Imperial College, London
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Control Theory and Dynamical Systems

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Introduction

The goal of control theory (or mathematical systems theory) is to influence dynamical systems such that they achieve a desired behavior, e.g. stability. This requires the analysis of the system behavior under admissible control actions and, possibly, under perturbations.

The main mathematical areas involved are
- ordinary/partial differential equations (possibly including algebraic constraints), discrete-time systems
- random systems
- optimization/optimal control
- numerics
- basic mathematics like linear algebra, complex and functional analysis, differential geometry, ...

Application areas

J. C. Maxwell, On governors, Proc. Royal Soc. (1868).

The origins and the main applications are in engineering, in particular,

- mechanics,
- electromagnetic and electrical systems
- heat transfer

Many other applications, e.g., in economics, biology, ...

Recent trend:

- (digitally) networked control systems

Here: deterministic finite dimensional systems

We consider

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)),\ u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in \Omega \text{ for } t \in \mathbb{R}\} \\ y(t) &= h(x(t), u(t))\end{aligned}$$

with control range $\Omega \subset \mathbb{R}^m$ and

$$\begin{aligned}x_{n+1} &= f(x_n, u_n),\ u = (u_n)_{n\in\mathbb{Z}} \in \{u : \mathbb{Z} \to \mathbb{R}^m \,|u_n \in \Omega \text{ for } n \in \mathbb{Z}\} \\ y_n &= h(x_n, u_n).\end{aligned}$$

Here $u(\cdot)$ is the input or control and $y(\cdot)$ is the output or observation. We assume that for every initial state $x(0) = x_0$ and every control function u there is a unique solution $\varphi(t, x_0, u), t \in \mathbb{R}$ ($\varphi(n, x_0, u), n \in \mathbb{Z}$, resp.).

Instability of the geostationary orbit of a satellite

Circular orbit at fixed altitude at 35600 km in the equator plane with the same velocity of rotation as the earth.
Influence the motion such that it returns to this orbit when it is slightly off.
Model (Trentelman et al.):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ [x_1(t)+R_0][x_4(t)+\omega]^2 - \frac{GM_E}{[x_1(t)+R_0]^2} + \frac{F_r(t)}{M_S} \\ x_4(t) \\ -\frac{2x_2(t)[x_4(t)+\omega]}{x_1(t)+R_0} + \frac{F_\theta(t)}{M_S[x_1(t)+R_0]} \end{bmatrix}$$

where R_0 radius of geostationary orbit,
x_1 deviation from R_0 and x_3 deviation from angle ω of earth rotation,
M_S, M_e mass of the satellite, earth
G earth's gravitational constant
$F_r(t), F_\theta(t)$ radial, angular force: **controls!** $y = x_3$ **Output.**
Equilibrium: $x_1 = x_2 = x_3 = x_4, F_r = 0, F_\theta = 0$, since $R_0\omega^2 - \frac{GM_E}{R_0^2} = 0$.

The stabilization problem

For initial states

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (x_1^0, x_2^0, x_3^0, x_4^0),$$

Kepler's law implies that the resulting orbit will be an ellipse with the earth in one of its focuses. The equilibrium position will not be asymptotically stable.

Problem: Find a feedback controller that generates a control input $u = (F_r, F_\theta)$ on the basis of the measured output $y = x_3$ such that the equilibrium position becomes locally asymptotically stable.

Linear control systems

Linear control systems have the form

$$\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t),$$

where A, B, C and D are matrices of appropriate dimension.
For the satellite model, linearize in the equilibrium. With $f : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4, g : \mathbb{R}^4 \to \mathbb{R}$, near $(0,0) \in \mathbb{R}^4 \times \mathbb{R}^2$

$$f(x,u) \approx f(0,0) + D_x f(0,0)x + D_u f(0,0)u.$$

Here $f(0,0) = 0$ and

$$D_x f(0,0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega R_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{R_0} & 0 & 0 \end{bmatrix}, D_u f(0,0) = \begin{bmatrix} 0 & 0 \\ \frac{1}{M_S} & 0 \\ 0 & 0 \\ 0 & \frac{1}{M_S R_0} \end{bmatrix}.$$

We obtain a linear control system with

$$A = D_x f(0,0), B = D_u f(0,0), C = [0,0,1,0], D = 0.$$

Stabilization

Local stabilization follows if the linearized system is stabilized.

For the linear system, use
- linear state feedback $u = Fx$, or
- a linear feedback controller

$$\begin{aligned} \dot{w}(t) &= Kw(t) + Ly(t) \\ u(t) &= Mw(t) + Ny(t) \end{aligned}$$

with appropriate matrices F, and K, L, M, N, resp.

Control-theoretic problems

The most fundamental dichotomy in problems and concepts concerns open-loop and closed-loop problems.

Open-loop problems are
- controllability
- observability
- optimal control

Closed-loop problems are
- stabilization
- tracking
- keeping the system in a “safe region”
- robustness against perturbations

2. Controllability and observability

Given a control system

$$\dot{x} = f(x, u),$$

is it possible to reach from any given initial state x_0 any final state x_1 using admissible control functions?
In particular, analyze this for linear control systems without and with control constraints.
For nonlinear systems, determine regions of complete controllability (control sets).

Given a measured output $y(t), t \in [0, T]$, determine the initial state $x(0)$ and the present state $x(t)$. When is this always possible (observability)?

3. Stabilization

If the system should operate in a steady state, it should return to it under (small) deviations from the steady state. This can be based on knowledge of the state or, under observability, on measured outputs.

The linear theory is clear cut. In particular, linear-quadratic optimal control yields an efficient method to construct stabilizing controls.

For nonlinear problems there is a diversity of concepts, some of which will be discussed.

4. Control sets, the control flow and relations to random systems

Open-loop control systems can be viewed as skew product dynamical systems, if one includes the control functions (together with the time shift) into the state.
- topologically transitive sets $\leftrightarrow$ control sets
- chain transitivity $\leftrightarrow$ chain control sets

The term $u(t)$ can also be interpreted as a deterministic perturbation. Additionally, one may replace $u(t)$ by a stochastic perturbation. This leads to connections between control systems and degenerate Markov diffusions and piecewise deterministic Markov processes, resp.

5. Bilinear control systems

This are systems of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t), \ (u_i(t))_{i=1,\ldots,m} \in \Omega \subset \mathbb{R}^m,$$

with matrices $A_0, A_1, \ldots, A_m$ (obtained by linearization w.r.t. x only).

The associated control flow is a linear skew product flow with chain transitive base space.

Selgrade's theorem together with the Morse spectrum yields a spectral decomposition which gives information on stability and stabilization.

6. Invariance entropy

The classical feedback concept supposes that the values $x(t) \in \mathbb{R}^n$ are available for the controller. If this information is sent over a digital channel, e.g. over the internet, this concept is no longer available: The communication may be constrained by finite bit rates, loss of information packages, delays etc.

I will concentrate on the problem of finite bit rates, where the concept of invariance entropy has been developed in order to determine the information that is actually necessary for control tasks, here, making a subset of the state space invariant. This has close connections to topological and measure theoretic entropy of dynamical systems.

In particular, a data rate theorem will be presented.

Some basic references for mathematical control theory

E. Sontag, Mathematical Control Theory, Springer 1998.
H.L. Trentelman, A.A. Stoorvogel, M. Hautus, Control Theory for Linear Systems, 2002.
J. M. Coron, Control and Nonlinearity, Amer. Math. Soc. 2007.
V. Jurdjevic, Geometric Control Theory, Cambridge University Press, 1997.
D. Hinrichsen, A.J. Pritchard, Mathematical Systems Theory I, Springer 2005.

For invariance entropy:
Ch. Kawan, Invariance Entropy for Deterministic Control Systems, Springer LNM Vol. 2089, 2013.

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Controllability and Observability

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Introduction

We will consider control systems in continuous or discrete time on a state space M given by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in \Omega \text{ for } t \in \mathbb{R}\} \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

or

$$\begin{aligned} x_{k+1} &= f(x_k, u_k), u \in \mathcal{U} = \{u : \mathbb{Z} \to \mathbb{R}^m \,|u_k \in \Omega \text{ for } k \in \mathbb{Z}\} \\ y_k &= h(x_k, u_k). \end{aligned}$$

We assume that for very initial state $x(0) = x_0 \in M$ and every control function u there is a unique solution $\varphi(t, x_0, u)$, $t \in \mathbb{R}$.
The reachable set from x_0 is

$$\mathcal{R}(x_0) = \{\varphi(t, x_0, u) \,|t > 0, u \in \mathcal{U}\}$$

and the controllable set for x_0 is

$$\mathcal{C}(x_0) = \{x \,|x_0 = \varphi(t, x, u), t > 0, u \in \mathcal{U}\}$$

Controls versus perturbations

The term $u(\cdot)$ in the system equations above may be interpreted as a control function, which may be chosen in order to achieve a desired behavior of the system.

A "dual" interpretation is obtained by considering $u(\cdot)$ as a **time dependent (deterministic) perturbation** acting on the system. Then a typical question is:
What is the "worst" behavior that may occur under such perturbations?

Later we will also see that there are relations to stochastic perturbations acting on the system.

Kalman Criterion

The system is called controllable if $\mathcal{R}(x_0) = M$ for all $x_0 \in M$
$(\Leftrightarrow \mathcal{C}(x_0) = M$ for all $x_0 \in M)$

Theorem (Kalman). A linear control system without control constraints

$$\dot{x} = Ax + Bu \qquad \text{where } A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times m}$$

is controllable in $\mathbb{R}^d$ iff $\mathrm{rank}[B, AB, \ldots, A^{d-1}B] = d$.
Then we also say that (A, B) is controllable.

Example (the linear oscillator is controllable)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - kx_2 + u \end{aligned}$$

Here $d = 2$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $[B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & -k \end{bmatrix}$.

For a subspace V the smallest A-invariant subspace $\supset V$ is given by

$$\langle A\,|V\,\rangle = V + AV + ... + A^{d-1}V.$$

The matrix

$$W_t := \int_0^t e^{As}BB^\top e^{A^\top s}ds$$

is positive semi-definite and for every $t > 0$

$$\langle A\,|\mathrm{im}B\,\rangle = \mathrm{im}W_t.$$

This follows (for the orthogonal complements) using the series expansion of e^{At}.

Furthermore, the reachable subspace from 0 satisfies

$$\mathcal{R}(0) = \langle A\,|\mathrm{im}B\,\rangle = \mathrm{im}[B\ AB\ \ldots\ A^{d-1}B].$$

Here "$\supset$" follows by choosing for $\langle A\,|\mathrm{im}B\,\rangle \ni x = W_t z$

$$u(\tau) = B^\top e^{A^\top \tau}z, \tau \in [0,t].$$

Example. $A = \begin{bmatrix} -2 & -6 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $[B, AB] = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ has rank 1.

Not controllable!

Observability

A control system is observable if for all $u \in \mathcal{U}$

$$h(\varphi(t, x_1, u)) = h(\varphi(t, x_2, u)) \text{ for } t \geq 0 \implies x_1 = x_2.$$

Theorem. A linear control system

$$\dot{x} = Ax + Bu, \; y = Cx,$$

is observable (on arbitrarily short time intervals) iff

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \dots \\ CA^{d-1} \end{bmatrix} = d.$$

Proof. By linearity, the control does not play a role. Hence observability means

$$Ce^{At}x_1 = Ce^{At}x_2, t \geq 0.$$

Local controllability by linearization

Let x^* be an equilibrium for a control system

$$\dot{x}(t) = f(x(t), u(t))$$

for a control $u^* \in \mathbb{R}^m$, i.e., $0 = f(x^*, u^*)$.
If f is C^1 and the linearized (autonomous) control system

$$\dot{y}(t) = \frac{\partial f(x^*, u^*)}{\partial x} y(t) + \frac{\partial f(x^*, u^*)}{\partial u} v(t)$$

is controllable, then the nonlinear control system is locally controllable near x^*.

Proof: Use the implicit function theorem.

