited system coincides with the maximal rate of divergence for the time-reversed system, provided that there exists a feedback K such that $\mathrm{Lie}(A-(\mathrm{Tr}(A)/d)\mathrm{Id}_d, BK-(\mathrm{Tr}(BK)/d)\mathrm{Id}_d)$, the Lie algebra generated by $A-(\mathrm{Tr}(A)/d)\mathrm{Id}_d$ and $BK-(\mathrm{Tr}(BK)/d)\mathrm{Id}_d$ is equal to $\mathrm{sl}(d,\mathbb{R})$. If $d\geq 3$, we slightly simplify the latter condition by merely asking that there exists a feedback K such that $\mathrm{Lie}(A,BK)$ is equal to $\mathrm{gl}(d,\mathbb{R})$. To prove that result, we first prove that $\mathrm{RD}(A,B) = \mathrm{RC}(-A,-B)$ if there exists a feedback K such that the projection on the real projective space $\mathbb{RP}^{d-1}$ of the bilinear system $\dot{x} = Ax + vBKx$, $x\in\mathbb{R}^d$, $v\in\mathbb{R}$, satisfies the Lie algebra rank condition. We denote by $\mathrm{PLARC}(A,B)$ the set of all such K. In the single-input case, we can refine the result by showing that if (A,B) is controllable then $\mathrm{PLARC}(A,B)$ is nonempty (and conversely if $d\geq 3$). Moreover, we show in a second step that $\mathrm{PLARC}(A,B)$ is nonempty if and only if $\mathrm{LARC}_0(A,B)$ is nonempty, the latter being defined as the set of feedbacks K such that $\mathrm{Lie}(A-(\mathrm{Tr}(A)/d)\mathrm{Id}_d, BK-(\mathrm{Tr}(BK)/d)\mathrm{Id}_d)$ is equal to $\mathrm{sl}(d,\mathbb{R})$.

Let us briefly describe the techniques used in the paper. In order to relate asymptotic properties of (1) and of the corresponding time-reversed system, one must take advantage of the linearity of the problem by analyzing the periodic trajectories of the projected control system on $\mathbb{RP}^{d-1}$. Thus, one is naturally led to consider the family of continuous linear flows on a vector bundle defined by the persistently excited systems associated with the feedbacks K. Such constructions have been used for bilinear control systems in Colonius and Kliemann [12] and for switched systems by Wirth in [22]. The crucial technical step consists of extending to the PE context the results of [12] asserting that if $K \in \mathrm{PLARC}(A,B)$ then periodic trajectories of the projected system corresponding to periodic PE-signals retain the asymptotic properties of the original system. This step exploits the properties of the *invariant control sets* to construct periodic trajectories of the projected system. Let us emphasize that this is based on the Lie algebra rank condition above and does not need specific values of μ and T, provided that $0 < \mu < T$. Finally, since our rates of convergence/divergence are defined in terms of Lyapunov exponents, we rely on tools from dynamical systems theory such as Morse spectrum and control sets, which are used for proving regularity properties for the functions $(A,B,K) \mapsto \mathrm{rc}(A,B,K)$ and $(A,B,K) \mapsto \mathrm{rd}(A,B,K)$.

Let us mention the recent contribution to the theory of linear control systems with general time-varying coefficients given by Anderson, Ilchmann and Wirth [4], also based on Lyapunov exponents. Our contribution is independent of their results, since persistently excited systems present distinctive features.

Before providing the structure of the paper, let us open some perspectives for future work related to the issues discussed here. First of all, it would be interesting to relate, in the multi-input case, the nonemptiness of $\mathrm{LARC}_0(A,B)$ with algebraic properties of the pair (A,B). Secondly, the understanding of the generalized pole-shifting property for $d \geq 3$ and in the multi-input case is still a widely open problem.

The contents of this paper are as follows: Section 2 provides the notion of persistently excited system as well as growth rates of solutions. In particular, the maximal rates of convergence and divergence are defined. Furthermore, Lie algebraic conditions are recalled for bilinear control systems in $\mathbb{R}^d$ and their projections onto projective space. Section 3 shows that the exponential growth rates can be determined by certain periodic trajectories of the projected systems. This is used in Section 4 to derive continuity properties of growth rates. In Section 5 the relation between maximal rates of convergence and divergence is explored and the main result is given and commented (Theorem 5.4). Finally, Section 6 gives a detailed analysis of the single-input case.

Acknowledgements It is a pleasure to acknowledge U. Helmke and P. Kokkonen for pointing out, respectively, the papers [13, 21] and [1, 15], which led us to Proposition 5.1. We also thank J-P. Gauthier and F. Wirth for several fruitful exchanges.

2 Problem formulation and preliminaries

In this section we formally introduce persistently excited linear systems and recall notions and facts concerning their stability properties. In particular, Lyapunov exponents and associated rates of convergence and divergence are recalled. Finally, accessibility properties of related control systems are discussed.

2.1 PE systems and (T,μ)-stabilizers

The following notion is fundamental for this paper.

Definition 2.1 ((T,μ)-signal) *Let μ and T be positive constants with $\mu < T$. A* (T,μ)-signal *is a measurable function $\alpha : \mathbb{R} \to [0,1]$ satisfying*

$$\int_t^{t+T} \alpha(s)ds \geq \mu \ \ \textit{for all } t \in \mathbb{R}\,. \tag{3}$$

We use $\mathcal{G}(T,\mu)$ to denote the set of all (T,μ)-signals.

Given two positive integers $d \geq 2$ and $m \geq 1$, let $M_{d,m}(\mathbb{R})$ be the set of $d \times m$ matrices with real entries and we use $M_d(\mathbb{R})$ to denote $M_{d,d}(\mathbb{R})$. We write $P_{d,m}$ for $M_d(\mathbb{R}) \times M_{d,m}(\mathbb{R})$.

Definition 2.2 (PE system) *Given two positive constants μ and T with $\mu < T$ and a pair $(A, B) \in P_{d,m}$, we define the* persistently excited system *(PE system for short) associated with T, μ, A, and B as the family of non-autonomous linear control systems*

$$\dot{x} = Ax + \alpha Bu, \quad \alpha \in \mathcal{G}(T, \mu). \tag{4}$$

Given a persistently excited system (4), we consider the following problem: Is it possible to stabilize (4) *uniformly* with respect to every (T, μ)-signal α, i.e., to find a matrix $K \in M_{m,d}(\mathbb{R})$ which makes the origin globally asymptotically stable for

$$\dot{x} = (A + \alpha(t)BK)x, \tag{5}$$

with K depending only on A, B, T and μ?

Note that (5) defines a linear continuous flow Φ on the vector bundle $\mathcal{G}(T, \mu) \times \mathbb{R}^d$, since $\mathcal{G}(T, \mu)$ is a shift-invariant (i.e., $\alpha(\cdot)$ is a (T, μ)-signal if and only if the same is true for $\alpha(t_0 + \cdot)$ for every $t_0 \in \mathbb{R}$), convex and weak-$\star$ compact subset of $L^\infty(\mathbb{R}, \mathbb{R})$ (see [12] for definitions).

Referring to $x(\cdot\,; t_0, x_0, A, B, K, \alpha)$ as the solution of (5) passing through x_0 at time t_0, we introduce the following definition.

Definition 2.3 ((T, μ)-stabilizer) *Let $T > \mu > 0$. The gain $K \in M_{m,d}(\mathbb{R})$ is said to be a (T, μ)-*stabilizer *for* (4) *if* (5) *is globally exponentially stable, uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$, i.e., there exist $C, \gamma > 0$ such that every solution $x(\cdot\,; t_0, x_0, A, B, K, \alpha)$ of* (5) *satisfies*

$$|x(t; t_0, x_0, A, B, K, \alpha)| \leq Ce^{-(t-t_0)\gamma}|x_0| \text{ for every } t \geq t_0.$$

The definition above is clearly independent of the choice of the norm on $\mathbb{R}^d$. In the following, we assume $|\cdot|$ to be a fixed norm in $\mathbb{R}^d$ and we denote by $\|\cdot\|$ the induced matrix norm.

Remark 2.4 Since $\mathcal{G}(T, \mu)$ is shift-invariant and compact, Fenichel's uniformity lemma (see [12, Lemma 5.2.7]) yields the following equivalent reformulation of the above definition: $K \in M_{m,d}(\mathbb{R})$ is a (T, μ)-*stabilizer* for (4) if, for every $\alpha \in \mathcal{G}(T, \mu)$, every solution $x(\cdot\,; t_0, x_0, A, B, K, \alpha)$ of (5) tends to zero as time goes to $+\infty$.

2.2 Convergence and divergence rates and a generalized pole-shifting property

Next we introduce a number of rates describing the stability properties of PE systems. Let $(A, B) \in P_{d,m}$, $K \in M_{m,d}(\mathbb{R})$, and $T > \mu > 0$. For $\alpha \in \mathcal{G}(T, \mu)$ and $0 \neq x_0 \in \mathbb{R}^d$ let

$$\lambda^+(x_0, A, B, K, \alpha) = \limsup_{t \to +\infty} \frac{1}{t} \log |x(t; 0, x_0, A, B, K, \alpha)|,$$
$$\lambda^-(x_0, A, B, K, \alpha) = \liminf_{t \to +\infty} \frac{1}{t} \log |x(t; 0, x_0, A, B, K, \alpha)|.$$

Set

$$\Lambda^+(A, B, K, \alpha) = \sup_{x_0 \neq 0} \lambda^+(x_0, A, B, K, \alpha), \qquad \Lambda^-(A, B, K, \alpha) = \inf_{x_0 \neq 0} \lambda^-(x_0, A, B, K, \alpha).$$

The *rate of convergence* and the *rate of divergence* associated with the family of systems $\dot{x} = (A + \alpha BK)x$, $\alpha \in \mathcal{G}(T, \mu)$, are defined as

$$\mathrm{rc}(A, B, K) = \inf_{\alpha \in \mathcal{G}(T,\mu)} \big(-\Lambda^+(A, B, K, \alpha) \big) \text{ and } \mathrm{rd}(A, B, K) = \inf_{\alpha \in \mathcal{G}(T,\mu)} \Lambda^-(A, B, K, \alpha)), \tag{6}$$

respectively. In particular $\mathrm{rc}(A,B,K)>0$ if and only if K is a (T,μ)-stabilizer for (4). Notice that, differently from [10], we omit here from the arguments of rc and rd the quantities T,μ (on which they actually depend), since we focus here on the dependence of these objects on A, B, and K.

Since each signal constantly equal to $\bar\alpha\in[\mu/T,1]$ is in $\mathcal{G}(T,\mu)$, one immediately gets the estimates

$$\mathrm{rc}(A,B,K)\le\min_{\bar\alpha\in[\mu/T,1]}\min(-\Re(\sigma(A+\bar\alpha BK))), \tag{7}$$

and

$$\mathrm{rd}(A,B,K)\le\min_{\bar\alpha\in[\mu/T,1]}\min(\Re(\sigma(A+\bar\alpha BK))), \tag{8}$$

where $\sigma(M)$ denotes the spectrum of a matrix M and $\Re(\zeta)$ the real part of a complex number ζ.

A linear change of coordinates $y=Px$, $v=Vu$ does neither affect $\Lambda^+(A,B,K,\alpha)$ nor $\Lambda^-(A,B,K,\alpha)$. Hence

$$\mathrm{rc}(A,B,K)=\mathrm{rc}(PAP^{-1},PBV^{-1},VKP^{-1}), \tag{9}$$

and

$$\mathrm{rd}(A,B,K)=\mathrm{rd}(PAP^{-1},PBV^{-1},VKP^{-1}), \tag{10}$$

for all invertible matrices $P\in M_d(\mathbb{R})$ and $V\in M_m(\mathbb{R})$.

Remark 2.5 Let P be a change of coordinates which brings the pair (A,B) into a controllability decomposition (A',B') of (A,B), namely,

$$A'=\begin{pmatrix}A_1 & A_2\\ 0 & A_3\end{pmatrix},\quad B'=\begin{pmatrix}B_1\\ 0\end{pmatrix}, \tag{11}$$

with (A_1,B_1) controllable. Then, by (9), (10), and a standard argument based on the variation of constant formula, one gets that, for every $K=(K_1\ K_2)\in M_{d,m}(\mathbb{R})$,

$$\mathrm{rc}(A,B,K)=\min(\mathrm{rc}(A_1,B_1,K_1),\min(-\Re(\sigma(A_3)))), \tag{12}$$

$$\mathrm{rd}(A,B,K)=\min(\mathrm{rd}(A_1,B_1,K_1),\min(\Re(\sigma(A_3)))). \tag{13}$$

Define the *maximal rate of convergence* associated with the PE system (4) as

$$\mathrm{RC}(A,B)=\sup_{K\in M_{m,d}(\mathbb{R})}\mathrm{rc}(A,B,K), \tag{14}$$

and similarly, the *maximal rate of divergence* as

$$\mathrm{RD}(A,B)=\sup_{K\in M_{m,d}(\mathbb{R})}\mathrm{rd}(A,B,K). \tag{15}$$

Because of (9) and (10), one has

$$\mathrm{RC}(A,B)=\mathrm{RC}(PAP^{-1},PBV^{-1}),\qquad \mathrm{RD}(A,B)=\mathrm{RD}(PAP^{-1},PBV^{-1}) \tag{16}$$

for all invertible matrices $P\in M_d(\mathbb{R})$ and $V\in M_m(\mathbb{R})$.

Thanks to Remark 2.5, one deduces for a controllability decomposition of the pair (A,B) as in (11) that

$$\mathrm{RC}(A,B)=\min(\mathrm{RC}(A_1,B_1),\min(-\Re(\sigma(A_3)))),\quad \mathrm{RD}(A,B)=\min(\mathrm{RD}(A_1,B_1),\min(\Re(\sigma(A_3)))). \tag{17}$$

Notice also that

$$\mathrm{RC}(A+\lambda\mathrm{Id}_d,B)=\mathrm{RC}(A,B)-\lambda,\qquad \mathrm{RD}(A+\lambda\mathrm{Id}_d,B)=\mathrm{RD}(A,B)+\lambda. \tag{18}$$

Remark 2.6 Let $(A, B) \in P_{d,m}$ for some $d, m \in \mathbb{N}$. A necessary condition for one of the quantities $\mathrm{RC}(A, B)$ or $\mathrm{RD}(A, B)$ to be infinite is that the pair (A, B) is controllable. This immediately follows from (17).

Remark 2.7 Let $m = 1$ and suppose that for A there exists $\bar{B}$ for which $(A, \bar{B})$ is controllable. Then $\mathrm{RC}(A, B)$ and $\mathrm{RD}(A, B)$ do not depend on B, as long as (A, B) is controllable. This follows from (16) and the fact that the controllability form of a single-input controllable system only depends on the matrix A.

Given a controllable pair (A, B), whether or not RC and RD are both infinite can be understood as whether or not a *generalized pole-shifting property* holds true for the PE system $\dot{x} = Ax + \alpha Bu$, $\alpha \in \mathcal{G}(T, \mu)$. One of the aims of the paper is to investigate up to which extent the unboundedness of RC and RD are equivalent properties. In the planar single-input case the two properties are equivalent, as recalled below ([10, Proposition 4.3]).

Proposition 2.8 *Let $d = 2$ and $m = 1$ and consider a PE system of the form $\dot{x} = Ax + \alpha Bu$, $\alpha \in \mathcal{G}(T, \mu)$. Then $\mathrm{RC}(A, B) = +\infty$ if and only if $\mathrm{RD}(A, B) = +\infty$.*

2.3 Projected dynamics on $\mathbb{RP}^{d-1}$

Let $\Pi : \mathbb{R}^d \to \mathbb{RP}^{d-1}$ be the canonical projection. Given a matrix $A \in M_d(\mathbb{R})$, we denote by ΠA the vector field on $\mathbb{RP}^{d-1}$ obtained by projection of the vector field $x \mapsto Ax$ onto $T\mathbb{RP}^{d-1}$, i.e., for every $q = \Pi x \in \mathbb{RP}^{d-1}$ with $x \in \mathbb{R}^d \setminus \{0\}$,

$$(\Pi A)(q) = d\Pi_x(Ax),$$

where $d\Pi_x : T_x\mathbb{R}^d \to T_q\mathbb{RP}^{d-1}$ denotes the differential of Π at x. Notice that $\Pi A = \Pi(A + \lambda \mathrm{Id})$ for every $\lambda \in \mathbb{R}$.

Given two matrices A_1 and A_2 in $M_d(\mathbb{R})$ and a set of admissible controls $\mathcal{U} \subset L^\infty(\mathbb{R}, [0, 1])$, we define three control systems as follows:

$$\dot{x} = A_1x + uA_2x, \qquad x \in \mathbb{R}^d, \quad u \in \mathcal{U}, \tag{19}$$

$$\dot{q} = (\Pi A_1)(q) + u(\Pi A_2)(q), \qquad q \in \mathbb{RP}^{d-1}, \quad u \in \mathcal{U}, \tag{20}$$

$$\dot{M} = A_1M + uA_2M, \qquad M \in M_d(\mathbb{R}), \quad u \in \mathcal{U}. \tag{21}$$

We say that $\{A_1, A_2\}$ satisfies the *Lie algebra rank condition* if the Lie algebra $\mathrm{Lie}(A_1, A_2)$ generated by A_1 and A_2 is equal to $M_d(\mathbb{R})$. Similarly, we say that $\{A_1, A_2\}$ satisfies the *projected Lie algebra rank condition* if $\{\Pi A_1, \Pi A_2\}$ satisfies the Lie algebra rank condition on $\mathbb{RP}^{d-1}$, i.e., $\mathrm{Lie}_q(\Pi A_1, \Pi A_2) = T_q\mathbb{RP}^{d-1}$ for every $q \in \mathbb{RP}^{d-1}$. This coincides with hypothesis (H) in [11].

Given a pair $(A, B) \in P_{d,m}$, let $\mathrm{LARC}(A, B)$ (respectively, $\mathrm{PLARC}(A, B)$) be the set of $K \in M_{m,d}(\mathbb{R})$ such that $\{A, BK\}$ satisfies the Lie algebra rank condition (respectively, the projected Lie algebra rank condition). We also find useful to introduce the set $\mathrm{LARC}_0(A, B)$ of $M_{m,d}(\mathbb{R})$ made of those feedbacks K such that $\mathrm{Lie}(A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d, BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d) = \mathrm{sl}(d, \mathbb{R})$.

The proof of the following lemma is trivial.

Lemma 2.9 *Let $A_1, A_2 \in M_d(\mathbb{R})$. Then $\Pi[A_1, A_2] = [\Pi A_1, \Pi A_2]$. As a consequence, the attainable set for (20) from every initial condition $q_0 = \Pi x_0 \in \mathbb{RP}^{d-1}$, is the projection on $\mathbb{RP}^{d-1}$ of the attainable set of (19) from x_0 and the evaluation at q_0 of the attainable set for (21) from the identity. Moreover, for every $(A, B) \in P_{d,m}$ and $\lambda \in \mathbb{R}$,*

$$\mathrm{LARC}(A + \lambda\mathrm{Id}, B) \subseteq \mathrm{LARC}_0(A, B) \subseteq \mathrm{PLARC}(A, B). \tag{22}$$

Remark 2.10 For $K \in M_{m,d}(\mathbb{R})$, define the *system group* G_K (respectively, G_K^0) as the orbit through the identity for system (21), with $A_1 = A$ and $A_2 = BK$ (respectively, $A_1 = A-(\mathrm{Tr}(A)/d)\mathrm{Id}_d$ and $A_2 = BK-(\mathrm{Tr}(BK)/d)\mathrm{Id}_d$). It is well known that G_K and G_K^0 are Lie subgroups of $M_d(\mathbb{R})$ with Lie algebras given by $\mathrm{Lie}(A, BK)$ and $\mathrm{Lie}(A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d, BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d)$, respectively. The actions of G_K and G_K^0 on $\mathbb{RP}^{d-1}$ coincide. Moreover, by the orbit theorem applied to the analytic system (20), such an action is transitive if and only if K is in $\mathrm{PLARC}(A, B)$. (For details, see [14, Section 3.5].) In particular, if K is in $\mathrm{PLARC}(A, B)$ then the Lie algebra $\mathrm{Lie}(A, BK)$ is irreducible, i.e., there does not exist a proper subspace of $\mathbb{R}^d$ which is invariant for all the elements of $\mathrm{Lie}(A, BK)$.

3 Growth rates and periodicity

We start this section by a controllability property for the induced system on projective space, which is useful for the subsequent discussion on growth rates.

Let us consider, for a moment, the system

$$\dot{x} = (A + v(t)BK)x, \tag{23}$$

where $K \in M_{m,d}(\mathbb{R})$ is a given feedback matrix and the role previously played by the (T, μ)-signal α is now taken by v, seen as a control parameter, with values in a closed subinterval I of $[0, 1]$ with nonempty interior (the *control range*). We assume that v belongs to $L^\infty(\mathbb{R}, I)$, without persistent excitation assumptions on it.

As noticed in Section 2, the homogeneous bilinear control system (23) in $\mathbb{R}^d$ induces a control system in the projective space $\mathbb{RP}^{d-1}$, given by

$$\dot{q} = (\Pi A)(q) + v(t)(\Pi BK)(q). \tag*{$(24)_I$}$$

Denote by $t \mapsto q(t; t_0, q_0, v)$ the trajectory of $(24)_I$ with initial condition $q(t_0) = q_0 \in \mathbb{RP}^{d-1}$ corresponding to the control $v \in L^\infty(\mathbb{R}, I)$.

We also find useful to introduce the notation $R(\cdot; t_0, v, A, B, K)$ for the principal fundamental solution of (23) associated with the control v, i.e., the solution $R(\cdot)$ of the Cauchy problem

$$\dot{R} = (A + v(t)BK)R, \quad R(t_0) = \mathrm{Id}_d. \tag{25}$$

The following controllability property motivates the role of the assumption that the feedback matrix K is in $\mathrm{PLARC}(A, B)$, which will appear repeatedly in the following sections.