A controllability result for nonlinear systems

Consider a control system

$$\dot{x} = f(x, u) \text{ with } u(t) = (u_i(t))_{i=1,\ldots,m} \in \Omega$$

with smooth vector fields $f(\cdot, u), u \in \Omega \subset \mathbb{R}^m$.
The Lie bracket of vector fields is the vector field

$$[g, h](x) = \frac{\partial h(x)}{\partial x} g(x) - \frac{\partial g(x)}{\partial x} h(x).$$

The Lie algebra generated by $F = \{f(\cdot, u) \,|\, u \in \Omega\}$ is the smallest vector space containing F closed under Lie brackets denoted by

$$\mathcal{LA}(F).$$

A controllability result for nonlinear systems

Theorem. Consider

$$\dot{x} = f(x,u), u \in \Omega,$$

where Ω is a neighborhood of $0 \in \mathbb{R}^m$. Let $F = \{f(\cdot,u) \,|u \in \Omega\}$. Assume that for some $x \in \mathbb{R}^d$

$$\{h(x) \,|h \in \mathcal{LA}(F)\} = \mathbb{R}^d.$$

Then the system is **locally accessible**, i.e. for every $T > 0$

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \varnothing \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \varnothing.$$

An Example

$$\dot{x}_1 = u(t),\ \dot{x}_2 = (x_1)^2 \text{ in } \mathbb{R}^2.$$

Then $\mathcal{R}(0,0) = \mathbb{R} \times \mathbb{R}_+$.
"$\subset$„ is clear. For "$\supset$„ let $u(t) \equiv \varepsilon^{-1}$ on $[0, t_0)$ and $u(t) \equiv 0$ for $t \geq t_0$,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t/\varepsilon \\ t^3/\left(3\varepsilon^2\right) \end{bmatrix}, t \in [0, t_0] \text{ and } \dot{x}_1 = 0 \text{ for } t > t_0.$$

Here $f_0 = \begin{bmatrix} 0 \\ (x_1)^2 \end{bmatrix}, f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f_{0*} = \frac{\partial f_0}{\partial x} = \begin{bmatrix} 0 & 0 \\ 2x_1 & 0 \end{bmatrix}, f_{1*} = 0.$

Thus

$$[f_0, f_1] = \begin{bmatrix} 0 \\ -2x_1 \end{bmatrix}, [[f_0, f_1], f_1] = -[f_0, f_1]_* f_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Hence $\{h(0)\,|\,h \in \mathcal{LA}(f_0, f_1)\} = \mathbb{R}^2$.
Observe that linearization in $(0,0)$ yields the non-controllable system

$$A = \frac{\partial f(0,0)}{\partial x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \frac{\partial f(0,0)}{\partial u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Sketch of the proof I

It is convenient to denote by $e^{th}x_0$ the solution of

$$\dot{x} = h(x), x(0) = x_0.$$

The condition

$$\{h(x) \,|h \in \mathcal{LA}(F)\} = \mathbb{R}^d.$$

implies that there are vector fields $h_1, \ldots, h_d \in F$ and a neighborhood of $0 \in \mathbb{R}^d$ such that for the map

$$\Phi_{h_0,\ldots,h_d} : \mathbb{R}^d \ni (t_1, \ldots, t_d) \mapsto e^{t_d h_d} \cdots e^{t_1 h_1} x$$

the Jacobi matrix satisfies: For every $\varepsilon > 0$ there are $t^0 = (t_1^0, \ldots, t_d^0)$ with $0 < t_i^0 < \varepsilon$ such that

$$\mathrm{rank}\,(\Phi_{h_0,\ldots,h_d})_*\,(t^0) = d.$$

This follows since for vector fields $h_0, \ldots, h_k$ and a slice $\Phi_{h_0,\ldots,h_k} : W \to \mathbb{R}^d$, $\mathrm{rank}\,(\Phi_{h_0,\ldots,h_k})_*\,(t^0) = k$, it follows that all vector fields in $\mathcal{LA}(h_0, \ldots, h_k)$ are tangential to this slice.

Sketch of the proof II

Let $h_i = f(\cdot, u_i) \in F$ with

$$\operatorname{rank}\left(\Phi_{h_0,\dots,h_d}\right)_*\left(t^0\right) = d.$$

Then one defines a control u by $u(t) = u_1$ on $[0, t_1], u(t) = u_2$ on $(t_1, t_1 + t_2]$ etc.

The inverse function theorem guarantees that the reachable set (by varying the t_i) has nonvoid interior.

Driftless control systems

A control-affine system is called driftless if it has the form

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \; u(t) = (u_i(t))_i \in \Omega \subset \mathbb{R}^m.$$

Corollary. A driftless control system with control range Ω being a neighborhood of 0 satisfies $\mathcal{R}(x) = \mathbb{R}^d$ for all $x \in \mathbb{R}^d$ if

$$\{ h(x) \,|\, h \in \mathcal{LA}(f_1, \ldots, f_m) \} = \mathbb{R}^d \text{ for all } x \in \mathbb{R}^d.$$

The **proof** uses that

$$\mathcal{LA}(f_1, \ldots, f_m) = \mathcal{LA}\left(\sum_{i=1}^{m} u_i f_i(\cdot) \,|\, (u_i) \in \Omega \right)$$

and that one can go forward “and backward” in time.

Controllability with control constraints

Example

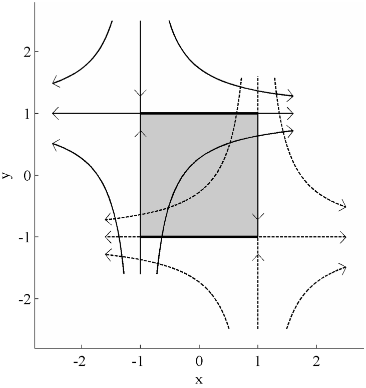
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \;\; u(t) \in \Omega = [-1, 1].$$

The phase portrait for $u \equiv 1$ is obtained from the one for $u \equiv 0$ by

$$\tilde{x} = x + 1, \tilde{y} = y - 1.$$

Hence the equilibrium for $u \equiv 1$ is $(-1, 1)$.
Then a maximal set of approximate controllability is

$$D = (-1, 1) \times [-1, 1].$$



Definition. A set $\varnothing \neq D \subset \mathbb{R}^d$ is a **control set** if it is maximal with
(i) for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$;
(ii) for all $x \in D$ one has $D \subset \overline{\mathcal{R}(x)}$.
It is called an invariant control set if for all $x \in D$ one has $\overline{D} = \overline{\mathcal{R}(x)}$.

Note. Local accessibiliy implies $\text{int}D \subset \mathcal{R}(x)$ for all $x \in D$.

The linear case

Theorem. Consider $\dot{x} = Ax + Bu,\ u(t) \in \Omega$, with (A, B) controllable and Ω a compact neighborhood of $0 \in \mathbb{R}^m$.
(i) There is a unique control set D with $\mathrm{int} D \neq \varnothing$, D is convex and $0 \in \mathrm{int} D$.
(ii) D is closed iff $\mathcal{R}(x) \subset D$ for all $x \in D$ iff D is an invariant control set.
(iii) D is bounded iff A is hyperbolic (i.e., $\mathrm{spec}(A) \cap \imath\mathbb{R} = \varnothing$) and $D = \mathbb{R}^d$ iff $\mathrm{spec}(A) \subset \imath\mathbb{R}$.

- Convexity of D follows by Lyapunov's convexity theorem.
- Null controllability holds iff $\mathrm{spec}(A) \subset \overline{\mathbb{C}_-}$.
- (iii) shows that complete controllability with bounded controls is exceptional.

Existence of invariant control sets

Proposition. Let M be a compact positively invariant set (i.e. $\varphi(t, x, u) \in M$ for all $x \in M, u \in \mathcal{U}$).
Then M contains an invariant control set.
If every point in M is locally accessible, then the number of invariant control sets in M is finite and each of them has nonvoid interior.

Proof: Existence follows using Zorn's lemma.

Note that the number of control sets in M with nonvoid interior may be infinite.

If $\mathrm{int} D \neq \varnothing$, then

$$D = \overline{\mathcal{R}(x)} \cap \mathcal{C}(x) \text{ for every } x \in \mathrm{int} D.$$

Continuous stirred tank reactor

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1 - x_2)e^{x_1} \\ -x_2 + \alpha(1 - x_2)e^{x_1} \end{bmatrix} + u(t)\begin{bmatrix} x_c - x_1 \\ 0 \end{bmatrix},$$

where x_1 is the temperature and x_2 is the product concentration, x_c is the coolant temperature and the control affects the heat transfer coefficient with parameters

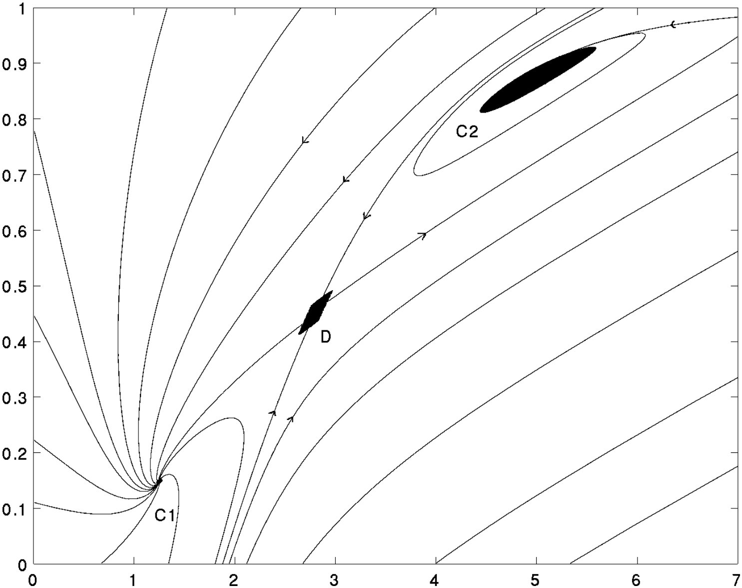
$$a = 0.95,\ \alpha = 0.05,\ B = 10.0,\ c_c = 1.0$$

and control range

$$\Omega = [-0.15, 0.15].$$

The uncontrolled system has an unstable fixed point at

$$(x_1^*, x_2^*) \sim (2.8, 0.45) \in D$$


C2
D
C1
0
0.1
0.2
0.3
0.4
0.5
0.6
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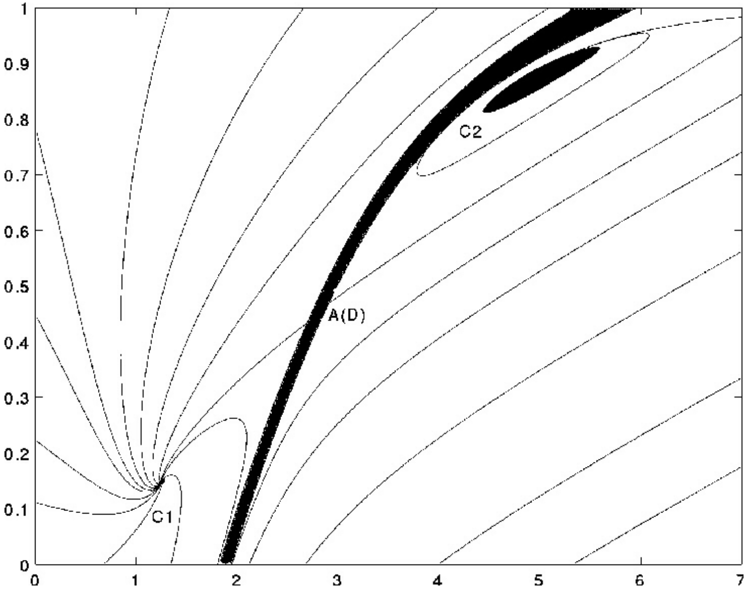
3
4
5
6
7


1
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0.4
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0
0
1
2
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4
5
6
7
C1
C2
A(D)

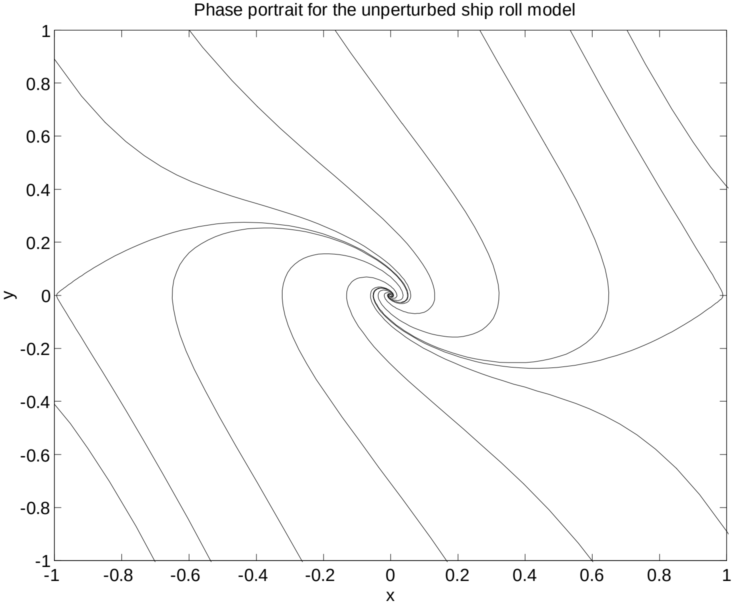
Ship roll motion

Thompson et al.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \alpha x^3 - \delta_1 y - \delta_2 y\,|y| + u(t) \text{ with } u(t) \in [-\rho, \rho]. \end{aligned}$$

nonlinear oscillator with linear ($\delta_1 > 0$) and quadratic ($\delta_2 > 0$) viscous damping
capsizing occurs for $|x| \geq 1/\sqrt{\alpha}$
choose $\alpha = 1.0, \delta_1 = \delta_2 = 1.0$
For $u \in (-\frac{2}{3}, \frac{2}{3})$ there are three fixed points, one is stable and two are unstable.

We consider the controlled (perturbed!) dynamics with control range $U = [-\rho, \rho]$.
Note the behavior of the control sets for $\rho > 0.3849 < \frac{2}{3}$.