Theorem 3.1 *Consider the projected system $(24)_I$, where $I = [\mu/T, 1] \subset (0, 1]$ and $K \in \mathrm{PLARC}(A, B)$. Then there exists a unique compact subset C of $\mathbb{RP}^{d-1}$ with nonempty interior having the following properties:*

(i) For all $q_0 \in C$, $t \geq 0$, and $v \in L^\infty(\mathbb{R}, I)$ one has $q(t; q_0, v) \in C$.

(ii) For every $q_- \in \mathrm{int}C$ there exists a time $\hat{\tau} > 0$ such that for all $q_0 \in \mathbb{RP}^{d-1}$ there is $v_0 \in L^\infty(\mathbb{R}, I)$ with

$$q(\tau; q_0, v) = q_- \textit{ for some } \tau \in [0, \hat{\tau}].$$

Proof. System (23) may be written in the form

$$\dot{x} = (A + v(t)BK)x = (A + BK + v'(t)BK)x, \quad v'(t) \in I' := [\mu/T - 1, 0].$$

The control range I' is compact and convex and contains 0, and the Lie algebra rank condition holds, since $K \in \mathrm{PLARC}(A,B)$. Hence the projected control system in $\mathbb{RP}^{d-1}$ satisfies the assumptions of [12, Theorem 7.3.3]. It follows that the control system $(24)_I$ has a unique invariant control set C, which is compact, has nonempty interior, and is contained in the closure of every attainable set of $(24)_I$. Recall that an invariant control set is characterized by condition (i) together with the property that every element of C is approximately controllable from every other element of C (cf. [12, Definition 3.1.3]). The proof is completed by noticing that [12, Lemma 3.2.21] implies assertion (ii) stating exact controllability to points in the interior of C. ■

We turn to growth rates for PE systems. Given $\alpha \in \mathcal{G}(T,\mu)$ and $x_0 \in \mathbb{R}^d \setminus \{0\}$, we say that (α, x_0) is #-admissible for A, B, K if there exists $\tau > 0$ such that both $t \mapsto \alpha(t)$ and $t \mapsto \Pi x(t; 0, x_0, A, B, K, \alpha)$ are τ-periodic. Corresponding rates of convergence and divergence are defined by replacing in the definitions of $\mathrm{rc}, \mathrm{RC}, \mathrm{rd}, \mathrm{RD}$ the class of trajectories $x(\cdot; 0, x_0, A, B, K, \alpha)$ corresponding to (T,μ)-signals by the subclass corresponding to pairs (α, x_0) that are #-admissible for A, B, K. More precisely, let

$$\mathrm{rc}_\#(A,B,K) := \inf -\lambda^+(x_0, A, B, K, \alpha) \text{ and } \mathrm{rd}_\#(A,B,K) := \inf \lambda^-(x_0, A, B, K, \alpha),$$

where in both cases the infimum is taken over all (α, x_0) which are #-admissible for A, B, K. If the considered A, B, K are clear from the context, we omit these arguments here and in other expressions. Furthermore, let

$$\mathrm{RC}_\#(A,B) := \sup_{K \in M_{m,d}(\mathbb{R})} \mathrm{rc}_\#(A,B,K) \text{ and } \mathrm{RD}_\#(A,B) := \sup_{K \in M_{m,d}(\mathbb{R})} \mathrm{rd}_\#(A,B,K).$$

Lemma 3.2 *Let $I := [\mu/T, 1]$ and K in $\mathrm{PLARC}(A,B)$. Consider the set C from Theorem 3.1 for the projected system $(24)_I$. Fix a point $\Pi x_- \in \mathrm{int} C$, and let $x_0 := e^{(T-\mu)(A+BK)} x_-$. Then for every $\varepsilon > 0$ there exists $\bar{\tau} > 0$ such that for every $t > \bar{\tau}$ and every $\alpha \in \mathcal{G}(T,\mu)$ there exists $\alpha_\# \in \mathcal{G}(T,\mu)$ with $(\alpha_\#, x_0)$ #-admissible for A, B, K satisfying*

$$\left| \lambda^+(x_0, A, B, K, \alpha_\#) - \frac{1}{t} \log |x(t; 0, x_0, A, B, K, \alpha)| \right| < \varepsilon. \tag{26}$$

Proof. First note that $\Pi x_0 \in C$, since Πx_- is in the invariant control set C and the control $u \equiv 1$ has values in $I = [\mu/T, 1]$.

For every $\alpha \in \mathcal{G}(T,\mu)$ and $t > 0$ consider the signal $\alpha^t_\#$ obtained through the following procedure: Let $\alpha^t_\#(s) = \alpha(s)$ for $s \in [0,t]$ and $\alpha^t_\#(s) = 1$ for $s \in (t, t+T-\mu]$. By Theorem 3.1 there exist a time $\hat{\tau}$ independent of $\alpha(\cdot)$ and t and a control $v^t : [0, \tau^{(t)}] \to [\mu/T, 1]$ with $\tau^{(t)} \le \hat{\tau}$ such that $\Pi x(\tau^{(t)}; 0, y^t, v^t) = \Pi x_-$, where $y^t := x(t+T-\mu; 0, x_0, \alpha^t_\#)$. The definition of $\alpha^t_\#$ is then concluded by taking

$$\alpha^t_\#(s) = \begin{cases} v^t(s-(T-\mu)) & \text{for} \quad s \in (t+T-\mu, t+T-\mu+\tau^{(t)}] \\ 1 & \text{for} \quad s \in (t+T-\mu+\tau^{(t)}, t+2(T-\mu)+\tau^{(t)}], \end{cases}$$

and extending $\alpha^t_\#$ periodically on $\mathbb{R}$ with period

$$T^{(t)} := t + 2(T-\mu) + \tau^{(t)}.$$

Then $\Pi x(T^{(t)}; x_0, A, B, K, \alpha^t_\#) = \Pi x_0$, hence periodicity in projective space holds. By construction, $\alpha^t_\# \in \mathcal{G}(T,\mu)$ and (α^t, x_0) is #-admissible for A, B, K. Periodicity in projective space and homogeneity of the evolution imply

$$\begin{aligned} \lambda^+\left(x_0, \alpha^t_\#\right) &= \lim_{k\to\infty} \frac{1}{kT^{(t)}} \log |x(kT^{(t)}; 0, x_0, \alpha^t_\#)| \\ &= \frac{1}{T^{(t)}} \log |x(T^{(t)}; 0, x_0, \alpha^t_\#)|. \end{aligned} \tag{27}$$

Notice now that for every $t > 0$

$$x(T^{(t)}; 0, x_0, \alpha^t_{\#}) = R^{(t)} x(t; 0, x_0, \alpha^t_{\#}),$$

where $R^{(t)} = R(T^{(t)}; t, \alpha^t_{\#}, A, B, K)$.

Since $T^{(t)} - t \leq 2(T - \mu) + \hat{\tau}$, Gronwall's lemma immediately yields the existence of $C_0 > 1$ independent of t and $\alpha(\cdot)$ such that $\|R^{(t)}\|, \left\|\left(R^{(t)}\right)^{-1}\right\| \leq C_0$ for all $t > 0$.

Then $\alpha^t_{\#}(s) = \alpha(s)$ for $s \in [0, t]$ implies

$$\left|\log\left|x(T^{(t)}; 0, x_0, \alpha^t_{\#})\right| - \log|x(t; 0, x_0, \alpha)|\right| < \log C_0.$$

It follows that

$$\begin{aligned}
&\left|\frac{1}{T^{(t)}} \log\left|x(T^{(t)}; 0, x_0, \alpha^t_{\#})\right| - \frac{1}{t}\log|x(t; 0, x_0, \alpha)|\right| \\
&\leq \frac{1}{T^{(t)}}\left|\log\left|x(T^{(t)}; 0, x_0, \alpha^t_{\#})\right| - \log\left|x(t; 0, x_0, \alpha^t_{\#})\right|\right| + \left|\frac{1}{T^{(t)}} - \frac{1}{t}\right| \left|\log\left|x(t; 0, x_0, \alpha^t_{\#})\right|\right| \\
&< \frac{1}{T^{(t)}}\left[\log C_0 + (2(T-\mu) + \hat{\tau})\frac{1}{t}\left|\log\left|x(t; 0, x_0, \alpha^t_{\#})\right|\right|\right].
\end{aligned}$$

Since $A + \alpha BK$ is uniformly bounded for $\alpha \in [0, 1]$, Gronwall's lemma again shows that

$$\frac{1}{t}\left|\log\left|x(t; 0, x_0, \alpha^t_{\#})\right|\right| < C_1$$

for a constant $C_1 > 0$, uniformly with respect to $t \geq 1$ and $\alpha \in \mathcal{G}(T, \mu)$. It follows that for $t \geq 1$

$$\left|\frac{1}{T^{(t)}} \log\left|x(T^{(t)}; 0, x_0, \alpha^t_{\#})\right| - \frac{1}{t}\log\left|x(t; 0, x_0, \alpha^t_{\#})\right|\right| < \frac{1}{t}\left(\log C_0 + (2T + \hat{\tau})C_1\right).$$

Assertion (26) then follows from (27) by taking $t \geq \bar{\tau} := 1 + \varepsilon^{-1}\left(\log C_0 + (2T + \hat{\tau})C_1\right)$. ■

The periodic approximation provided by Lemma 3.2 gives the following approximation result for the rate of convergence.

Proposition 3.3 *Let (A, B) be in $P_{d,m}$. For every $K \in \mathrm{PLARC}(A, B)$, we have $\mathrm{rc}(A, B, K) = \mathrm{rc}_{\#}(A, B, K)$.*

Proof. For every K the inequality

$$\mathrm{rc}(A, B, K) \leq \mathrm{rc}_{\#}(A, B, K) \tag{28}$$

is trivially satisfied. In order to prove the converse inequality, we fix $K \in \mathrm{PLARC}(A, B)$, a constant $m \in \mathbb{R}$ such that $\mathrm{rc}(A, B, K) < m$, and we show that $\mathrm{rc}_{\#}(A, B, K) < m$. By definition, there exist a (T, μ)-signal α_0 and a vector x_0 such that

$$\lambda^+(x_0, A, B, K, \alpha_0) = \limsup_{t \to +\infty} \frac{1}{t}\log|x(t; 0, x_0, A, B, K, \alpha_0)| > -m. \tag{29}$$

For a given function α_0 the maximal Lyapunov exponent $\Lambda^+(A, B, K, \alpha_0)$ is attained on every basis of $\mathbb{R}^d$ (see [7, Chapter 2]). Since, moreover, $\mathrm{int} C$ is nonvoid, the set

$$\left\{e^{(T-\mu)(A+BK)}x \mid \Pi x \in \mathrm{int} C\right\}$$

contains a basis of $\mathbb{R}^d$. This implies that in (29) the point x_0 can be chosen in this set. Now we can apply Lemma 3.2 with $\varepsilon = \frac{1}{2}(\lambda^+(x_0, A, B, K, \alpha_0) + m)$, $\alpha = \alpha_0$, and t large enough such that

$$\frac{1}{t} \log |x(t; 0, x_0, \alpha_0)| > -m + \varepsilon.$$

Thus there is $\alpha^t_\#$ such that $(x_0, \alpha^t_\#)$ is #-admissible with

$$\lambda^+ \left(x_0, A, B, K, \alpha^t_\#\right) > -m.$$

It follows that $\mathrm{rc}_\#(A, B, K) < m$. ■

Next we analyze the relations between convergence and divergence rates using time reversal in PE systems. The time reversed system corresponding to a non-autonomous control system of the type $\dot{x} = Ax + \alpha(t)Bu$ is $\dot{x} = -Ax - \alpha(-t)Bu$. This justifies the notation $\alpha_-(t) = \alpha(-t)$ for every signal α. Moreover, it is clear that the two systems have the same accessibility properties and, in particular, that $\mathrm{PLARC}(A, B) = \mathrm{PLARC}(-A, -B)$.

The rates of convergence and divergence for #-admissible pairs satisfy the following property under time reversal.

Proposition 3.4 *Let (A, B) be in $P_{d,m}$ and $K \in M_{m,d}(\mathbb{R})$. Then* $\mathrm{rd}_\#(-A, -B, K) = \mathrm{rc}_\#(A, B, K)$.

Proof. Note that (α, x_0) is #-admissible for A, B, K if and only if (α_-, x_0) is #-admissible for $-A, -B, K$ and $-\lambda^+(x_0, A, B, K, \alpha) = \lambda^-(x_0, -A, -B, K, \alpha_-)$.

Then, by taking the infimum with respect to all #-admissible pairs (α, x_0) for A, B, K one concludes

$$\begin{aligned}
\mathrm{rc}_\#(A, B, K) &= \inf\{-\lambda^+(x_0, A, B, K, \alpha) \mid (\alpha, x_0) \text{ \#-admissible for } A, B, K\} \\
&= \inf\{\lambda^-(x_0, -A, -B, K, \alpha_-) \mid (\alpha, x_0) \text{ \#-admissible for } -A, -B, K\} \\
&= \mathrm{rd}_\#(-A, -B, K).
\end{aligned}$$

■

Next we provide a property dealing with the rate of divergence and #-admissible pairs. The proof is an adaptation of [11, §3.4].

Proposition 3.5 *Let (A, B) be in $P_{d,m}$ and $K \in \mathrm{PLARC}(A, B)$. Then* $\mathrm{rd}_\#(A, B, K) = \mathrm{rd}(A, B, K)$.

Proof. The inequality $\mathrm{rd}(A, B, K) \leq \mathrm{rd}_\#(A, B, K)$ holds trivially. In order to show the converse define for $t > 0$

$$S^t(A, B, K) = \{R(t; 0, \alpha, A, B, K) \mid \alpha \in \mathcal{G}(T, \mu),\ \alpha \ t\text{-periodic}\}.$$

Thus $S^t(A, B, K)$ consists of fundamental solutions for $\dot{x} = A + \alpha BK$ evaluated at time t (see (25)). Note that not every function α satisfying the persistent excitation condition (3) on $[0, t]$ may be extended t-periodically to a (T, μ)-signal in $\mathcal{G}(T, \mu)$. Set

$$\delta(A, B, K) = \limsup_{t \to +\infty} \frac{1}{t} \sup_{g \in S^t(A,B,K)} \log \|g\|, \qquad \delta^*(A, B, K) = \liminf_{t \to +\infty} \frac{1}{t} \inf_{g \in S^t(A,B,K)} \log m(g),$$

where $m(\cdot)$ is the *conorm* corresponding to the vector norm $|\cdot|$, defined by

$$m(g) = \min\{|gx| \mid |x| = 1\}.$$

Let us prove that $S^t(-A,-B,K) = \{g^{-1} \mid g \in S^t(A,B,K)\} = \big(S^t(A,B,K)\big)^{-1}$ for every $t>0$. This follows by time reversal: Set $V(s) = R(s,\alpha,-A,-B,K)$ for $s \in \mathbb{R}$, with α t-periodic and in $\mathcal{G}(T,\mu)$. Then, $V(0) = \mathrm{Id}$ and

$$\dot{V} = (-A-\alpha BK)V.$$

Fix $s \geq 0$ and let $G_s(t) = V(t)V(s)^{-1}$. Then, for $t \in [0,s]$, $G_s(s-t) = R(t,\alpha_s,A,B,K)$, where $\alpha_s(\tau) := \alpha(s-\tau)$, $\tau \in \mathbb{R}$. Therefore, for every $s \geq 0$, $V(s)^{-1} = R(s,\alpha_s,A,B,K)$. Hence, $V(t) = R(t,\alpha_-,A,B,K)^{-1}$, since $\alpha_t = \alpha_-$ by t-periodicity. Hence

$$\delta(-A,-B,K) = \limsup_{t\to+\infty} \frac{1}{t} \sup \log\|g\|, \qquad \delta^*(-A,-B,K) = \liminf_{t\to+\infty} \frac{1}{t} \inf \log m(g),$$

where the supremum and the infimum are taken over all $g \in S^t(-A,-B,K) = (S^t(A,B,K))^{-1}$. Using as in [11, Lemma 3.1] the relation $\|g\| = m(g^{-1})^{-1}$, we have

$$\inf_{g\in S^t(A,B,K)} \log m(g) = -\sup_{g\in (S^t(A,B,K))^{-1}} \log\|g\|.$$

Dividing by t and taking the $\liminf$ as t goes to $+\infty$ one finds

$$\delta^*(A,B,K) = -\delta(-A,-B,K). \tag{30}$$

Recall that by Proposition 3.4

$$\mathrm{rd}_\#(A,B,K) = \mathrm{rc}_\#(-A,-B,K). \tag{31}$$

Let us next prove that

$$\delta^*(A,B,K) \leq \mathrm{rd}(A,B,K). \tag{32}$$

For $t>0$ define

$$Q^t(A,B,K) = \{R(t,\alpha,A,B,K) \mid \alpha \in \mathcal{G}(T,\mu)\},$$

and set

$$\delta^*(t) := \inf_{g\in S^t(A,B,K)} \log m(g), \quad \delta_1^*(t) := \inf_{g\in Q^t(A,B,K)} \log m(g).$$

We next prove that

$$\delta^*(A,B,K) = \liminf_{t\to+\infty} \delta^*(t)/t = \liminf_{t\to+\infty} \delta_1^*(t)/t. \tag{33}$$

The first equality is clear by definition, and we trivially have $\delta^*(t) \geq \delta_1^*(t)$ for every $t>0$, since $S^t(A,B,L) \subset Q^t(A,B,L)$. Let us now show that there exist $C,\bar{t}>0$ such that $\delta^*(t) - \delta_1^*(t) \leq C$ for every $t \geq \bar{t}$.

Abbreviate $R(t,\alpha) := R(t,\alpha,A,B,K)$ and pick any $g := R(t,\alpha) \in Q^t(A,B,K)$ corresponding to $\alpha \in \mathcal{G}(T,\mu)$ and $t \geq T$. We modify α in order to get a periodic (T,μ)-signal $\tilde{\alpha} \in \mathcal{G}(T,\mu)$ as follows:

$$\tilde{\alpha}(s) = \begin{cases} \alpha(s) & \text{for} \quad s \in [0,t-T] \\ 1 & \text{for} \quad s \in (t-T,t) \end{cases}$$

and extend $\tilde{\alpha}$ t-periodically to $\mathbb{R}$. Notice that $\tilde{g} := R(t,\tilde{\alpha}) \in S^t(A,B,K)$ and

$$g = R(T,\alpha(\cdot+t-T))R(t-T,\alpha), \qquad \tilde{g} = e^{T(A+BL)}R(t-T,\alpha).$$

We conclude

$$\left\|g^{-1}\right\| = \left\|R(t-T,\alpha)^{-1}R(T,\alpha(\cdot+t-T))^{-1}\right\| \leq \left\|e^{T(A+BL)}\right\| \left\|\tilde{g}^{-1}\right\| \left\|R(T,\alpha(\cdot+t-T))^{-1}\right\|.$$

Since $\left\|R(T,\alpha(\cdot+t-T))^{-1}\right\|$ is bounded, uniformly with respect to α and $t \geq T$, there is a constant $C > 0$ such that $\|g^{-1}\| \leq C_0\|\tilde{g}^{-1}\|$. Then clearly

$$\log m(g) = -\log\|g^{-1}\| \geq -\log\|\tilde{g}^{-1}\| - C = \log m(\tilde{g}) - C \geq \delta_1^*(t) - C.$$

Taking the infimum over $g \in Q^t(A,B,L)$ one gets equality (33).

If $K \in \mathrm{PLARC}(A,B)$ we conclude the proof of the proposition by showing that

$$\mathrm{rc}_\#(-A,-B,K) \leq -\delta(-A,-B,K).$$

Combining it with (30), (31), (32) this will prove, as desired,

$$\mathrm{rd}_\#(A,B,K) = \mathrm{rc}_\#(-A,-B,K) \leq -\delta(-A,-B,K) = \delta^*(A,B,K) \leq \mathrm{rd}(A,B,K). \tag{34}$$

Then equalities hold here. We emphasize that the assumption that $K \in \mathrm{PLARC}(A,B)$ is only used at this stage of the argument.

For the sake of notational simplicity, let us prove the equivalent inequality $\mathrm{rc}_\#(A,B,K) \leq -\delta(A,B,K)$ (recall that $\mathrm{PLARC}(A,B) = \mathrm{PLARC}(-A,-B)$). Since $\mathrm{int}C$ is nonempty, there is a basis of $\mathbb{R}^d$, say $e_1,\dots,e_d$ with $\Pi e_1,\dots,\Pi e_d \in \mathrm{int}C$. Let $\kappa > 0$ be such that, for every $d \times d$ matrix g, the following inequality holds true

$$\|g\| \leq \kappa \max_{i=1,\dots,d} |ge_i|.$$

Then, clearly, for every $t > 0$ and every $g \in S^t(A,B,K)$,

$$\frac{1}{t}\log\|g\| \leq \frac{1}{t}\log\left(\max_{i=1,\dots,d}|ge_i|\right) + \frac{1}{t}\log\kappa.$$

Then, for every $\varepsilon > 0$, there exist $i \in \{1,\dots,d\}$, $t > 0$ arbitrarily large, and $\alpha \in \mathcal{G}(T,\mu)$ such that α is t-periodic and

$$\frac{1}{t}\log|x(t;0,e_i,A,B,K,\alpha)| > \delta(A,B,K) - \varepsilon.$$

Applying Lemma 3.2 with $x_0 = e_i$ and α, t, ε as above, we obtain that there exists $\alpha_\#$ such that $(\alpha_\#, e_i)$ is $\#$-admissible for A,B,K and

$$\lim_{s\to+\infty}\frac{1}{s}\log|x(s;0,e_i,A,B,K,\alpha_\#)| > \delta(A,B,K) - 2\varepsilon.$$

We deduce that $\mathrm{rc}_\#(A,B,K) \leq -\delta(A,B,K)$, as required. ■

Corollary 3.6 *If K belongs to* $\mathrm{PLARC}(A,B)$ *then*

$$\mathrm{rc}(A,B,K) = \mathrm{rd}(-A,-B,K) = -\delta(A,B,K).$$

Proof. Since $K \in \mathrm{PLARC}(A,B)$,

$$\mathrm{rc}(A,B,K) = \mathrm{rc}_\#(A,B,K) = \mathrm{rd}_\#(-A,-B,K) = \mathrm{rd}(-A,-B,K), \tag{35}$$

where the first equality follows from Proposition 3.3, the second equality from Proposition 3.4, and the last equality from Proposition 3.5 applied to the pair $(-A,-B)$. Since equalities hold in (34), we also have

$$\mathrm{rd}(-A,-B,K) = -\delta(A,B,K).$$

■

Remark 3.7 Corollary 3.6 and the definition of δ imply that $\mathrm{rc}(A,B,K)$ is also equal to the supremal Bohl exponent

$$\sup_{\alpha\in\mathcal{G}(T,\mu)} \limsup_{s,t-s\to+\infty} \frac{1}{t-s}\log\|R(t;s,\alpha,A,B,K)\|.$$

This is analogous to [12, Theorem 7.2.20].