Phase portrait for the unperturbed ship roll model
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
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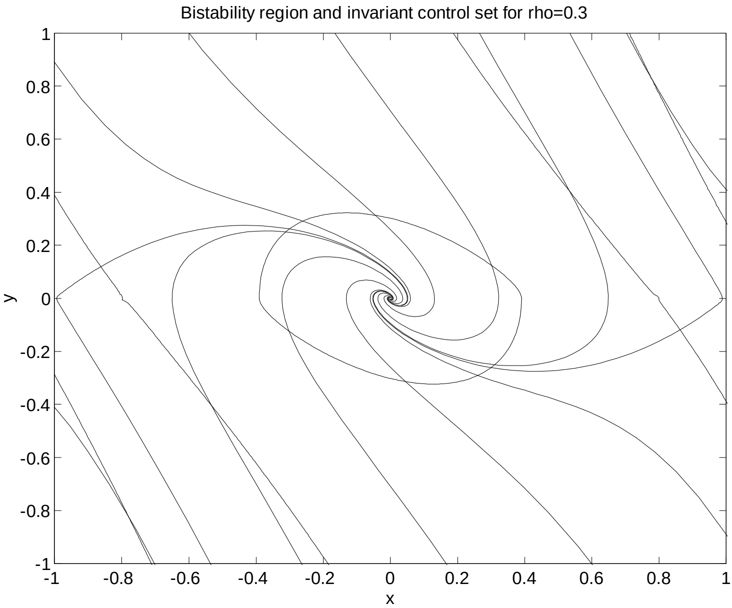
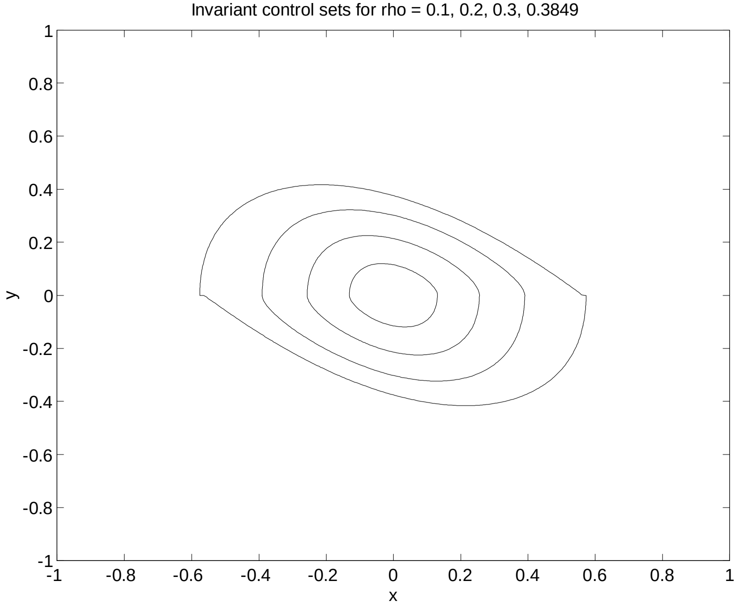
0
0.2
0.4
0.6
0.8
1



Bistability region and invariant control set for rho=0.3
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



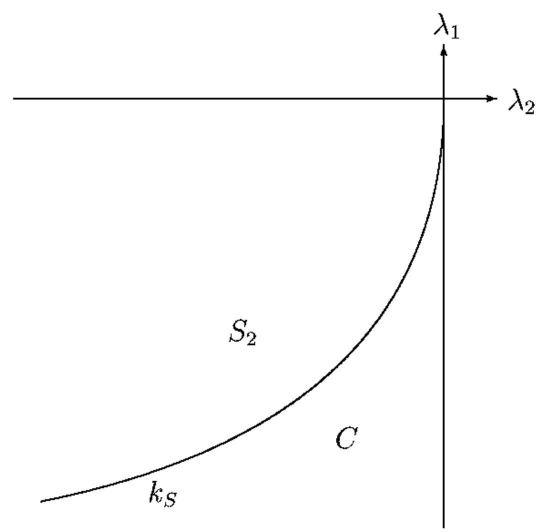
Invariant control sets for rho = 0.1, 0.2, 0.3, 0.3849
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Takens-Bogdanov System

$$\dot{x} = y$$
$$\dot{y} = \lambda_1 + \lambda_2 x + x^2 + xy$$

Universal unfolding of a singularity with zero as eigenvalue that is geometrically simple and algebraically double
A model for
- motion of a thin panel in a flow field (Holmes),
- nonlinear behavior of solar gravity modes (Merryfield et al.),
- shock wave phenomena (Keyfitz),
- competing species in population dynamics (Burchard)
Depending on λ_1, λ_2 there are two fixed points: a saddle and a stable focus (S_2), or a stable focus with homoclinic orbit (k_S), or two fixed points with unstable limit cycle (C).

bifurcation diagram

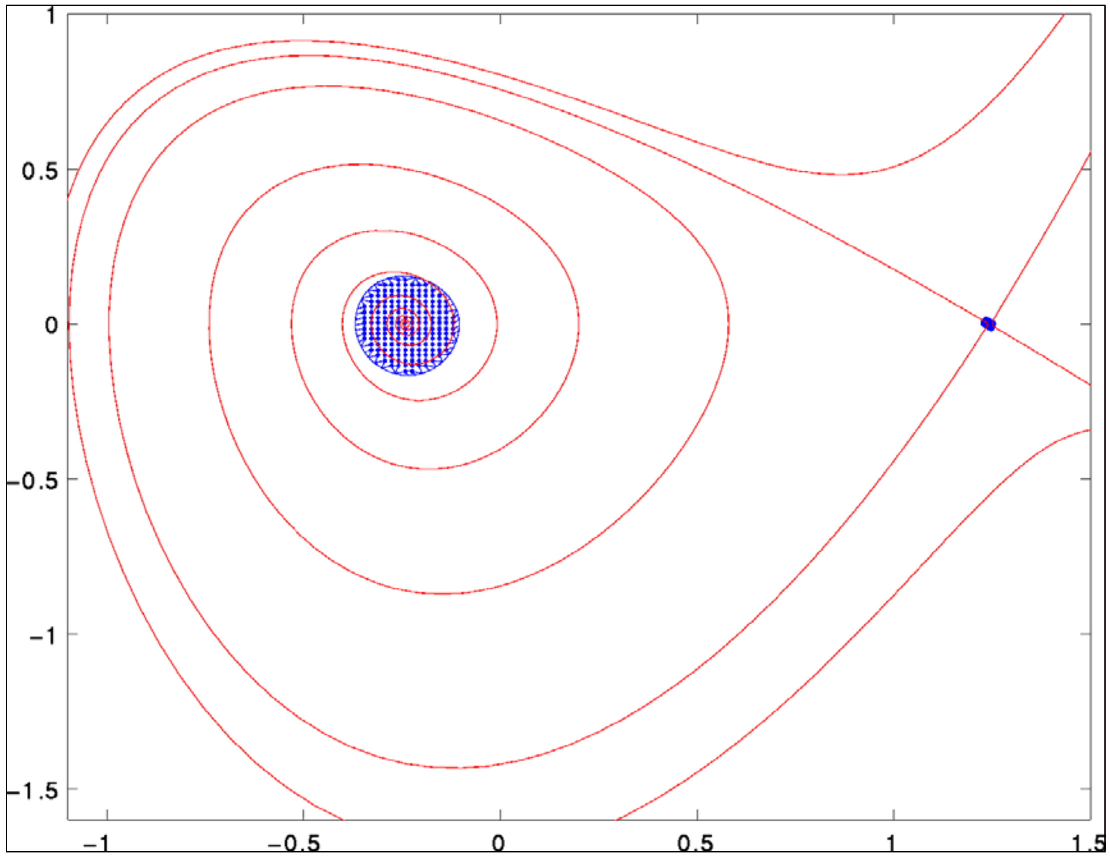


S_2 two fixed points: saddle and stable focus
k_S stable focus, homoclinic orbit
C two fixed points, unstable limit cycle

Controlled Takens-Bogdanov system

In area S_2 with $\lambda_1 = -0.3,\ \lambda_2 = -1.0$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \lambda_1 + \lambda_2 x + x^2 + xy + u(t) \text{ with } u(t) \in [-\rho, \rho]. \end{aligned}$$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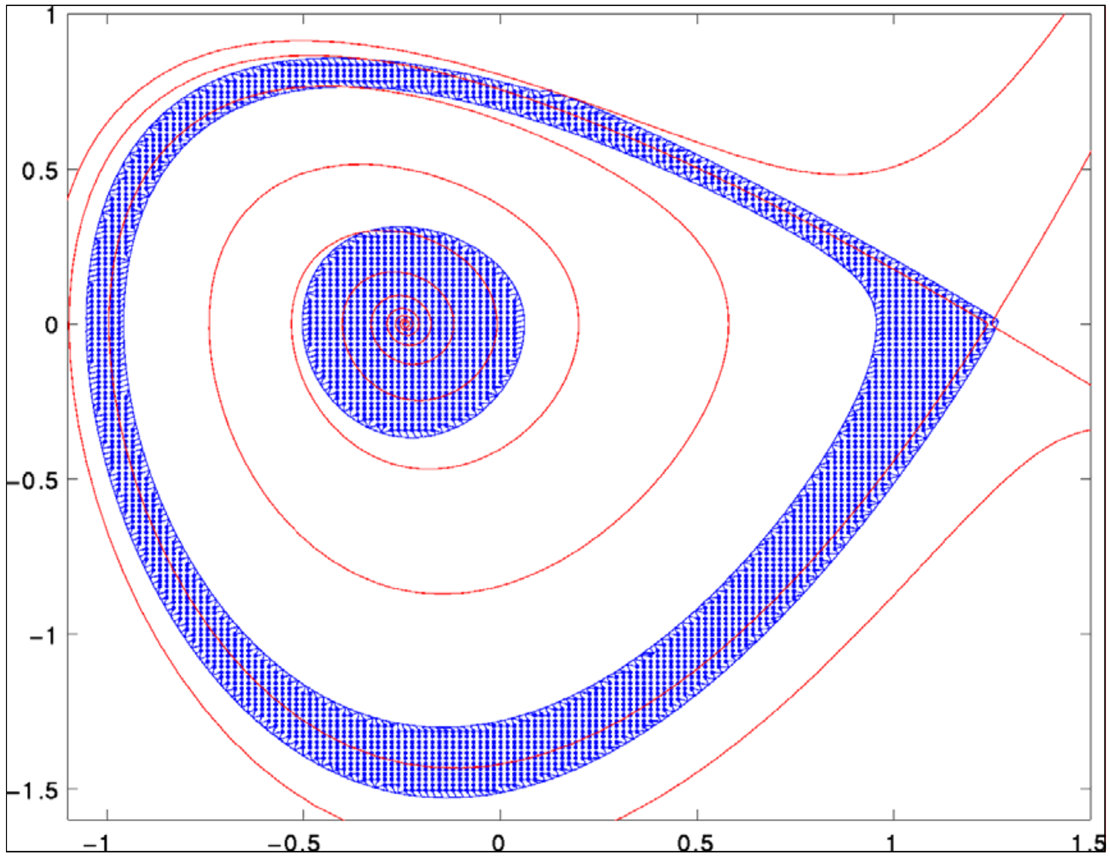
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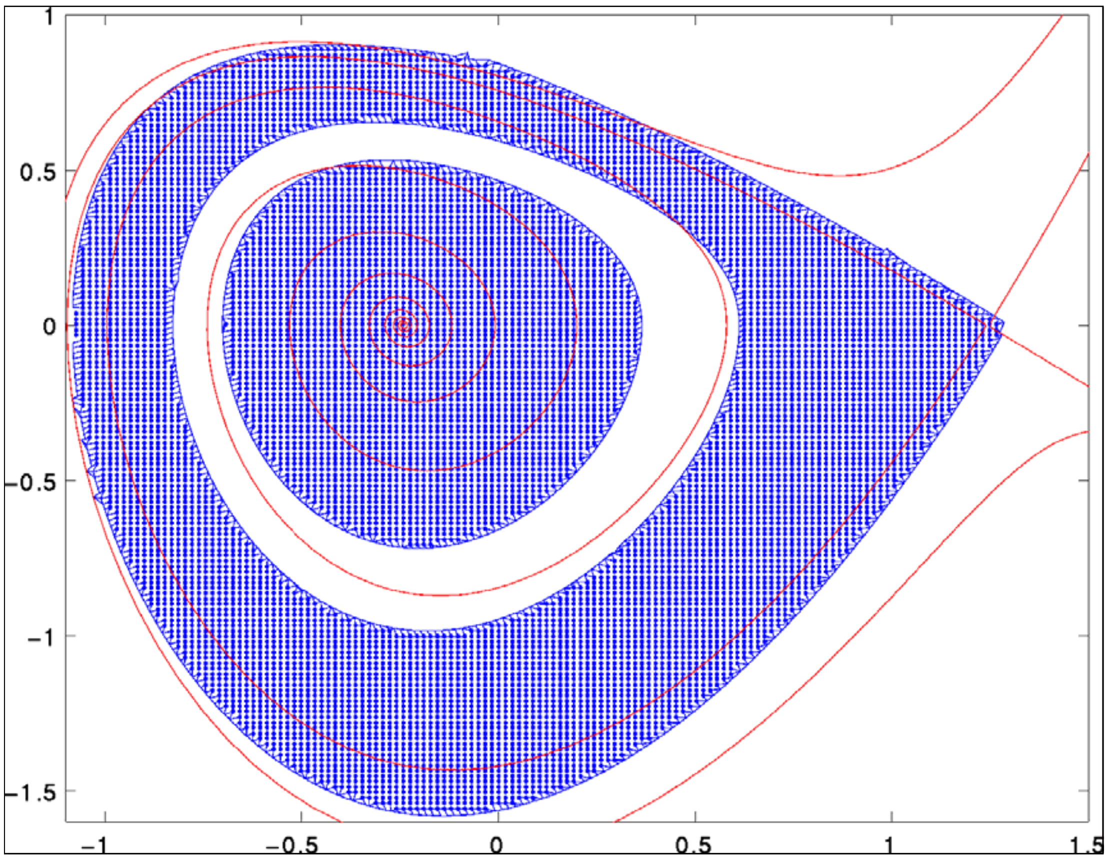