4 Continuity properties of the growth rates

We investigate in this section the continuity properties of the convergence and divergence rates, i.e. of the maps $(A, B, K) \mapsto \mathrm{rc}(A, B, K)$ and $(A, B, K) \mapsto \mathrm{rc}(A, B, K)$ defined in (6). This issue can be restated as the study of the continuity properties of the maximal and minimal Lyapunov exponents of system (4) with respect to (A, B, K).

Denote by θ the flow

$$\theta_t : \alpha \mapsto \alpha(t + \cdot), \qquad t \in \mathbb{R},$$

defined on $\mathcal{G}(T, \mu)$. Clearly, the periodic points of the shift are the periodic (T, μ)-signals.

Lemma 4.1 *The periodic (T, μ)-signals are dense in $\mathcal{G}(T, \mu)$ for the weak-$\star$ topology.*

Proof. Let $\alpha \in \mathcal{G}(T, \mu)$. We construct a sequence of periodic (T, μ)-signals α_k weak-$\star$ converging to α. Define

$$\alpha_k(t) = \begin{cases} 1 & \text{for} \quad [-T + \mu - k, -k), \\ \alpha(t) & \text{for} \quad t \in [-k, k], \\ 1 & \text{for} \quad t \in (k, , k + T - \mu], \end{cases}$$

and extend on $\mathbb{R}$ by $2(k + T - \mu)$-periodicity. Then α_k belongs to $\mathcal{G}(T, \mu)$. Take $y \in L^1(\mathbb{R}, \mathbb{R})$ and let $\varepsilon > 0$. There exists $k_\varepsilon \in \mathbb{N}$ such that for all $k \geq k_\varepsilon$

$$\int_{\mathbb{R} \setminus [-k,k]} |y(t)|\, dt < \varepsilon.$$

Then, for $k > k_\varepsilon$,

$$\left| \int_{\mathbb{R}} y(t)\alpha_k(t)dt - \int_{\mathbb{R}} y(t)\alpha(t)dt \right| \leq \int_{\mathbb{R}} |y(t)|\, |\alpha_k(t) - \alpha(t)|\, dt \leq \int_{\mathbb{R} \setminus [-k,k]} |y(t)|\, dt < \varepsilon.$$

■

Since $\mathcal{G}(T, \mu)$ is compact connected metrizable for the weak-$\star$ topology, the above lemma yields that the flow θ in the base $\mathcal{G}(T, \mu)$ is chain transitive and the flows $\Phi = \Phi(A, B, K)$ on $\mathcal{G}(T, \mu) \times \mathbb{R}^d$ depend continuously on (A, B, K). Thus the flow Φ satisfies the assumptions in [12, Corollary 5.3.11] and upper semicontinuity of the supremal spectral growth rates follows. More precisely, for a sequence $(A^n, B^n, K^n) \to (A, B, K)$, denoting by λ^n any Lyapunov exponent associated with (A^n, B^n, K^n), one has

$$\sup\{\lambda \in \mathbb{R} \,|\, \text{ there are Lyapunov exponents } \lambda^n \text{ with } \lambda^n \to \lambda\} \leq \sup_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 0} \lambda^+(x_0, A, B, K, \alpha).$$

(Here, in addition to [12, Corollary 5.3.11], it is used that the supremum of the Morse spectrum coincides with the supremum over all Lyapunov exponents, [12, Theorem 5.1.6].) In the same way, one obtains that

$$\inf\{\lambda \in \mathbb{R} \,|\, \text{ there are Lyapunov exponents } \lambda^n \text{ with } \lambda^n \to \lambda\} \geq \inf_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 0} \lambda^-(x_0, A^0, B^0, K^0, \alpha).$$

An immediate consequence is that

$$\limsup_{n \to \infty} \sup_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 1} \lambda^+(x_0, A^n, B^n, K^n, \alpha) \leq \sup_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 0} \lambda^+(x_0, A^0, B^0, K^0, \alpha).$$

In other words, the maximal Lyapunov exponent is upper semicontinuous or, equivalently, the rate of convergence $\mathrm{rc}(A, B, K)$ is lower semicontinuous with respect to (A, B, K). Analogously,

$$\liminf_{n \to \infty} \inf_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 1} \lambda^-(x_0, A^n, B^n, K^n, \alpha) \geq \inf_{\alpha \in \mathcal{G}(T,\mu), x_0 \neq 1} \lambda^-(x_0, A^0, B^0, K^0, \alpha),$$

Hence the minimal Lyapunov exponent is lower semicontinuous or, equivalently, the rate of divergence $\mathrm{rd}(A, B, K)$ is lower semicontinuous with respect to (A, B, K).

Theorem 4.2 *(i) The functions* $\mathrm{rc}, \mathrm{rd} : P_{d,m} \times M_{m,d}(\mathbb{R}) \to \mathbb{R}, (A, B, K) \mapsto \mathrm{rc}(A, B, K)$ *are lower semicontinuous. (ii) The restrictions of* rc *and* rd *to the set of all* (A, B, K) *with* $K \in \mathrm{PLARC}(A, B)$ *are also upper semicontinuous, and hence continuous there.*

Proof. Assertion (i) has been established above. We show upper semicontinuity of $\mathrm{rc}(A, B, K)$, i.e., lower semicontinuity of the maximal Lyapunov exponent on $\{(A, B, K) \mid K \in \mathrm{PLARC}(A, B)\}$.

Consider a sequence $(A^n, B^n, K^n) \to (A^0, B^0, K^0)$. We have to show that

$$\limsup_{n\to\infty} \mathrm{rc}(A^n, B^n, K^n) \leq \mathrm{rc}(A^0, B^0, K^0), \tag{36}$$

that is

$$\liminf_{n\to\infty} \sup_{\alpha\in\mathcal{G}(T,\mu),x_0\neq 0} \lambda^+(x_0, A^n, B^n, K^n, \alpha) \geq \sup_{\alpha\in\mathcal{G}(T,\mu),x_0\neq 0} \lambda^+(x_0, A^0, B^0, K^0, \alpha).$$

Let $\varepsilon > 0$. Since $\mathrm{rc}(A^0, B^0, K^0) = \mathrm{rc}_\#(A^0, B^0, K^0)$, there exists a $\#$-admissible pair $(\alpha_\varepsilon, x_{0,\varepsilon})$ such that

$$\left| \mathrm{rc}(A^0, B^0, K^0) - \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon| \right| < \varepsilon,$$

where τ_ε is the period of the trajectory on $\mathbb{RP}^{d-1}$ associated with α_ε and starting from $\Pi x_{0,\varepsilon}$ and $\nu_\varepsilon \in \mathbb{R}$ satisfies $R_\varepsilon(\tau_\varepsilon)x_{0,\varepsilon} = \nu_\varepsilon x_{0,\varepsilon}$, with $R_\varepsilon(\cdot) = R(\cdot; 0, \alpha_\varepsilon, A^0, B^0K^0)$ defined according to (25).

Furthermore, recall that eigenvalues depend continuously on the matrix (this follows, e.g., from [20, Lemma A.4..1]). For $n \in \mathbb{N}$, let $R^n(\cdot) = R(\cdot; 0, \alpha_\varepsilon, A^n, B^nK^n)$. Then there exists a sequence $(\nu_\varepsilon^n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ converging to ν_ε as n tends to infinity such that ν_ε^n is an eigenvalue of $R^n(\tau_\varepsilon)$ for $n \in \mathbb{N}$. One therefore has for n large enough that

$$\left| \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon| - \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon^n| \right| < \varepsilon.$$

Furthermore, there is $x_\varepsilon^n \in \mathbb{S}^{d-1}$ in the generalized real eigenspace associated with $(\nu_\varepsilon^n, \overline{\nu_\varepsilon^n})$ such that

$$|R^n(k\tau_\varepsilon)x_\varepsilon^n| = |\nu_\varepsilon^n|^k, \qquad \text{for every } k \in \mathbb{N}.$$

This implies

$$\sup_{\alpha\in\mathcal{G}(T,\mu),x_0\neq 0} \lambda^+(x_0, A^n, B^n, K^n, \alpha) \geq \lambda^+(x_\varepsilon^n, A^n, B^n, K^n, \alpha_\varepsilon) = \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon^n|,$$

and hence for n large enough

$$\mathrm{rc}(A^n, B^n, K^n) \leq \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon^n| \leq \frac{1}{\tau_\varepsilon} \log |\nu_\varepsilon| + \varepsilon \leq \mathrm{rc}(A^0, B^0, K^0) + 2\varepsilon.$$

Since ε is arbitrary, assertion (36) follows.

Finally, upper semicontinuity of $\mathrm{rd}(A, B, K)$ is a consequence of Corollary 3.6. ∎

5 Properties of maximal growth rates

Before stating the main result of the paper, we need the following proposition.

Proposition 5.1 *Let $(A, B) \in P_{d,m}$. The following statements are equivalent:*

(i) $\mathrm{PLARC}(A, B)$ *is nonempty;*

(ii) $\mathrm{LARC}_0(A, B)$ *is nonempty;*

(iii) $\mathrm{LARC}_0(A, B)$ *is dense in* $M_{m,d}(\mathbb{R})$*;*

(iv) $\mathrm{PLARC}(A, B)$ *is dense in* $M_{m,d}(\mathbb{R})$*.*

Moreover, if $d \geq 3$, then the above statements are also equivalent to the following ones:

(v) $\mathrm{LARC}(A, B)$ *is nonempty;*

(vi) $\mathrm{LARC}(A, B)$ *is dense in* $M_{m,d}(\mathbb{R})$*.*

Proof. Notice that the implication $(iii) \Rightarrow (iv)$ follows directly from Lemma 2.9, while $(iv) \Rightarrow (i)$ is trivial.

We next prove that (i) implies (ii). Let $K^\star$ be in $\mathrm{PLARC}(A, B)$. Since $\mathbb{RP}^{d-1}$ is compact, there exists an open neighborhood V_0 of $K^\star$ contained in $\mathrm{PLARC}(A, B)$. For every $K \in V_0$, let G_K^0 and L_K^0 be, respectively, the group and the Lie algebra generated by $A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d$ and $BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d$. As noticed in Remark 2.10, G_K^0 acts transitively on $\mathbb{RP}^{d-1}$, which implies that L_K^0 belongs, up to similarity, to the list given in [13, Theorem 19] (based on results given in [21]). The important consequence for the present proof is that all Lie algebras in that list are, up to similarity, either equal to $\mathrm{sl}(d)$ or $\mathrm{spin}(9, 1)$ or contained in $\mathrm{so}(d)$. Therefore, to close the argument, it is enough to find $K \in V_0$ such that no matrix similar to $BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d$ belongs to $\mathrm{so}(d)$ nor $\mathrm{spin}(9, 1)$. Notice that the spectrum of any matrix similar to an element of $\mathrm{so}(d)$ is contained in the imaginary axis, while the eigenvalues of a matrix similar to an element of $\mathrm{spin}(9, 1)$ are symmetric with respect to the origin, in the sense that if λ is an eigenvalue then $-\lambda$ is as well, with the same algebraic multiplicity as λ (a proof of this fact is provided in Appendix). It is therefore enough to find $K \in V_0$ and a nonzero real number λ so that λ is an eigenvalue of $BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d$ and either $-\lambda$ is not or its multiplicity is different from the one of λ.