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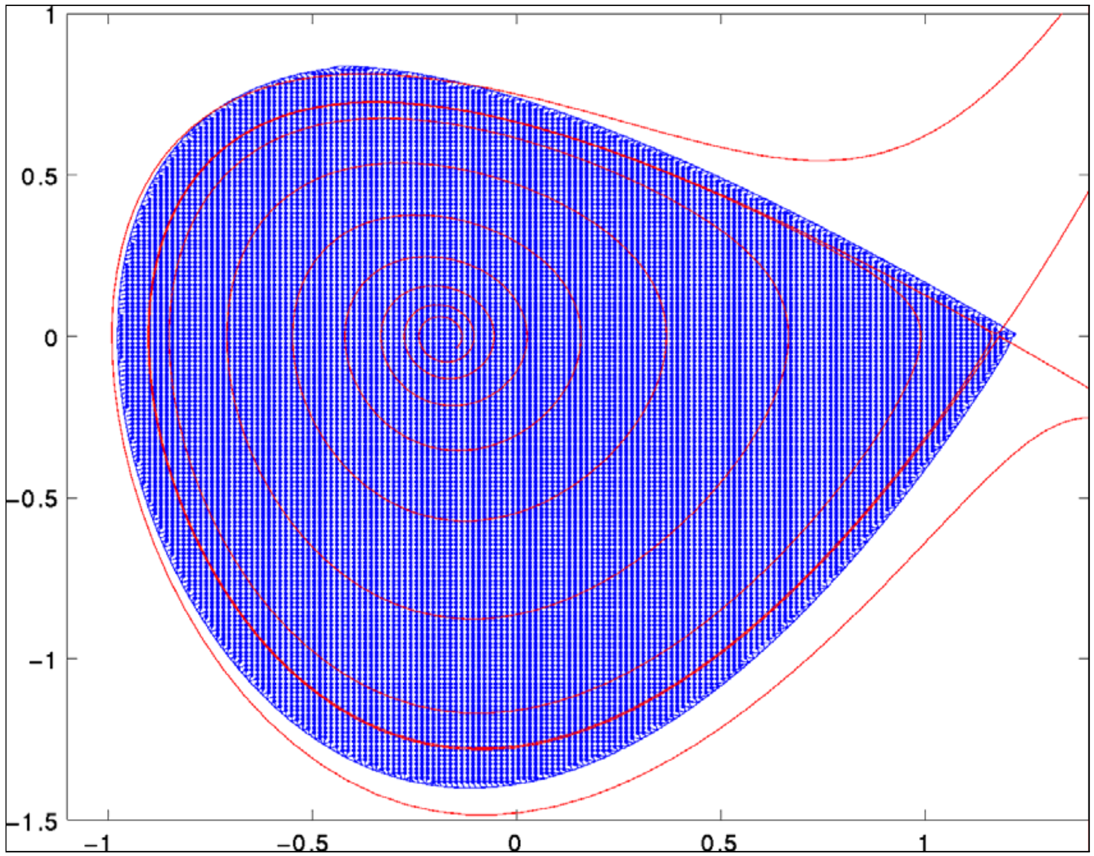

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−1
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0.5
1

Concluding remarks

For linear control systems without control constraints, controllability and observability are determined by linear algebra.

In the presence of control constraints, controllability is exceptional, instead subsets of controllability (control sets) play an important role for linear systems.

For nonlinear systems, Lie-algebraic arguments are used to guarantee local accessibility, which can be used together with control sets.

Imperial College, London
June 2018

Stabilization

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Introduction

Stabilization is one of the major themes in control theory. Very often, a primary goal is to ensure stability (or to improve stability properties), since otherwise the system may just explode.
Let us start with linear systems

$$\dot{x}(t) = Ax(t) + Bu(t),\ \ u(t) \in \mathbb{R}^m.$$

Controllability guarantees that one can reach $0 \in \mathbb{R}^d$ (in finite time) from each $x_0 \in \mathbb{R}^d$ by an appropriate control $u_{x_0}(\cdot)$.
However, if A has eigenvalues with positive real parts, the solution will diverge under arbitrarily small perturbations:

$$\varphi(t, x_0 + \varepsilon x_1, u_{x_0}) = \varepsilon \underbrace{e^{At} x_1}_{\to\infty \text{ gener.}} + \underbrace{e^{At} x_0 + \int_0^t e^{A(t-s)} B u_{x_0}(s) ds}_{\to 0}.$$

A remedy is to use feedbacks:

State feedback: Find a matrix F such that with $u = Fx$

$$\dot{x}(t) = Ax(t) + BFx(t) = (A + BF)x(t).$$

is (asymptotically) stable.

Some observations:
(i) By coordinate transformation we may assume that

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \text{ with } (A_1, B_1) \text{ controllable.}$$

(ii) For scalar control and (A, b) controllable, we may assume

$$A = \begin{bmatrix} 0 & 1 & . & . & 0 \\ . & . & . & & . \\ . & . & & & 1 \\ \alpha_0 & \alpha_1 & . & . & \alpha_{n-1} \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \end{bmatrix}$$

with $\chi_A(z) = z^n - \alpha_{n-1} z^{n-1} - ... - \alpha_1 z - \alpha_0$.

(iii) This can be stabilized by

$$f = (\beta_0 - \alpha_0, \beta_1 - \alpha_1, \ldots, \beta_{n-1} - \alpha_{n-1}) \in \mathbb{R}^{1 \times d},$$

since

$$A + bf = A + \begin{bmatrix} 0 \\ . \\ . \\ 1 \end{bmatrix} (\beta_0 - \alpha_0, \ldots, \beta_{n-1} - \alpha_{n-1}) = \begin{bmatrix} 0 & 1 & . & 0 \\ . & . & . & . \\ . & . & & . \\ \beta_0 & \beta_1 & . & \beta_{n-1} \end{bmatrix}.$$

with $\chi_A(z) = z^n - \beta_{n-1} z^{n-1} - ... - \beta_1 z - \beta_0$.

(iv) (Heymann's Lemma) Let (A, B) be controllable and $b = Bv \neq 0$. Then there is F such that

$$(A + BF, b) \text{ is controllable.}$$

(ii) - (iv) imply that every controllable pair is stabilizable. Use (i) to get

Theorem. For (A, B) let χ be a normed polynomial with $\deg\chi = \dim \langle A \,|\mathrm{im} B \rangle$. Then there exists a feedback F s.t.

$$\chi_{A+BF} = \chi \cdot \chi_{A_3}.$$

This is known as the **pole shifting theorem**.

The theorem also shows that stabilizability is equivalent to asymptotic null controllability.

Laplace-transforms and poles

For initial condition $x(0) = 0$, take Laplace transforms

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t) dt, \;\; \hat{x}(s) = \int_0^\infty e^{-st} x(t) dt.$$

By partial integration

$$\dot{x}^\wedge(s) = \int_0^\infty e^{-st} \dot{x}(t) dt = s\hat{x}(s) = s \int_0^\infty e^{-st} x(t) dt = s\hat{x}(s).$$

Thus

$$\hat{x}(s) = (sI - A)^{-1} B \hat{u}(s).$$

The eigenvalues of A are the poles of $(sI - A)^{-1}B$.

Stabilization via outputs

Consider $\dot{x} = Ax + Bu, \ y = Cx$.
Static output feedback: With $u = Fy = FCx$

$$\dot{x}(t) = Ax(t) + BFCx = (A + BFC)x(t).$$

Example

$$\dot{x}_1 = x_2, \dot{x}_2 = u \,, \ y = x_1 .$$

This system is controllable and observable, but there is no (as.) stabilizing feedback $k : \mathbb{R} \to \mathbb{R}$

$$\dot{x}_1 = x_2, \dot{x}_2 = k(y) = k(x_1).$$

In fact,

$$V(x_1, x_2) = (x_2)^2 - 2\int_0^{x_1} k(s)ds$$

is constant along trajectories with $V(0,0) = 0$ and $V(0,\alpha) = \alpha^2 \neq 0$ for $\alpha \neq 0$.

Dynamic observers

Instead of this static output feedback use dynamic output feedback:

Separate the output stabilization problem into two subproblems:

(i) find a stabilizing state feedback;

(ii) estimate the state and use this estimate in (i).

A dynamic observer

ad (ii) For $\dot{x} = Ax + Bu, y = Cx$ find L such that $A + LC$ is stable.
Then, by linearity, the dynamic observer

$$\dot{z} = (A + LC)z - Ly + Bu$$

satisfies

$$\|z(t) - x(t)\| \longrightarrow 0 \text{ for } t \longrightarrow \infty.$$

In fact: the error $e(t) = z(t) - x(t)$ converges to 0, since

$$\begin{aligned}
\dot{e} &= \dot{z} - \dot{x} = (A + LC)z - Ly + Bu - Ax - Bu \\
&= (A + LC)z - LCx - Ax \\
&= (A + LC)(z - x) \\
&= (A + LC)e.
\end{aligned}$$

Dynamic compensator

Theorem. If (A, B) and $(A^\top, C^\top)$ are stabilizable (i.e., asymptotic null controllability and asymptotic observability hold), then there are F and L such that following the dynamic output feedback stabilizes the system,

$$u = Fz,$$

where

$$\dot{z} = (A + LC)z + BFz - LCx.$$

Compensator

We use the estimate $z(t)$ instead of the state $x(t)$ in the state feedback and assume that (A, B) and $(A^\top, C^\top)$ are stabilizable.

Then the system is stabilized by $u = Fz$, since the following coupled system is stable,

$$\begin{aligned} \dot{x} &= Ax + BFz \\ \dot{z} &= (A + LC)z + BFz - LCx. \end{aligned}$$

In fact, it turns out that the system matrix

$$\begin{bmatrix} A & BF \\ -LC & A + LC + BF \end{bmatrix}$$

is stable.

Linear-quadratic optimal control

This is an efficient (and intensely studied) method to construct stabilizing feedbacks. Consider

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) + Du(t). \end{aligned}$$

Here $z(t)$ is the output which is to be controlled. This can be done by minimizing for given initial state x_0 over u

$$J(x_0; u) = \int_0^\infty \left[\|Cx(t)\|^2 + \|Du(t)\|^2 \right] dt.$$

More generally, minimize with $Q \geq 0$ and $N > 0$,

$$J(x_0; u) = \int_0^\infty \left[x(t)^\top Q x(t) + u(t)^\top N u(t) \right] dt.$$

For $Q > 0$, $x(t) \to 0$ for $t \to \infty$ if there is u with $J(x_0; u) < \infty$.

Goal: Show that the optimal controls can be written as feedback $u = Fx$.

This problem is closely related to positive semidefinite solutions of the algebraic matrix Riccati equation

$$A^\top P + PA - PBB^\top P + Q = 0. \qquad \text{(ARE)}$$

A typical result:

Theorem. Assume that (A, B) is stabilizable and $\mathrm{spec}(A) \cap \imath\mathbb{R} = \varnothing$.
(i) There is a smallest positive semidefinit solution P^- of ARE.
(ii) For every input u

$$J(x_0; u) = x_0^\top P^- x_0 + \int_0^\infty \left\| u(t) + B^\top P^- x(t) \right\|^2 dt.$$

(iii) The optimal input is given by the feedback

$$u(t) = -B^\top P^- x(t).$$

The **proof** uses the finite time problem and completion of squares.

An example

Stabilize an inverted pendulum on a flying quadrocopter.

The complete system is described by a 16-dimensional system of differential equations (12 for the quadrocopter + 4 for the pendulum) with 4 control inputs.
After simplification to 13 dimensions and linearization in the equilibrium a linear-quadratic optimal control problem is solved.
Critical is the measurement of the states which is done by an infrared motion tracking system.

HEHN AND D'ANDREA, IEEE TRANS. AUT. CONTROL (2011)

Further problems

The H^∞-problem for

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ z &= Cx + Du \end{aligned}$$

Goal: Given $\gamma > 0$ find F such that $A + BF$ is stable and (for $x_0 = 0$)

$$\|z\|_2 \leq \gamma \|d\|_2 \text{ for all perturbations } d \in L^2(0, \infty, \mathbb{R}^\ell).$$

This is possible for $\gamma > \|G_F\|$ with

$$G_F : L^2(0,\infty) \to L^2(0,\infty), d(\cdot) \mapsto z(\cdot) = \int_0^\cdot Ce^{(A+BF)(t-\tau)} Ed(\tau)\, \mathrm{d}\tau.$$

(well defined for $A + BF$ stable)
This again leads to LQ-optimal control (without positive definiteness).

Note that for stable A and

$$G : L^2(0,\infty) \to L^2(0,\infty), d(\cdot) \mapsto z(\cdot) = \int_0^{\cdot} Ce^{A(t-\tau)} Ed(\tau)\, \mathrm{d}\tau$$

and

$$G(s) = C(sI - A)^{-1} E$$

one has

$$\|G\| = \sup \left\{ \frac{\|G(d)\|_2}{\|d\|_2} \,\middle|\, 0 \neq d \in L^2 \right\} = \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|,$$

where $\|G(i\omega)\|$ denotes the largest singular value. This is the H^∞-norm of matrix-valued functions which are holomorphic on the open right half plane.

Nonlinear stabilization at an equilibrium

Consider

$$\dot{x}(t) = f(x(t), u(t))$$

and let x^* be an equilibrium $f(x^*, u^*) = 0$. Linearization in (x^*, u^*) yields

$$\dot{y}(t) = f_x(x^*, u^*)y(t) + f_u(x^*, u^*)v(t)$$

and write $A = f_x(x^*, u^*)$ and $B = f_u(x^*, u^*)$.
Then a stabilizing feedback F for the linearized system is locally stabilizing for the nonlinear system

$$\dot{x}(t) = f(x(t), F(x(t) - x^*)).$$

(use a Lyapunov function)

Brockett's necessary condition

Theorem. Consider $\dot{x} = f(x, u)$, $u \in U$ open. If there is a locally stabilizing continuous feedback $F : \mathbb{R}^d \to U$, then $f(\mathbb{R}^d, U)$ is a neighborhood of 0.