With no loss of generality, we assume that $B = \begin{pmatrix} \mathrm{Id}_m \\ 0 \end{pmatrix}$ and thus

$$F(K) := BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d = \begin{pmatrix} K_1 - (\mathrm{Tr}(K_1)/d)\mathrm{Id}_m & K_2 \\ 0 & -(\mathrm{Tr}(K_1)/d)\mathrm{Id}_{d-m} \end{pmatrix},$$

where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ is an arbitrary matrix of $M_{m,d}(\mathbb{R})$.

If $m = d$, then clearly $\{F(K) \mid K \in V_0\}$ is an open nonempty subset of $\mathrm{sl}(d, \mathbb{R})$ and one concludes.

If $m = d - 1$, set $K_t = K^\star + t\,\mathrm{diag}(\mathrm{Id}_{d-1}, -(d-1))$ for $t \in \mathbb{R}$. Clearly $K_t \in V_0$ for t small enough. For $t \neq 0$ small enough, $-(\mathrm{Tr}(K_1^\star)/d) - (d-1)t$ is a nonzero real eigenvalue of $F(K_t)$ and the other eigenvalues are of the type $\lambda - \mathrm{Tr}(K_1^\star)/d + t/d$ with λ eigenvalue of $K_1^\star$. Then, if $\mathrm{Tr}(K_1^\star)/d + (d-1)t$ is an eigenvalue of $F(K_t)$, then there exists an eigenvalue λ of $K_1^\star$ so that

$$\lambda - 2\frac{\mathrm{Tr}(K_1^\star)}{d} = \left(d - 1 - \frac{1}{d}\right)t.$$

Hence, for $t \neq 0$ small enough, $\mathrm{Tr}(K_1^\star)/d + (d-1)t$ cannot be an eigenvalue of $F(K_t)$ and we are done.

Assume now that $m \le d-2$. Then there exists $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \in V_0$ such that $k_1 := -\mathrm{Tr}(K_1)/d \neq 0$ and the eigenvalues of $K_1 - (\mathrm{Tr}(K_1)/d)\mathrm{Id}_m$ and k_1 are two by two distinct. As a consequence, k_1 is a nonzero eigenvalue of $F(K)$ of multiplicity at least two and the multiplicity of $-k_1$ as an eigenvalue of $F(K)$ is at most one. As noticed above, this allows to conclude that any matrix similar to $F(K)$ does not belong to $\mathrm{so}(d)$ if $d \neq 10$ and neither to $\mathrm{so}(10)$ nor $\mathrm{spin}(9,1)$ if $d = 10$.

We conclude the proof of the first part of the statement by showing that (ii) implies (iii). Assume that there exists $K_0 \in M_{m,d}(\mathbb{R})$ such that $\mathrm{Lie}(A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d, BK_0 - (\mathrm{Tr}(BK_0)/d)\mathrm{Id}_d) = \mathrm{sl}(d,\mathbb{R})$. Let us select a basis of $\mathrm{sl}(d,\mathbb{R})$ made of iterated Lie brackets of $A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d$ and $BK_0 - (\mathrm{Tr}(BK_0)/d)\mathrm{Id}_d$, denoted by $Q_1(K_0), \dots, Q_{d^2-1}(K_0)$.

For every $K \in M_{m,d}(\mathbb{R})$ and $j = 1, \dots, d^2-1$, denote by $Q_j(K)$ the iterated Lie bracket of $A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d$ and $BK - (\mathrm{Tr}(BK)/d)\mathrm{Id}_d$ obtained by replacing K_0 by K in the Lie bracket expression of $Q_j(K_0)$.

Consider each $Q_j(K)$ as a row vector of $\mathbb{R}^{d^2-1}$ (using, for instance, the representation on the basis $Q_1(K_0), \dots, Q_{d^2-1}(K_0)$) and define $f(K) = \det(Q_1(K), \dots, Q_{d^2-1}(K))$. Since f is an analytic function of the entries of K and $f(K_0) \neq 0$, we deduce that f cannot vanish on any nonempty open subset of $M_{m,d}(\mathbb{R})$. This proves the density of $\mathrm{LARC}_0(A,B)$.

Let us now prove that (iv) implies (vi) when $d \ge 3$. The proof of the proposition is then concluded by noticing that (vi) trivially implies (v), which implies (i) by Lemma 2.9. Let $K \in \mathrm{LARC}_0(A,B)$. Notice in particular that $B \neq 0$. By a perturbative argument, one gets that

$$\mathrm{Lie}(A - (\mathrm{Tr}(A)/d)\mathrm{Id}_d, B\tilde{K} - (\mathrm{Tr}(B\tilde{K})/d)\mathrm{Id}_d) = \mathrm{sl}(d,\mathbb{R}),$$

for every $\tilde{K}$ in a neighborhood of K, and thus we may assume that $\mathrm{Tr}(BK) \neq 0$. We now show that $L_K = \mathrm{Lie}(A, BK) = M_n(\mathbb{R})$. Notice that the map $\Phi : M \mapsto M - (\mathrm{Tr}(M)/d)\mathrm{Id}_d$ is surjective from L_K to $\mathrm{sl}(d,\mathbb{R})$. Indeed, by hypothesis $\mathrm{sl}(d,\mathbb{R}) = \mathrm{Lie}(\Phi(A), \Phi(BK))$ and $\Phi : L_K \to \mathrm{sl}(d,\mathbb{R})$ is a Lie algebra homomorphism. Hence, L_K has codimension at most 1 in $M_d(\mathbb{R})$. Using the fact that $d \ge 3$ and a theorem of Amayo (see [1] and also [15, Theorem 1.1]), one gets that the only Lie subalgebra of $M_d(\mathbb{R})$ of codimension 1 is $\mathrm{sl}(d,\mathbb{R})$. Since $\mathrm{Tr}(BK) \neq 0$ then $L_K \neq \mathrm{sl}(d,\mathbb{R})$ and therefore L_K must be equal to $M_d(\mathbb{R})$. ■

Remark 5.2 The hypothesis $d \ge 3$ is essential in the proof of the implication $(iv) \Rightarrow (vi)$, where the fact that $\mathrm{sl}(d,\mathbb{R})$ is simple for $d \ge 3$ is crucial. Indeed, the latter is not true when $d = 2$ and the implication $(iv) \Rightarrow (vi)$ is not true, as illustrated by Proposition 5.3 below.

In the context of control theory, it seems reasonable to address the issue of a possible relationship between nonemptiness of $\mathrm{LARC}(A,B)$ (or $\mathrm{PLARC}(A,B)$) and controllability of the pair (A,B).

More precisely, is it true that if $\mathrm{PLARC}(A,B)$ is nonempty then (A,B) is controllable? The next proposition shows that the answer is yes, except in trivial situations.

Proposition 5.3 *Let $(A,B) \in P_{d,m}$ be a non-controllable pair such that* $\mathrm{PLARC}(A,B)$ *is nonempty. Then $B = 0$, $d = 2$ and the eigenvalues of A are non-real.*

Proof. Consider a controllability decomposition of the pair (A,B) of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where $(A_1, B_1) \in P_{r,m}$ is controllable with controllability index $r < d$ and $A_3 \in M_{d-r}(\mathbb{R})$.

Let $K \in \mathrm{PLARC}(A,B)$. It is clear that every matrix C in the Lie algebra $\mathrm{Lie}(A, BK)$ is of the form

$$C = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix},$$

with $C_1 \in M_r(\mathbb{R})$, $C_3 \in M_{d-r}(\mathbb{R})$.

If $r > 0$ then the Lie algebra $\mathrm{Lie}(A, BK)$ is reducible, contradicting $K \in \mathrm{PLARC}(A, B)$, as noticed in Remark 2.10. This proves that $r = 0$, hence $B = 0$.

Then the tangent space to any orbit of $\dot{q} = (\Pi A)q$, $q \in \mathbb{RP}^{d-1}$, is of dimension at most one, implying that $d = 2$. Then A does not have an invariant space of dimension one, hence the result.

■

Thanks to the results of the previous sections, we are ready to state the main result of the paper, equality between maximal growth rates for PE systems.

Theorem 5.4 *Let $(A, B) \in P_{d,m}$ and assume that $\mathrm{LARC}_0(A, B)$ is nonempty. Then for persistently excited systems of the form (4) the maximal rate of convergence defined in (14) and the maximal rate of divergence defined in (15) satisfy*

$$\mathrm{RC}(A, B) = \mathrm{RD}(-A, -B).$$

Proof. According to Proposition 5.1, one can assume that $\mathrm{PLARC}(A, B)$ is non empty. By definition of $\mathrm{RC}(A, B)$, for every $\xi < \mathrm{RC}(A, B)$ there exists $K_\xi \in M_{d,m}(\mathbb{R})$ such that $\mathrm{rc}(A, B, K_\xi) > \xi$. According to Proposition 5.1 given below, the nonemptiness of $\mathrm{PLARC}(A, B)$ is equivalent to its density in $M_{m,d}(\mathbb{R})$. As a consequence, there exists a sequence $(K_\xi^n)_{n\in\mathbb{N}}$ in $\mathrm{PLARC}(A, B)$ converging to K_ξ. Theorem 4.2(i)) shows lower semicontinuity of $\mathrm{rc}(A, B, \cdot)$ on $M_{d,m}(\mathbb{R})$ and hence $\mathrm{rc}(A, B, K_\xi^n) > \xi$ for n large enough. Hence

$$\mathrm{RC}(A, B) = \sup_{K\in\mathrm{PLARC}(A,B)} \mathrm{rc}(A, B, K) = \sup_{K\in\mathrm{PLARC}(A,B)} \mathrm{rc}_\#(A, B, K),$$

where the last equality follows from Proposition 3.3. Proposition 3.4 then implies that $\mathrm{RC}(A, B) = \sup_{K\in\mathrm{PLARC}(A,B)} \mathrm{rd}_\#(-A, -B, K)$

Now by the lower semicontinuity of $\mathrm{rd}(A, B, \cdot)$ on $M_{m,d}(\mathbb{R})$ we obtain as claimed

$$\begin{aligned}\sup_{K\in\mathrm{PLARC}(A,B)} \mathrm{rd}_\#(-A, -B, K) &= \sup_{K\in\mathrm{PLARC}(A,B)} \mathrm{rd}(-A, -B, K)\\ &= \sup_{K\in M_{m,d}(\mathbb{R})} \mathrm{rd}(-A, -B, K) = \mathrm{RD}(-A, -B).\end{aligned}$$

■

Remark 5.5 According to Proposition 5.1, if one assumes that $d \geq 3$, then the nonemptiness hypothesis on $\mathrm{LARC}_0(A, B)$ can be weakened to the nonemptiness of $\mathrm{LARC}(A, B)$.

6 The single-input case

In this section we assume $m = 1$ and we use b to denote the $d\times 1$ matrix B. Let, moreover, $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$.

6.1 Conditions for K to belong to $\mathrm{PLARC}(A, b)$

Given a controllable pair $(A, b) \in P_{d,1}$, let $v(A, b)$ and $P(A, b)$ be, respectively, the unique row vector in $M_{1,d}(\mathbb{R})$ and the unique invertible matrix so that $(J_d + e_d v(A, b), e_d)$ is the controllability form of $(A - \mathrm{Tr}(A)\mathrm{Id}, b)$ (cf. [20]) and $P(A, b)$ is the corresponding change of coordinates, i.e.,

$$P(A, b)^{-1}(J_n + e_d v(A, b))P(A, b) = A - \mathrm{Tr}(A)\mathrm{Id}, \qquad P(A, b)^{-1}e_d = b.$$

Note that $v(A,b)e_d = 0$ by construction.

We now provide the main result of the section, which ensures that $\mathrm{PLARC}(A,b)$ is dense if the pair (A,b) is controllable.

Theorem 6.1 *Let $(A,b) \in P_{d,1}$ be a controllable pair. There exists $c > 0$ such that, for every $K \in M_{1,d}(\mathbb{R})$ for which the eigenvalues of $A + bK$ have either all real part smaller than $-c$ or all real part larger than c, it follows that $K \in \mathrm{LARC}(A - \mathrm{Tr}(A)\mathrm{Id}, b)$. In particular, $\mathrm{PLARC}(A,b)$ is dense in $M_{1,d}(\mathbb{R})$.*

Remark 6.2 Theorem 6.1 does not generalize to the multi-input case under the sole condition that (A,B) is controllable. Think for instance of the system $(A,B) = (0_d, \mathrm{Id}_d)$. Then clearly $\mathrm{RC}(A,B) = +\infty$ and $\mathrm{PLARC}(A,B) = \emptyset$.

The proof of Theorem 6.1 is based on the following two technical results.

Proposition 6.3 *Let $(A,b) \in P_{d,1}$ be a controllable pair and $v(A,b)$ and $P(A,b)$ be defined as above. Fix $K \in M_{1,d}(\mathbb{R})$. For $j \geq 0$, set $K_j = K(J_d + e_d v(A,b))^j$, and $r_j = K(J_d + e_d v(A,b))^j e_d$. Then $KP(A,b) \in \mathrm{LARC}(A - \mathrm{Tr}(A)\mathrm{Id}, b)$ if $r_j \neq 0$ for every $j = 0, \dots, d-1$ and $K_0, \dots, K_{d-1}$ are linearly independent. In particular, $\mathrm{LARC}(J_d, e_d)$ contains all line vectors with coefficients in $\mathbb{R} \setminus \{0\}$.*

Proof. In this proof we abbreviate $v := v(A,b)$ and $P := (A,b)$. Since

$$\mathrm{LARC}(J_d + e_d v, e_d) = \mathrm{LARC}(P(A - \mathrm{Tr}(A)\mathrm{Id})P^{-1}, Pb) = \mathrm{LARC}(A - \mathrm{Tr}(A)\mathrm{Id}, b)P^{-1} \tag{37}$$

we can assume without loss of generality that (A,b) is in controllable form and that $\mathrm{Tr}(A) = 0$, i.e., $A = J_d + e_d v$, $b = e_d$, $v\, e_d = 0$.

For $j \geq 0$, define

$$f_{d-j} = (J_d + e_d v)^j e_d.$$

We next prove that the rank-1 matrices $f_j K_l$, $j = 1, \dots, d$, $l = 0, \dots, d-1$, belong to

$$\mathcal{L} := \mathrm{Lie}(J_d + e_d v, e_d K).$$

Notice that for $j, l \geq 0$

$$K_l f_{d-j} = r_{l+j}.$$

Straightforward computations yield for $j \leq d$ and $l \geq 0$

$$\begin{aligned} [J_d + e_d v, f_j K_l] &= f_j K_{l+1} - f_{j-1} K_l, \\ [f_j K_l, [J_d + e_d v, f_j K_l]] &= -2 r_{d+l+1-j} f_j K_l - r_{d+l-j}(f_{j-1} K_l + f_j K_{l+1}). \end{aligned}$$

Hence, if $f_j K_l$ is in $\mathcal{L}$ and r_{d+l-j} is different from zero, then $f_j K_{l+1}$ and $f_{j-1} K_l$ also belong to $\mathcal{L}$.

By a trivial induction one deduces that $f_j K_l \in \mathcal{L}$ for $j \leq d$, $l \geq 0$ and $1 \leq j - l \leq d$, since $e_d K = f_d K_0 \in \mathcal{L}$ and $r_0, \dots, r_{d-1}$ are not zero.

We prove by induction on $m = l - j$ that the following property holds true:

$(\mathcal{P}_m)$ For every $j \leq d$ and $l \geq 0$ such that $j - l \geq m$ we have $f_j K_l \in \mathcal{L}$.

We proved $\mathcal{P}_m$ for m up to -1. Assume that P_m holds true for $m \geq -1$. Notice that for $j \leq d-1$ and $l \leq d-2$ one has

$$[f_j K_l, f_{j+1} K_{l+1}] = r_{d-1-(j-l)} f_j K_{l+1} - r_{d+1-(j-l)} f_{j+1} K_l$$

and $r_{d-1-(j-l)} \neq 0$. If $l - j = m$ then $f_{j+1} K_l \in \mathcal{L}$ and we conclude that also $f_j K_{l+1}$ is in $\mathcal{L}$, i.e., $\mathcal{P}_{m+1}$ holds true.

We have proved, as claimed, that $f_j K_l \in \mathcal{L}$ for $j = 1, \dots, d$ and $l = 0, \dots, d-1$. Since $f_1, \dots, f_d$ and $K_0, \dots, K_{d-1}$ are linearly independent, respectively by construction and by hypothesis, then it follows that $\mathcal{L} = M_d(\mathbb{R})$, concluding the proof of the proposition. ∎

Proposition 6.4 *If all the real parts of the eigenvalues of $J_d + e_d K$ are nonzero and have the same sign, then $|k_{d-m}| \geq c_0 |k_{d-m+1}|(d-m)/(m+1)$ for every $m = 0, \dots, d-1$, with $k_{d+1} := 1$ and $c_0 := \min_{\lambda \in \sigma(J_d + e_d K)} |\Re(\lambda)|$.*

Proof. Denote by $\lambda_1, \dots, \lambda_d$ the eigenvalues of $J_d + e_d K$. Notice that $|k_d| = |\lambda_1 + \cdots + \lambda_d| = |\Re(\lambda_1)| + \cdots + |\Re(\lambda_d)| \geq d\, c_0$.

For $m = 1, \dots, d$,

$$|k_{d-m+1}| = \left| \sum_{j_1 < \cdots < j_m} \lambda_{j_1} \cdots \lambda_{j_m} \right| = \sum_{j_1 < \cdots < j_m} |\Re(\lambda_{j_1} \cdots \lambda_{j_m})|.$$

Since

$$\sum_{j_1 < \cdots < j_m} \left(\lambda_{j_1} \cdots \lambda_{j_m} \sum_{j \neq j_1, \dots, j_m} \lambda_j \right) = (m+1) \sum_{j_1 < \cdots < j_{m+1}} \lambda_{j_1} \cdots \lambda_{j_{m+1}},$$

we have

$$\begin{aligned}
(m+1)|k_{d-m}| &= \left| \sum_{j_1 < \cdots < j_m} \left(\lambda_{j_1} \cdots \lambda_{j_m} \sum_{j \neq j_1, \dots, j_m} \lambda_j \right) \right| \\
&\geq \sum_{j_1 < \cdots < j_m} \sum_{j \neq j_1, \dots, j_m} |\Re(\lambda_{j_1} \cdots \lambda_{j_m})| |\Re(\lambda_j)| \\
&\geq \sum_{j_1 < \cdots < j_m} (n-m) |\Re(\lambda_{j_1} \cdots \lambda_{j_m})| c_0 = (d-m) c_0 |k_{d-m+1}|.
\end{aligned}$$

This concludes the proof. ■

We can now prove Theorem 6.1.

Proof of Theorem 6.1. According to (37), one must prove the assertion for $(A, b) = (J_d + e_d v, e_d)$ with $v e_d = 0$.

Let then $K \in M_{1,d}(\mathbb{R})$ be such that the eigenvalues of $J_d + e_d(v + K)$ have either all real part smaller than $-c$ or all real part larger than c for some $c > 0$ to be chosen later. Applying Proposition 6.4 one gets that for every $m = 1, \dots, d-1$,

$$|k_{d-m} + v_{d-m}| \geq c |k_{d-m+1} + v_{d-m+1}|(d-m)/(m+1),$$

and $|k_d| \geq cd$, where $v = (v_1, \dots, v_{d-1}, 0)$. By taking c large enough with respect to $|v|$ we have

$$|k_{d-m}| \geq \frac{c}{2d} |k_{d-m+1}|, \qquad m = 1, \dots, d-1. \tag{38}$$

In particular, $k_m \neq 0$ for $m = 1, \dots, d$. According to Proposition 6.3 the proof is completed if we show that $r_m = K(J_d + e_d v)^m e_d \neq 0$, $m = 0, \dots, d-1$, and $K_0, \dots, K_{d-1}$ are linearly independent.

By definition of r_m a trivial induction yields

$$r_m = k_{d-m} + \sum_{j < d-m} \alpha_j^m k_j$$

for some constants α_j^m independent of K. Fix $m \in \{0, \dots, d-m\}$. If all the corresponding α_j^m are equal to zero then we are done, otherwise let $\bar{\jmath}$ be the smallest index j so that $\alpha_j^m \neq 0$. Then

$$r_m = \alpha_{\bar{\jmath}}^m k_{\bar{\jmath}} (1 + \xi)$$

with $|\xi| = O(1/c)$ according to (38). The assertion follows from c large enough.

Similarly, one has that $K = k_1(e_1^T + \Xi)$ with $|\Xi| = O(1/c)$. One immediately deduces that the for $m = 0, \ldots, d-1$, $K_m = k_1(e_{m+1}^T + O(1/c))$. Hence, $K_0, \ldots, K_{d-1}$ are linearly independent for c large enough.

The last part of the statement follows from Lemma 2.9 and Proposition 5.1. ■

6.2 Maximal growth rates in the single-input case

All controllable single-input systems have the same controllability form. Hence $\mathrm{RC}(A, b) = \mathrm{RC}(A, b')$ for (A, b) and (A, b') controllable (see Remark 2.5). In particular we can define $\mathrm{RC}(A)$ as the value $\mathrm{RC}(A, b)$ corresponding to the case where (A, b) is controllable, and similarly for $\mathrm{RC}_\#(A)$, $\mathrm{RD}_\#(A)$, $\mathrm{RD}(A)$.