Example (Brockett's nonholonomic integrator)

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

This is a simple model for a vehicle with angle $\theta = x_1$ in forward direction and position

$$(z_1, z_2) = (x_2 \cos\theta + x_3 \sin\theta, x_2 \sin\theta - x_3 \cos\theta).$$

No point $(0, 0, \varepsilon)$ with $\varepsilon \neq 0$ is in the image of f.
On the other hand, the system is asymptotically null controllable.

Control-Lyapunov functions

Asymptotic controllability to an equilibrium and stabilization can be dealt with using control-Lyapunov functions which decrease along trajectories for appropriate controls..

Roughly,
- asymptotic controllability to an equilibrium holds if there exists a continuous control-Lyapunov function
- stabilizability with continuous feedback holds if there exists a smooth control-Lyapunov function.

cf. Sontag (1998), Coron (2007).

Coron's return method: time-varying feedbacks

Theorem. Consider a driftless control system in $\mathbb{R}^d$

$$\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x)$$

and assume that

$$\{g(x) \,|g \in \mathcal{LA}(f_1, \dots, f_m)\} = \mathbb{R}^d \text{ for all } x \neq 0.$$

Then for every $T > 0$ there exists $u \in C^\infty(\mathbb{R}^d \times \mathbb{R})$ with

$$u(0,t) = 0,\ \ u(x, t+T) = u(x,t) \text{ for all } t \in \mathbb{R}, x \in \mathbb{R}^d$$

such that 0 is globally asymptotically stable for

$$\dot{x} = \sum_{i=1}^{m} u_i(x,t) f_i(x).$$

The proof constructs periodic trajectories near 0 with controllable linearization. Coron (1992), (2007).

Example

Nonholonomic integrator

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \dot{x}_3 = x_1 u_2 - x_2 u_1.$$

Here

$$f_1(x) = \begin{bmatrix} 1 \\ 0 \\ -x_2 \end{bmatrix}, f_2(x) = \begin{bmatrix} 0 \\ 1 \\ x_1 \end{bmatrix}.$$

Brockett's necessary condition is violated, but the Lie algebra rank condition is satisfied. Hence it can be globally asymptotically stabilized by means of periodic time-varying feedback.

Stabilization with piecewise constant controls

Continuous stirred tank reactor

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 - a(x_1 - x_c) + B\alpha(1 - x_2)e^{x_1} \\ -x_2 + \alpha(1 - x_2)e^{x_1} \end{bmatrix} + u(t) \begin{bmatrix} x_c - x_1 \\ 0 \end{bmatrix},$$

where x_1 is the temperature and x_2 is the product concentration, x_c is the coolant temperature and the control affects the heat transfer coefficient with parameters

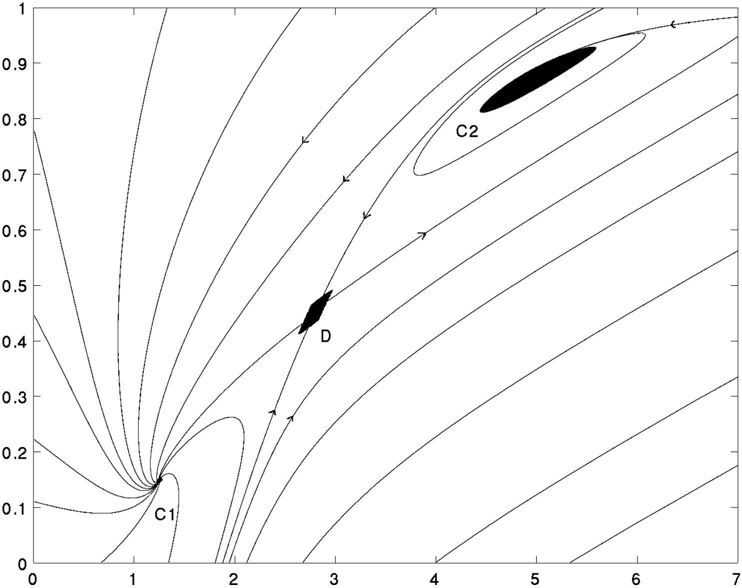
$$a = 0.95,\ \alpha = 0.05,\ B = 10.0,\ c_c = 1.0$$

and control range

$$\Omega = [-0.15, 0.15].$$

The uncontrolled system has an unstable fixed point at

$$(x_1^*, x_2^*) \sim (2.8, 0.45) \in D$$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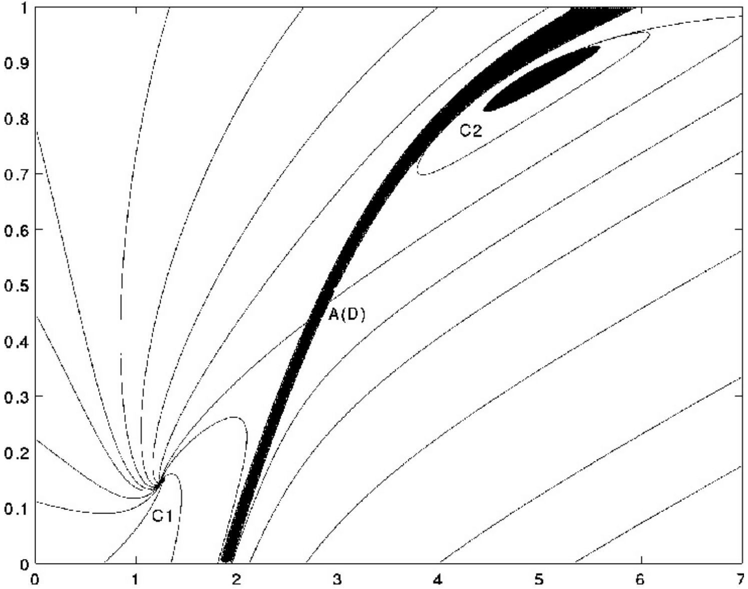
C2
D
C1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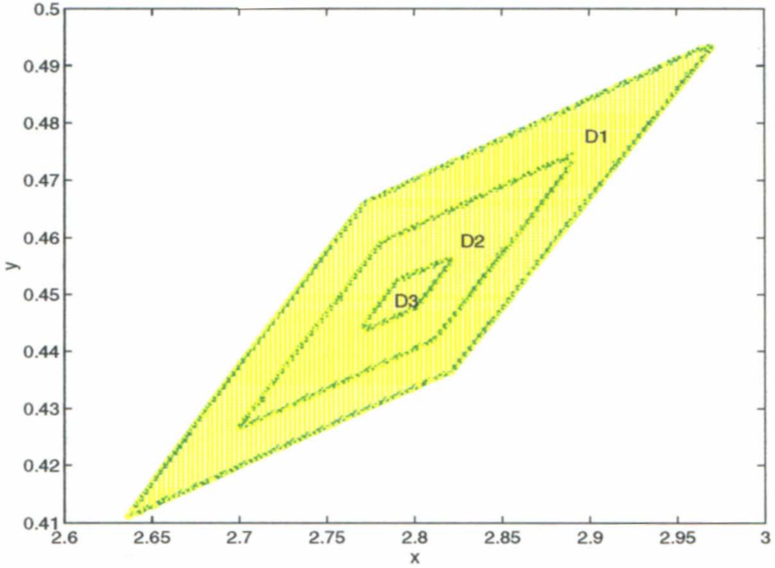
4
5
6
7
C2
A(D)
C1

Stabilization by piecewise constant feedbacks

Let D be the control set around the equilibrium. Define

$$\begin{aligned} R_0 &: = \{x \,|\, \varphi(t,x,\rho) \in D \text{ for some } t>0\}, \\ R_1 &: = \{x \,|\, \varphi(t,x,-\rho) \in R_0 \cup D \text{ for some } t>0\} \setminus R_0 \\ R_2 &: = \{x \,|\, \varphi(t,x,\rho) \in R_0 \cup R_1 \cup D \text{ for some } t>0\} \setminus (R_0 \cup R_1) \\ & \quad \text{etc.} \end{aligned}$$


0.5
0.49
0.48
0.47
0.46
0.45
0.44
0.43
0.42
0.41
y
2.6
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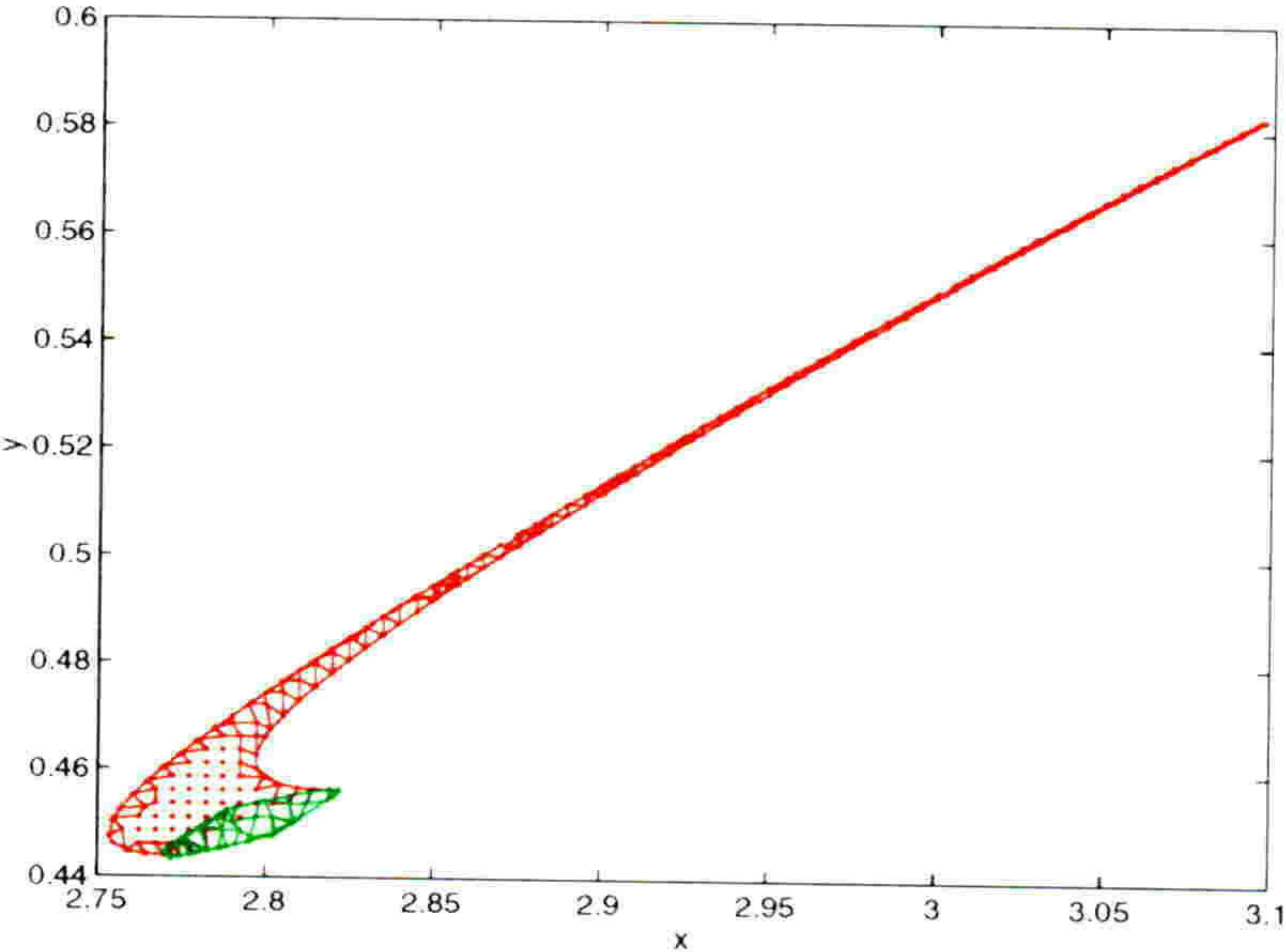
2.75
2.8
2.85
2.9
2.95
3
x
D1
D2
D3