Theorem 6.5 *Let A be such that there exists b with (A, b) controllable. Then*

$$\mathrm{RC}(A) = \mathrm{RD}(-A), \tag{39}$$

and in particular $\mathrm{RC}(A) = +\infty$ *if and only if* $\mathrm{RD}(-A) = +\infty$. *Moreover, there exists $c > 0$ depending on A but not on T, μ such that if* $\mathrm{RC}(A) > c$ *or* $\mathrm{RD}(A) > c$ *then* $\mathrm{RC}(A) = \mathrm{RC}_\#(A) = \mathrm{RD}_\#(-A) = \mathrm{RD}(-A)$.

Proof. According to Theorem 6.1, $\mathrm{PLARC}(A, b)$ is dense. Theorem 5.4 then implies (39).

Let now $c > 0$ be as in the statement of Theorem 6.1. Assume that $\mathrm{RC}(A) > c$ (the case $\mathrm{RD}(A) > c$ being entirely similar). Hence the set $\Xi_c = \{K \in M_{1,d}(\mathbb{R}) \mid \mathrm{rc}(A, B, K) > c\}$ is nonempty. By taking $\bar{\alpha} = 1$ in (7), condition $\mathrm{rc}(A, b, K) > c$ implies that the eigenvalues of $A + bK$ have real part smaller than $-c$. Hence, Theorem 6.1 implies that $\Xi_c \subset \mathrm{PLARC}(A, b)$. By equation (35) one deduces that $\mathrm{RC}(A), \mathrm{RC}_\#(A), \mathrm{RD}_\#(-A), \mathrm{RD}(-A)$ are all equal. ■

Combining the above result with equation (18) one gets the following corollary.

Corollary 6.6 *Let A be such that there exists b with (A, b) controllable and $-A + \frac{2}{d}\mathrm{Tr}(A)\mathrm{Id}_d$ is similar to A. Then*

$$\mathrm{RC}(A) = \mathrm{RD}(A) - \frac{2}{d}\mathrm{Tr}(A),$$

and in particular $\mathrm{RC}(A) = +\infty$ *if and only if* $\mathrm{RD}(A) = +\infty$.

Remark 6.7 When restricted to the case $d = 2$, Corollary 6.6 implies (and actually improves) Proposition 2.8, established in [10], because every traceless 2×2 matrix A is similar to its opposite $-A$.

Remark 6.8 Notice that a matrix A diagonalizable over $\mathbb{C}$ is similar to $-A$ if and only if $\mathrm{Tr}(A^k) = 0$ for every odd integer k. In particular, every skew-symmetric matrix A for which (A, b) is controllable for some b verifies the hypotheses of Corollary 6.6 and we conclude that $\mathrm{RC}(A) = \mathrm{RD}(A)$ for such matrices. The same is true for J_d, since every nilpotent matrix is similar to its opposite, and even more can be established, as stated below.

Proposition 6.9 *Let $\mathcal{T}$ be equal to the diagonal matrix* $\mathrm{diag}(1, -1, \ldots, (-1)^{d+1})$. *For every $K \in M_{1,d}(\mathbb{R})$ set $K_- = (-1)^d K\mathcal{T}$. Then* $\mathrm{rc}(J_d, e_d, K) = \mathrm{rd}(J_d, e_d, K_-)$ *for every $K \in M_{1,d}(\mathbb{R})$ with components in $\mathbb{R} \setminus \{0\}$.*

Proof. Notice that

$$\mathcal{T}^{-1} = \mathcal{T}, \quad J_d = -\mathcal{T} J_d \mathcal{T}, \quad \mathcal{T} e_d = (-1)^{d+1} e_d.$$

According to (9), with $P = \mathcal{T}$, one gets

$$\mathrm{rc}(J_d, e_d, K) = \mathrm{rc}(\mathcal{T} J_d \mathcal{T}^{-1}, \mathcal{T} e_n, K\mathcal{T}^{-1}) = \mathrm{rc}(-J_d, (-1)^{d+1} e_d, (-1)^d K_-).$$

Proposition 6.3 guarantees that $K_- \in \mathrm{PLARC}(J_d, e_d)$ and then Corollary 3.6 implies that

$$\mathrm{rc}(-J_d, (-1)^{d+1} e_d, (-1)^d K_-) = \mathrm{rd}(J_d, (-1)^d e_d, (-1)^d K_-).$$

Since $\mathrm{rd}(A, b, L) = \mathrm{rd}(A, \xi b, \xi^{-1} L)$ for every controllable pair (A, b) and every $\xi \neq 0$, we conclude. ■

7 Appendix

In this appendix, we prove the following lemma which is used in the proof of Proposition 5.1.

Lemma 7.1 *The eigenvalues of any element of spin$(9,1)$ are symmetric with respect to the origin, i.e., if λ is an eigenvalue then $-\lambda$ is as well with the same algebraic multiplicity as λ.*

Proof. Proving the result amounts showing that, for every $M \in \mathrm{spin}(9,1)$, the characteristic polynomial $P_M(X)$ of M can be written as $P_M(X) = Q(X^2)$ where Q is a unitary polynomial of degree five.

Recall that $\mathrm{spin}(9,1) = \{M \in M_{10}(\mathbb{R}) \mid M^T \mathrm{Id}_{(9,1)} + \mathrm{Id}_{(9,1)} M = 0\}$, where $\mathrm{Id}_{(9,1)} = \mathrm{diag}(\mathrm{Id}_9, -1)$. Rewrite $\mathrm{Id}_{(9,1)} = \mathrm{Id}_{10} - 2e_{10}e_{10}^T$ with $e_{10} = (0 \ \cdots \ 0\ 1)^T \in \mathbb{R}^{10}$. Let $M \in \mathrm{spin}(9,1)$ and set $v = M^T e_{10}$. Then, one immediately deduces that $\frac{M+M^T}{2} = v e_{10}^T + e_{10} v^T$ and then there exists a skew-symmetric matrix $A \in M_{10}(\mathbb{R})$ such that

$$M = A + v e_{10}^T + e_{10} v^T.$$

By using the definition of v, one deduces from the above equality that $Ae_{10} = (v^T e_{10}) e_{10}$. Since A is skew-symmetric, one gets that $Ae_{10} = 0$ and $v^T e_{10} = 0$. Therefore, there exists a skew-symmetric matrix $A_1 \in M_9(\mathbb{R})$ and $v_1 \in \mathbb{R}^9$ such that

$$M = \begin{pmatrix} A_1 & v_1 \\ v_1^T & 0 \end{pmatrix}.$$

Set $\alpha = \|v_1\| \geq 0$. Up to similarity with a matrix $U = \begin{pmatrix} U_1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $U_1 \in SO(9)$ and $U_1^T v_1 = \alpha f_1$ with $f_1 = (1\ 0\ \cdots\ 0)^T \in \mathbb{R}^9$, one can assume with no loss of generality that $v_1 = \alpha f_1$. If A_2 is the 8×8 skew-symmetric obtained from A_1 by removing the first line and the first column, one obtains after computations, that

$$P_M(X) = X P_{A_1}(X) + \alpha^2 P_{A_2}(X).$$

Since A_1 and A_2 are skew-symmetric, one has that $P_{A_1}(X)$ and $P_{A_2}(X)$ are equal to $XQ_1(X^2)$ and $Q_2(X^2)$ respectively, where Q_1 and Q_2 are unitary polynomials of degree four and one concludes by setting $Q(X) = XQ_1(X) + \alpha^2 Q_2(X)$. ■

References

[1] R. K. Amayo. Quasi-ideals of Lie algebras. II. *Proc. London Math. Soc. (3)*, 33(1):37–64, 1976.

[2] B. Anderson. Exponential stability of linear equations arising in adaptive identification. *IEEE Trans. Automat. Control*, 22(1):83–88, 1977.

[3] B. Anderson, R. Bitmead, C. Johnson, P. Kokotovic, R. Kosut, I. Mareels, L. Praly, and B. Riedle. *Stability of adaptive systems: Passivity and averaging analysis*. MIT Press Series in Signal Processing, Optimization, and Control, 8. MIT Press, Cambridge, MA, 1986.

[4] B. D. O. Anderson, A. Ilchmann, and F. R. Wirth. Stabilizability of linear time-varying systems. *Systems and Control Letters*, 62(9):747–755, 2013.

[5] S. Andersson and P. Krishnaprasad. Degenerate gradient flows: a comparison study of convergence rate estimates. In *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, volume 4, pages 4712–4717. IEEE, 2002.

[6] R. Brockett. The rate of descent for degenerate gradient flows. In *Proceedings of the 2000 MTNS*, 2000.

[7] L. Cesari. *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*. Springer-Verlag, 1971.

[8] A. Chaillet, Y. Chitour, A. Loría, and M. Sigalotti. Uniform stabilization for linear systems with persistency of excitation: the neutrally stable and the double integrator cases. *Math. Control Signals Systems*, 20(2):135–156, 2008.

[9] Y. Chitour, G. Mazanti, and M. Sigalotti. Stabilization of persistently excited linear systems. In *Hybrid Systems with Constraints*, Automation – Control and industrial engineering series, pages 85–120. Wiley-ISTE, London, UK, 2013.

[10] Y. Chitour and M. Sigalotti. On the stabilization of persistently excited linear systems. *SIAM J. Control Optim.*, 48(6):4032–4055, 2010.

[11] F. Colonius and W. Kliemann. Minimal and maximal Lyapunov exponents of bilinear control systems. *J. Differential Equations*, 101(2):232–275, 1993.

[12] F. Colonius and W. Kliemann. *The Dynamics of Control*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 2000.

[13] G. Dirr and U. Helmke. Accessibility of a class of generalized double-bracket flows. *Commun. Inf. Syst.*, 8(2):127–145, 2008.

[14] V. Jurdjevic. *Geometric control theory*, volume 52 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[15] E. V. Kissin. On normed Lie algebras with sufficiently many subalgebras of codimension 1. *Proc. Edinburgh Math. Soc. (2)*, 29(2):199–220, 1986.

[16] A. Loria, A. Chaillet, G. Besançon, and Y. Chitour. On the PE stabilization of time-varying systems: open questions and preliminary answers. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*, pages 6847–6852. IEEE, 2005.

[17] M. Lovera and A. Astolfi. Global spacecraft attitude control using magnetic actuators. In *Advances in dynamics and control*, volume 2 of *Nonlinear Syst. Aviat. Aerosp. Aeronaut. Astronaut.*, pages 1–13. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[18] G. Mazanti, Y. Chitour, and M. Sigalotti. Stabilization of two-dimensional persistently excited linear control systems with arbitrary rate of convergence. *SIAM J. Control Optim.*, 51(2):801–823, 2013.

[19] A. P. Morgan and K. S. Narendra. On the stability of nonautonomous differential equations $\dot{x} = [A + B(t)]x$ with skew-symmetric matrix $B(t)$. *SIAM J. Control Optim.*, 15(1):163–176, 1977.

[20] E. D. Sontag. *Mathematical Control Theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998.

[21] H. Völklein. Transitivitätsfragen bei linearen Liegruppen. *Arch. Math. (Basel)*, 36(1):23–34, 1981.

[22] F. Wirth. A converse Lyapunov theorem for linear parameter-varying and linear switching systems. *SIAM J. Control Optim.*, 44(1):210–239, 2005.