0.6
0.58
0.56
0.54
0.52
0.5
0.48
0.46
0.44
y
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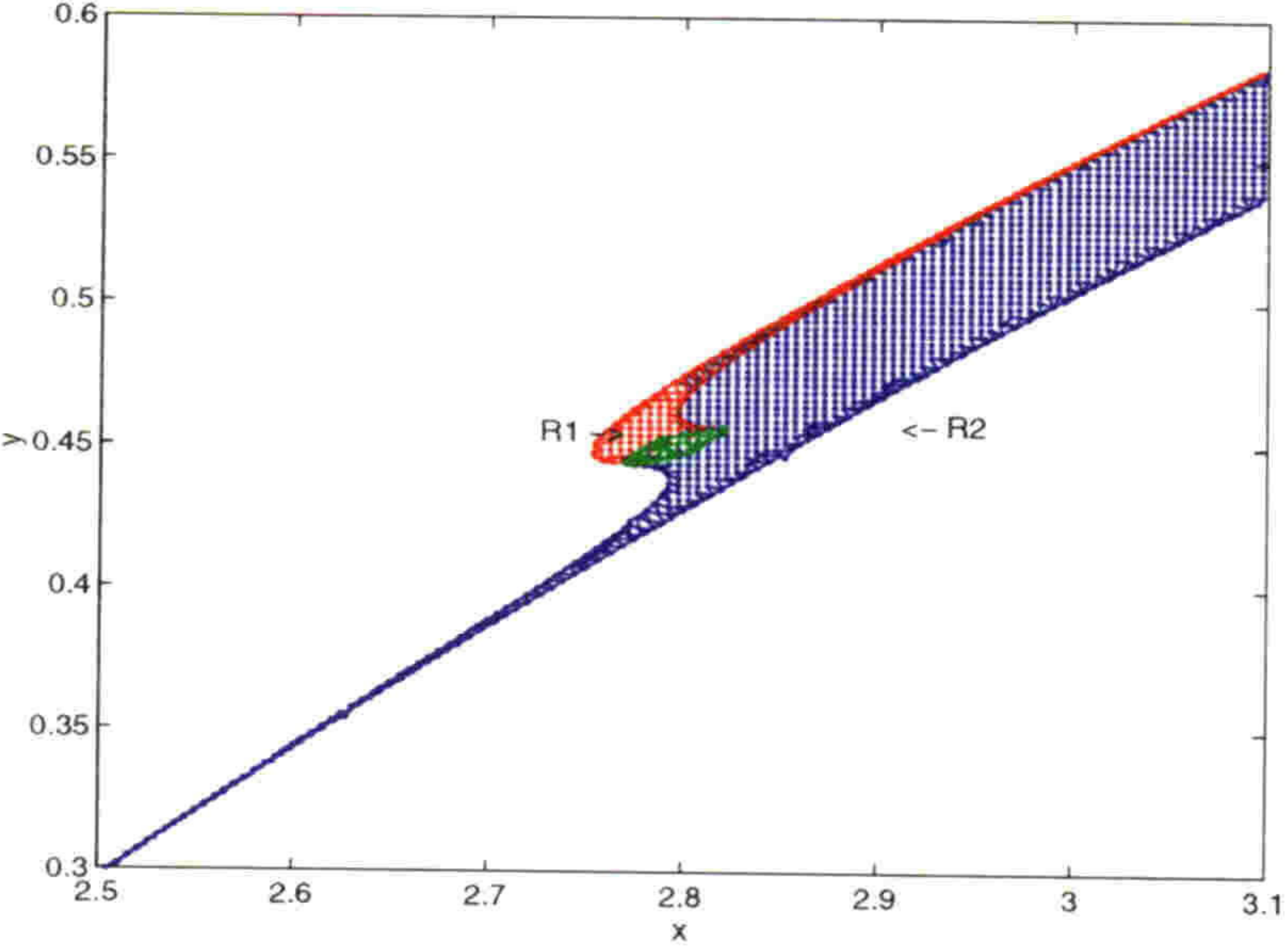
2.8
2.85
2.9
2.95
3
3.05
3.1
x



0.6
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0.5
0.45
0.4
0.35
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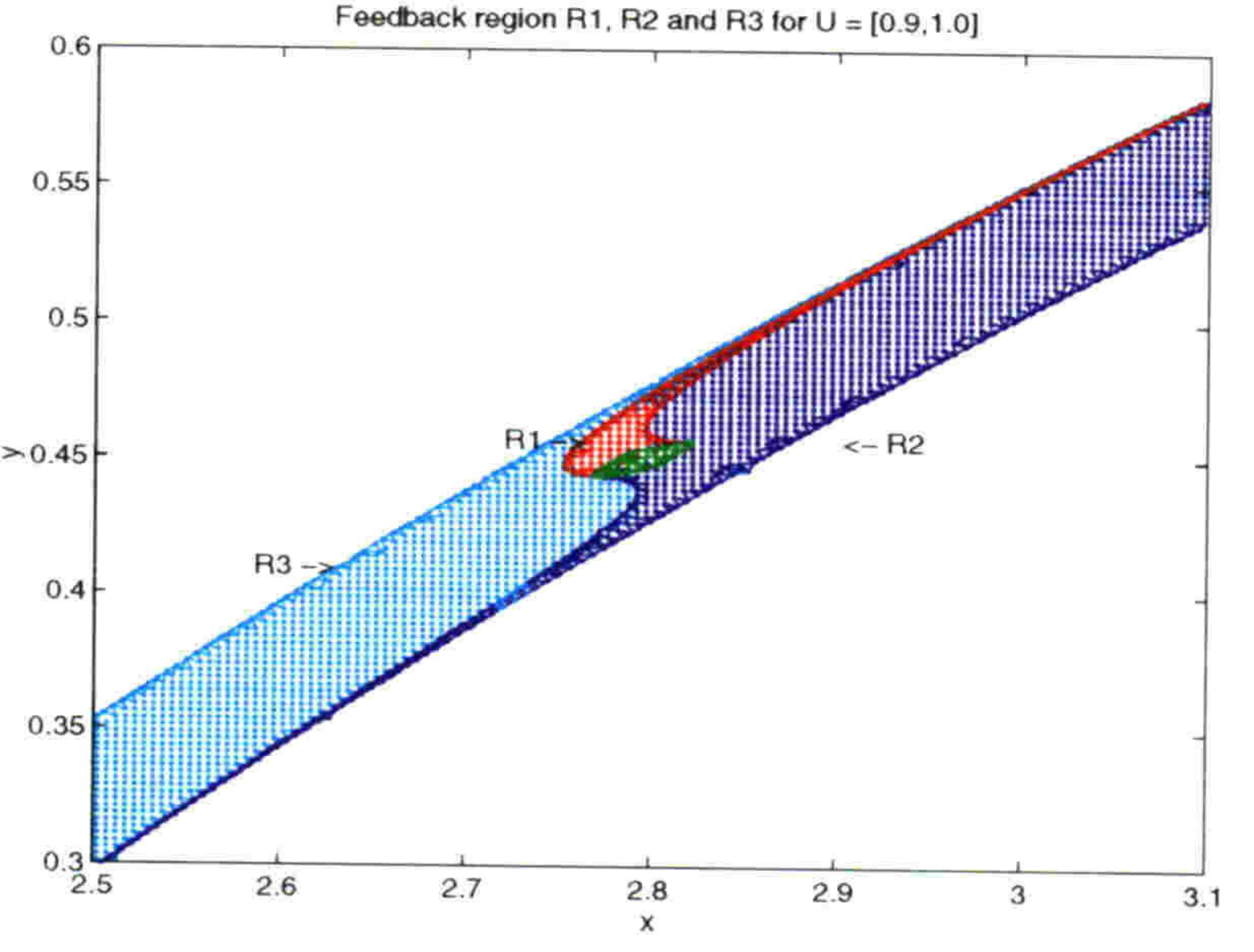
y
2.5
2.6
2.7
2.8
2.9
3
3.1
x
R1 ->
<- R2



Feedback region R1, R2 and R3 for U = [0.9,1.0]
0.6
0.55
0.5
0.45
0.4
0.35
0.3
y
2.5
2.6
2.7
2.8
2.9
3
3.1
x
R1 ->
<- R2
R3 ->

Final remarks

Since asymptotic stabilization is a basic problem in control, there is a multitude of algorithms to achieve it, in addition to the examples presented here.

- Backstepping
- Model-predictive control (receding horizon optimal control)
- ...

Note that in applications stability is only one goal among others including, in particular, robustness properties with respect to perturbations.

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June 2018

Control sets, the control flow and relations to random systems

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We will associate to control-affine systems a continuous dynamical system which allows us to use methods from the theory of dynamical systems on metric spaces in order to obtain results on controllability properties.

This is, in particular, based on ideas due to C. Conley involving chain transitivity, Morse decompositions and attractor-repeller pairs.

Discrete-time systems

Consider

$$x_{k+1} = f(x_k, u_k),\ \ u_k \in \Omega,\ \text{ for } k \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous on metric spaces M and Ω.
A control function u is an element of $\Omega^{\mathbb{N}}$ (or $\Omega^{\mathbb{Z}}$), the solutions are $\varphi(k, x, u)$.
The shift $\theta : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ is $\qquad \theta((u_k)_{k\geq 0}) = (u_{k+1})_{k\geq 0}$.
Define the skew product map

$$S : \Omega^{\mathbb{N}} \times M \to M, S(u, x) = (\theta u, f(x, u_0)).$$

Then

$$S^k(u, x) = (\theta^k u, \varphi(k, x, u))$$

and φ is a cocycle, i.e.,

$$\varphi(k + \ell, x, u) = \varphi(k, \varphi(\ell, x, u), \theta^\ell u),\ \text{ for } k, \ell \in \mathbb{N}$$

Proposition. The shift θ and the map S define continuous dynamical systems. If Ω is compact, also $\Omega^{\mathbb{N}}$ is compact.

Proof. Compactness of $\Omega^{\mathbb{N}}$ follows by Tychonov. Continuity of θ follows since the sets

$$W = W_0 \times W_1 \times \cdots \times W_N \times \Omega \times \cdots \subset \Omega^{\mathbb{N}}$$

with $W_i \subset U$ open form a subbasis of the product topology and the preimages

$$\theta^{-1} W = \Omega \times W_0 \times W_1 \times \cdots \times W_N \times \Omega \times \cdots$$

are open. S is continuous by continuity of f.

Continuous-time systems

Consider control-affine systems

$$\begin{aligned}\dot{x}(t) &= f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \\ u &\in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \subset \mathbb{R}^m\}\end{aligned}$$

with trajectories $\varphi(t, x, u), t \in \mathbb{R}$. A special case are bilinear systems

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t), \text{ with } A_i \in \mathbb{R}^{d \times d}.$$

Define the shift on $\mathcal{U}$ by $(\theta_t u)(s) = u(t+s), \ s \in \mathbb{R}$. Then

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d, \ (t, u, x) \to \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u))$$

is a skew product flow,

$$\varphi(t+s, x, u) = \varphi(t, \varphi(s, x, u), \theta_s u) \text{ for } t, s \in \mathbb{R},$$

hence

$$\Phi(t+s, x, u) = (\theta_{t+s} u, \varphi(t+s, x, u)) = \Phi_t \circ \Phi_s(u, x).$$

The shift

Proposition. Assume that $\Omega \subset \mathbb{R}^m$ is convex and compact.
(i) Then $\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \mid u(t) \in \Omega \subset \mathbb{R}^m\}$ is weak* compact and metrizable in $L^\infty = \left(L^1\right)^*$.
(ii) The shift θ is continuous, the periodic points are dense.

Proof. (i) $\mathcal{U}$ is a convex, bounded closed subset of L^∞, hence by Alaoglu's Theorem compact and metrizable. The periodic functions are dense: Let $u \in \mathcal{U}$ and $\varepsilon > 0$.

$$\forall x \in L^1 \; \exists T > 0 : \int_{\mathbb{R}\setminus[-T,T]} \|x(t)\| \, dt < \varepsilon / \mathrm{diam}\Omega.$$

Define $u_p(t) = u(t)$ on $[-T, T]$ and extend periodically. Then

$$\left| \int_{\mathbb{R}} [u(t) - u_p(t)]^\top x(t) dt \right| \leq \mathrm{diam}\Omega \int_{\mathbb{R}\setminus[-T,T]} \|x(t)\| \, dt.$$

(ii) Continuity of the shift in the L^1-topology on $\mathcal{U}$ follows, since the shift in L^1 is continuous.

Shadowing

Da Silva and Kawan DCDS (2016) have shown that the shift on $\mathcal{U}$ satisfies the following shadowing property:

For every $\varepsilon > 0$ there is $\delta > 0$ such that for every sequence $(u^k)_{k\in\mathbb{Z}}$ in $\mathcal{U}$ with $d(\theta_1 u^k, u^{k+1}) \leq \delta$ there is $u \in \mathcal{U}$ with

$$d(\theta_k u, u^{k+1}) \leq \varepsilon.$$

If the chain $(u^k)_{k\in\mathbb{Z}}$ is periodic, u can be chosen as a periodic function.

The Control Flow

Theorem. For a control affine system with compact and convex control range Ω, the control flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^d \to \mathcal{U} \times \mathbb{R}^d,\ (t, u, x) \to \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u))$$

is continuous.

Proof for bilinear control systems $\dot{x} = A_0 x + \sum_{i=1}^m u_i(t) A_i x$.
Let $t^n \to t^0, u^n \to u^0$ and $x^n \to x^0$ and abbreviate $\varphi^n(t) = \varphi(t^n, x^n, u^n)$.
Using Arzela-Ascoli, let $\varphi^n(\cdot) \to \psi(\cdot)$. Then on $[0, t^0 + 1]$

$$\varphi^n(t) = x^n + \int_0^t A_0 \varphi^n(s) + \sum_{i=1}^m u_i^n(s) A_i \left[\varphi^n(s) - \psi(s)\right] + \sum_{i=1}^m u_i^n(s) A_i \psi(s) ds$$

and by weak* convergence $\int_0^t \sum_i u_i^n(s) A_i \psi(s) ds \to \int_0^t \sum_i u_i^0(s) A_i \psi(s) ds$.
Hence by Gronwall $\psi = \varphi^0$.

Relations to controllability

A flow Φ on a compact metric space X is topologically transitive if there is $x \in X$ with

$$X = \{y = \lim_{k\to\infty} \Phi(t_k, x) \,|\, t_k \to \infty\}.$$

It is topologically mixing if for all open $V, W \subset X$ there is $T > 0$ with

$$\Phi(T, V) \cap W \neq \varnothing.$$

Note: Topological mixing $\Rightarrow$ topological transitive.

Recall: A control set D is a maximal set such that for all $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D, t \geq 0$, and

$$D \subset \overline{\mathcal{R}(x)} \text{ for all } x \in D.$$

A point x is locally accessible if for all $T > 0$

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \varnothing \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \varnothing.$$

Relations to controllability

The lift of a control set D with nonvoid interior is

$$\mathrm{cl}\{(u,x) \in \mathcal{U} \times M \,|\, \varphi(t,x,u) \in \mathrm{int}D \text{ for all } t \in \mathbb{R}\}.$$

Theorem. Assume local accessibility.
(i) The lift of a control set D with nonvoid interior is a maximal topologically mixing set for the control flow.
(ii) Conversely, every maximal topologically transitive set whose projection to M has nonvoid interior is the lift of a control set.

Proof. (i) Needs a subbasis of the topology on $\mathcal{U}$.
(ii). Use local accessibility!

Chain transitivity

Let Φ be a continuous flow on a compact metric space X.
For $\varepsilon, T > 0$ an (ε, T)-chain ζ from $x \in X$ to $y \in X$ is given by

$$n \in \mathbb{N}, x_0 = x, x_1, \ldots, x_n = y, T_0, T_1, \ldots, T_{n-1} > T$$

such that

$$d(\Phi(T_i, x_i), x_{i+1}) < \varepsilon \text{ for all } i.$$

A set $K \subset M$ is chain transitive if for all $x, y \in K$ and all $\varepsilon, T > 0$ there is an (ε, T)-chain from x to y.
A maximal chain transitive set is called chain recurrent component.

Remark. Conley's Fundamental Theorem implies that the control flow Φ is gradient-like outside the maximal chain transitive sets $\mathcal{E}_i$.

Example. A homolinic orbit together with the equilibrium.

We return to control systems.

Definition. A **chain control set** $E \subset M$ is a maximal set with
(i) for all $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$;
(ii) for all $x, y \in E$ and all $\varepsilon, T > 0$ there is a controlled (ε, T)-chain from x to y given by

$$n \in \mathbb{N}, x_0 = x,\ x_1, \ldots, x_n = y,\ u_0, \ldots, u_{n-1} \in \mathcal{U}, T_0, \ldots, T_{n-1} > T \text{ with}$$

$$\mathrm{d}(\varphi(T_i, x_i, u_i), x_{i+1}) < \varepsilon \text{ for all } i.$$

Chain control sets

We return to control-affine systems

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)),\ u \in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m) \,|u(t) \in \Omega\}$$

with control flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M,\ (t, u, x) \to (\theta_t u, \varphi(t, x, u)).$$

Theorem. For every compact chain control set E the lift

$$\mathcal{E} := \{(u, x) \in \mathcal{U} \times M \,|\varphi(t, x, u) \in E,\ t \in \mathbb{R}\}$$

is a chain recurrent component for the control flow Φ and conversely.

For the **proof** observe that the projection of a chain transitive set for Φ to M yields controlled (ε, T)-chains. For the converse one has to construct (ε, T)-chains in $\mathcal{U} \times M$ from controlled (ε, T)-chains.

Alternative characterization

A **Morse decomposition** of a flow is given by $\{\mathcal{M}_i \,|\, i = 1, \ldots, \ell\}$ with nonvoid, pairwise disjoint and compact isolated invariant sets s.t.
(i) $\forall x \in X : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$;
(ii) there are no cycles.

Example

$$\dot{x} = x(x-1)(x-2)^2(x-3).$$

Morse decompositions are e.g.

$$\begin{aligned} \mathcal{M}_1 &= \{0\} \preceq \mathcal{M}_2 = [1,3] \\ \mathcal{M}_1 &= \{0\} \preceq \mathcal{M}_3 = \{1\} \succeq \mathcal{M}_2 = [2,3] \\ \mathcal{M}_1 &= \{0\} \cup [2,3] \preceq \mathcal{M}_2 = \{1\}. \end{aligned}$$

with finest Morse decomposition

$$\{0\} \preceq \{1\} \succeq \{2\} \succeq \{3\}.$$

Morse decompositions and chain transitivity

Theorem. If for a flow on a compact metric space the number of chain recurrent components is finite, this corresponds to the finest Morse decomposition.

In particular, if the number of chain control sets in a compact invariant set is finite, this corresponds to the finest Morse decomposition of the control flow.

Parameter dependence

Under appropriate compactness assumptions, chain control sets depend upper semicontinuously on parameters, and control sets depend lower semicontinuously on parameters (in the Hausdorff metric).

Theorem. Fix α_0 and suppose that D^{α_0}.is a control set such that $\mathrm{cl}D^{\alpha_0} = E^{\alpha_0}$ is a chain control set. Then there are control sets D^{α} and chain control sets E^{α} with

$$\lim_{\alpha\to\alpha_0} \mathrm{cl}D^{\alpha} = \mathrm{cl}D^{\alpha_0} = E^{\alpha_0} = \lim_{\alpha\to\alpha_0} E^{\alpha}.$$

Thus we see that abrupt changes in the behavior can be expected only if control sets and chain control sets are different.

Chain control sets versus control sets I

Next we turn to conditions which ensure that a chain control set is the closure of a control set.

Theorem. Consider different control ranges $U^\rho = \rho \cdot U$ with $\rho \geq 0$, and assume the following ρ-**inner-pair condition**:
For all x, all $\rho' > \rho \geq 0$ and all $u \in \mathcal{U}^\rho$ there is $T > 0$ with

$$\varphi(T, x, u) \in \mathrm{int}\mathcal{R}^{\rho'}(x).$$

Then for all but at most countably many ρ-values and all control sets

$$\mathrm{cl}D^\rho = E^\rho.$$

Gayer (2003): The ρ-inner pair condition holds for all systems

$$\ddot{x} + g(t, x, \dot{x}) = h(t, x, \dot{x})u(t)$$

with g and h T-periodic in t and $h(t, x, \dot{x}) > 0$.
For the proof one plans a trajectory and solves for the control u.

Chain control sets versus control sets II

An alternative are hyperbolicity conditions for the control flow which imply the shadowing property.

FC/Du (2003), da Silva and Kawan (2016)

Stochastic Perturbations: Degenerate Markov Diffusions

Consider

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} \xi_i(t,\omega) f_i(x) \text{ on } M$$

with $\xi = h(\eta)$ and **background noise**

$$\mathrm{d}\eta = g_0(\eta)dt + \sum_{j=1}^{l} g_j(\eta) \circ dW_j \text{ on } N \text{ compact,}$$

η stationary, ergodic, and $h : N \to U$ surjective, $U \subset \mathbb{R}^m$ compact, convex.

Associated deterministic system

Consider the control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \quad \text{on } M,\ u \in \mathcal{U},$$

where

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \,|u(t) \in U \text{ for all } t, \text{ locally integrable}\}$$

for a given $U \subset \mathbb{R}^m$ with trajectories $\varphi(t, x_0, u)$, $t \in \mathbb{R}$. Let

$$\mathcal{R}_{\leq T}(x) = \{\varphi(t, x, u) \,|t \in [0, T] \text{ and } u \in \mathcal{U}\}.$$

Associated deterministic system

Consider the control system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x) \quad \text{on } M, \ u \in \mathcal{U},$$

where

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m \,|\, u(t) \in U \ \text{ for all } t, \text{ locally integrable}\}$$

for a given $U \subset \mathbb{R}^m$ with trajectories $\varphi(t, x_0, u)$, $t \in \mathbb{R}$. Let

$$\mathcal{R}_{\leq T}(x) = \{\varphi(t, x, u) \,|\, t \in [0, T] \text{ and } u \in \mathcal{U}\}.$$

Local accessibility means

$$\mathrm{int}\mathcal{R}_{\leq T}(x) \neq \emptyset \text{ and } \mathrm{int}\mathcal{C}_{\leq T}(x) \neq \emptyset \text{ for all } x \text{ and } T > 0.$$

A set $D \subset M$ is a **control set** if it is maximal with

$$D \subset \overline{\mathcal{R}(x)} \text{ for all } x \in D.$$

A set $C \subset M$ is an **invariant control set** if $\overline{C} = \overline{\mathcal{R}(x)}$ for all $x \in C$.

Analyze the pair process

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} h_i(\eta) f_i(x)$$
$$\mathrm{d}\eta = g_0(\eta)\mathrm{d}t + \sum_{j=1}^{l} g_j(\eta) \circ \mathrm{d}W_j$$

Assume

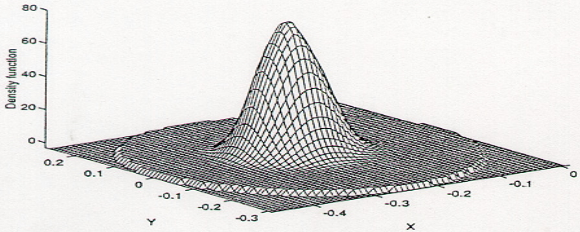
$$\dim \mathcal{L}\mathcal{A}\{g_1, ..., g_l\}(q) = \dim N \text{ for all } q \in N,$$

$$\dim \mathcal{L}\mathcal{A}\left\{\left[\begin{array}{c} f_0 + \sum h_i(w) f_i \\ g_0 + \sum w_j g_j \end{array}\right], w \in \mathbb{R}^l\right\}\left[\begin{array}{c} x \\ q \end{array}\right] = \dim M + \dim N.$$

Theorem The supports of the ergodic measures μ_i are $C_i \times N$ with C_i the invariant control sets. The μ_i are unique and have C^∞ densities. All other points are transient.

Stroock/Varadhan, Kunita, Kliemann 1987, Arnold/Kliemann 1987, FC/Kliemann 2008

Typical stationary density on an invariant control set



Invariant density, $\rho = 0.05$

support indicated by drop in grid.

Piecewise Deterministic Markov Processes

Let $E = \{0, 1, ..., m\}$ and for any $i \in E$ let $F^i : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth (C^∞) vector field with corresponding flow $\Phi^i_t(x)$, $t \geq 0$.

A **Piecewise Deterministic Markov Process (PDMP)** has the form $Z_t = (X_t, Y_t)$ living on $\mathbb{R}^d \times E$ where the continuous component X_t evolves according to a flow Φ^i_t; the component on E determines which of the flows Φ^i_t is active with random switching times.

Davis (1993)

Piecewise Deterministic Markov Processes

Choice of the flow $\Phi^i, i \in E = \{0, 1, ..., m\}$ on $M \subset \mathbb{R}^d$: Let

$$x \mapsto Q(x) = (Q(x))_{i,j} : \mathbb{R}^d \to \mathbb{R}^{(m+1)\times(m+1)}$$

be continuous with $Q(x)$ irreducible and aperiodic for all x.

Piecewise Deterministic Markov Processes

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be continuous with $Q(x)$ irreducible and aperiodic for all x.

Random switching times T_n: Determined by a homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity λ, and $U_n = T_n - T_{n-1}$.

Piecewise Deterministic Markov Processes

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be continuous with $Q(x)$ irreducible and aperiodic for all x.
Random switching times T_n: Determined by a homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity λ, and $U_n = T_n - T_{n-1}$.
The discrete-time process: Let $\tilde{Z}_n = \left(\tilde{X}_n, \tilde{Y}_n\right)$ on $M \times E$ be recursively defined by

$$\tilde{X}_{n+1} = \Phi^{\tilde{Y}_n}(U_{n+1}, \tilde{X}_n)$$

$$\mathbb{P}\left[\tilde{Y}_{n+1} = j \,\middle|\, \tilde{X}_{n+1}, \tilde{Y}_n = i\right] = Q(\tilde{X}_{n+1})_{i,j}.$$

The continuous-time process (by interpolation):

$$Z_t = \left(\Phi^{\tilde{Y}_n}(t - T_n, \tilde{X}_n), \tilde{Y}_n\right) \text{ for } t \in [T_n, T_{n+1}].$$

The associated deterministic control system

Recall that the flows Φ^i are given by the vector fields F^i.

$$\dot{x} = \sum_{i=0}^{m} v_i(t) F^i(x), \qquad t \geq 0.$$

with

$$v(t) = (v_i(t)) \in \left\{ v \in \mathbb{R}^{m+1} \,\middle|\, \sum_{i=0}^{m} v_i = 1, v_i \in \{0,1\} \right\}.$$

Up to closure, the trajectories of this system coincide with those of the control-affine system

$$\dot{x} = F^0(x) + \sum_{i=1}^{m} u_i(t)[F^i(x) - F^0(x)]$$

with controls taking values in

$$U = \left\{ u \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} u_i \leq 1, u_i \in [0,1] \right\}.$$

A Decisive Lemma for PDMP

Lemma

For all $T > 0, x \in M, i \in E, \delta > 0$ and every trajectory $\varphi(\cdot, x, u)$ of the control system one finds for start in x and $i \in E$ that there is $\varepsilon > 0$ such that

$$\mathbb{P}_{x,i}\left[\sup_{t\in[0,T]} \|X_t - \varphi(t,x,u)\| \leq \delta\right] \geq \varepsilon.$$

Benaïm, Le Borgne, Malrieu and Zitt (2015)

In the terminology of Arnold and Kliemann (1983) this is a **tube lemma** connecting the stochastic system and the control system.

Corollary

Let C be an invariant control set with nonvoid interior and let $x \in M$ with $\overline{O^{+}(x)} \cap C \neq \varnothing$.

Then there are $T > 0$ and $\varepsilon > 0$ with

$$\mathbb{P}_{x,i}\left[X_T \in \mathrm{int} C\right] \geq \varepsilon \text{ for all } i \in E.$$

This follows since then x can be steered into the interior of C in finite time.

Characterization of the supports of invariant measures for Piecewise Deterministic Markov Processes (PDMP)

Theorem

Assume that the control system is locally accessible on a compact positively invariant set M.

(i) Then for every ergodic measure μ of the process (Z_t) there is a compact invariant control set C with $\mathrm{supp}\mu = C \times E$.

(ii) Conversely, let C be a compact invariant control set. Then there exists an ergodic measure μ with support equal to $C \times E$ and every invariant measure with support contained in $C \times E$ has support equal to $C \times E$.

This is also true for the discrete-time process $(\tilde{Z}_n)$.

Theorem

Assume that for some x in a compact invariant control set C the Lie algebra $\mathcal{LA}(F^0, \dots, F^m)$ has full rank at x.

Then there is a unique invariant measure μ with $\mathrm{supp}\mu = C \times E$ (hence μ is ergodic) and there are $c > 0$ and $0 < \rho < 1$ such that for all $(x, i) \in C \times E$ and $A \subset C$

$$|\mathbb{P}_{x,i}[\tilde{Z}_n \in A] - \mu(A)| \leq c\rho^n, n \in \mathbb{N}.$$

An Example: Lotka-Volterra model with hunting and resting

The model:

$$\boxed{\begin{array}{c} \dot{x} = \alpha x \left(1 - \frac{1}{K}\right) x - \beta xy \\ \dot{y} = -\beta xy + \gamma(L - y) \end{array}}$$

$\frac{1}{\beta}$ corresponds to the hunting time of the predator y,
$\frac{1}{\gamma}$ corresponds to the resting time of the predator y,
normalized via $\frac{1}{\beta} + \frac{1}{\gamma} = 1$.
Coexistence and extinction as hunting (and resting) time undergoes random fluctuations.

Horsthemke, Lefever (84), FC, de la Rubia, Kliemann (96)

The Lotka-Volterra model as a PDMP

The model:

$$\boxed{\begin{array}{c}\dot{x} = \alpha x \left(1 - \frac{1}{K}\right) x - \beta xy \\ \dot{y} = -\beta xy + \gamma(\beta)(L - y)\end{array}}$$

with the normalization $\frac{1}{\beta} + \frac{1}{\gamma(\beta)} = 1$. For

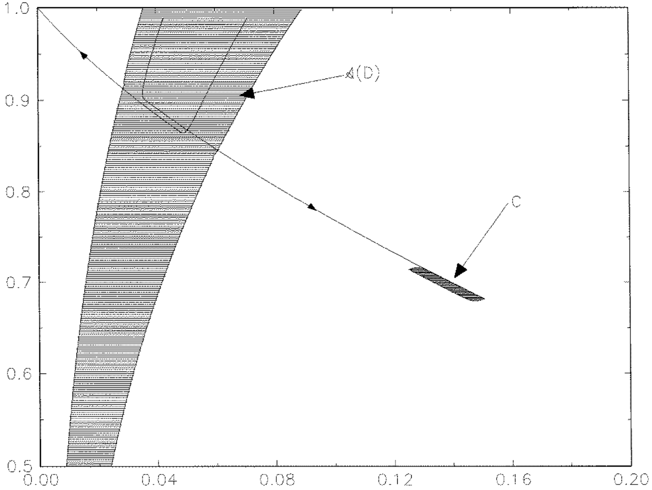
$$K = 0.5, L = 1.0, \alpha = 4.0, \beta > 4.0$$

the rectangle $[0, K] \times [0, L]$ is invariant and the fixed points are $(0, L)$(stable), an unstable and a stable fixed point.

Let β switch randomly between $\beta = 4.1$ and $\beta = 4.2$. Thus $E = \{0, 1\}$,

$$\begin{aligned} F^0 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \alpha x \left(1 - \frac{1}{K}\right) x - 4.1xy \\ -4.1xy + \gamma(4.1)(L - y) \end{bmatrix}, \\ F^1 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \alpha x \left(1 - \frac{1}{K}\right) x - 4.2xy \\ -4.2xy + \gamma(4.2)(L - y) \end{bmatrix} \end{aligned}$$

Two invariant measures with supports given by $\{(0, L)\}$ and C.



1.0
0.9
0.8
0.7
0.6
0.5
0.00
0.04
0.08
0.12
0.16
0.20
A(D)
C

Other (more realistic) Lotka-Volterra systems with random switching have been analyzed in detail by

Benaïm and Lobry, Lotka-Volterra with randomly fluctuating environments or “How switching between beneficial environments can make survival harder”,

Annals of Applied Probability (2016).

PDMP for a particle in a double well potential

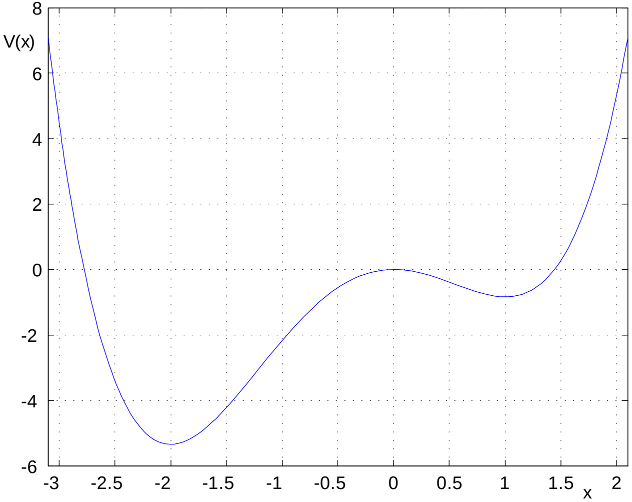
Consider

$$\ddot{x} + \gamma x + \frac{dV}{dx}(x) = 0$$

with

$$V(x) = \frac{1}{2}x^4 + \frac{2}{3}x^3 - 2x^2 \pm \rho x$$

PDMP for a particle in a double well potential



$V(x)$ with $\rho = 0$

PDMP with a double well potential

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\gamma\; y - x(2x^2 + 2\,x - 4) \pm \rho\end{aligned}$$

with $\gamma = 0.1$ and random switching between $\pm\rho$. Here $E = \{0, 1\}$ and

$$\begin{aligned}F^0 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) + \rho \end{bmatrix}, \\ F^1 \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) - \rho \end{bmatrix}.\end{aligned}$$

The associated control system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -\gamma\, y - x\,(2x^2 + 2\,x - 4) \end{bmatrix} + \begin{bmatrix} 0 \\ u(t) \end{bmatrix}, u(t) \in [-\rho, \rho].$$

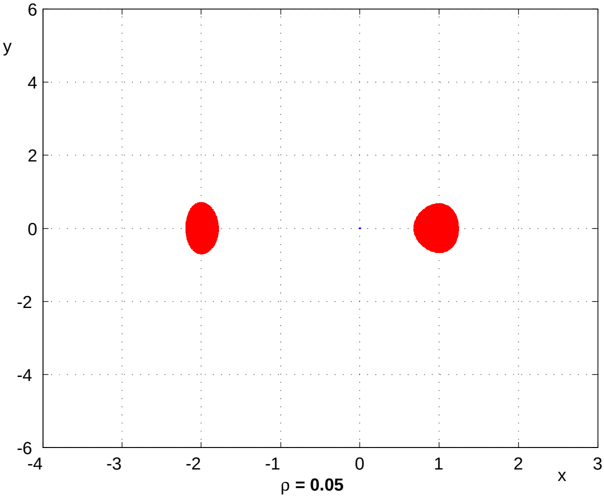
PDMP with a double well potential

For $\rho = 0.05$ there are two invariant control sets $C_1^{0.05}$ and $C_2^{0.05}$ that contain the stable fixed points $(1, 0)$ and $(-2, 0)$, respectively, of the uncontrolled equation and one non-invariant control set $D^{0.05}$ containing the hyperbolic fixed point $(0, 0)$ of the uncontrolled equation.

Increasing the control range, one finds that the control sets $C_1^{\rho_0}$ and D^{ρ_0} merge for some ρ_0 close to 0.085 and form one variant control set.

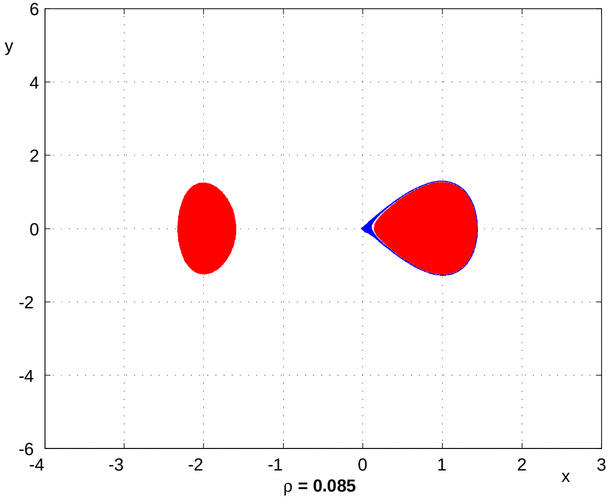
This determines the number of invariant measures for the PDMP and their supports.
Computations: Tobias Gayer with GAIO

Bifurcations: PDMP with a double well potential



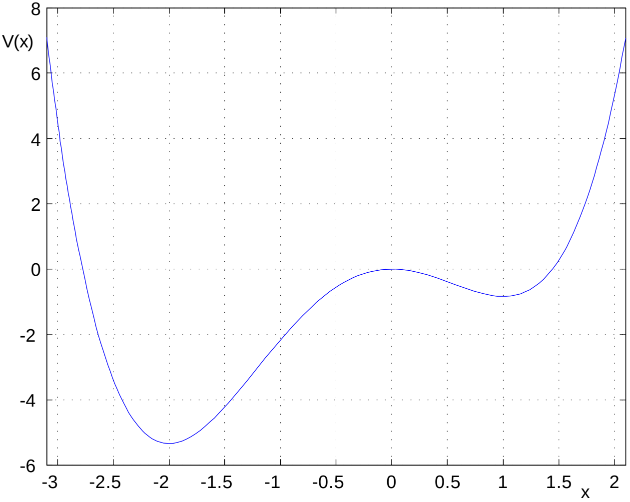
Supports of two invariant measures

PDMP with a double well potential

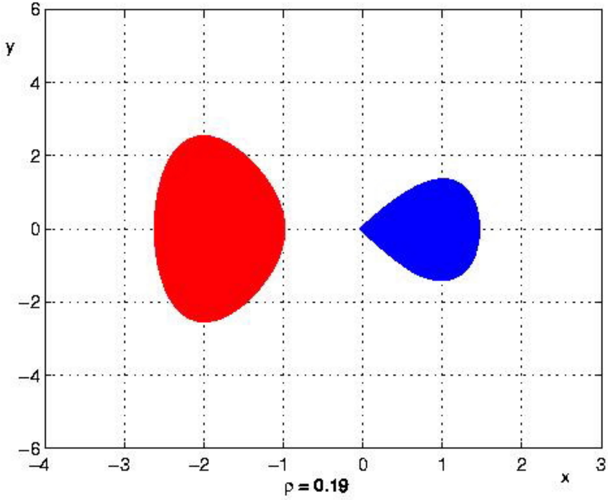


Supports of two invariant measures

PDMP with a double well potential

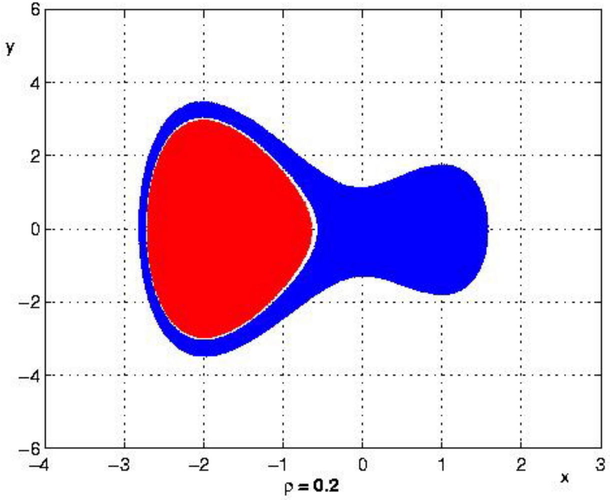


Bifurcations: PDMP with a double well potential



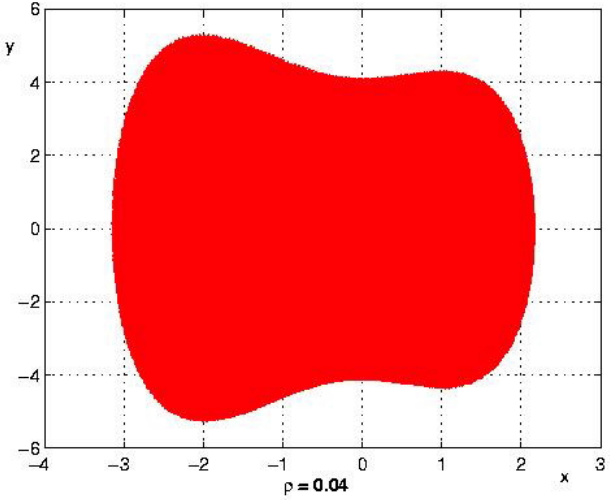
Support of a single invariant measure (in red)

Bifurcations: PDMP with a double well potential



Support of a single invariant measure (in red)

Bifurcations: PDMP with a double well potential



Support of a single invariant measure (in red)

Final Remarks

The concept of control flow allows us to consider the theory of (open loop) control systems as a chapter in the theory of dynamical systems. The control term can also be interpreted as a deterministic perturbation. As a random perturbation, one obtains that for degenerate Markov diffusions and for Piecewise Deterministic Markov Processes (with continuous trajectories) the supports of the invariant measures can be characterized by controllability properties.

In general, PDMP may also allow random jumps. Although control systems allowing discontinuous trajectories have been analyzed in the literature, their controllability properties are apparently unknown.

Imperial College, London
June 2018

Spectral Theory for Bilinear Control Systems

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Universität Augsburg

Introduction

A bilinear control systems has the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} u_i(t) A_i x(t) = A(u)x, \;\; u(t) = (u_i(t))_{i=1,\ldots,m} \in \Omega,$$

with $d \times d$-matrices $A_0, A_1, \ldots, A_m \in \mathbb{R}^{d\times d}$ and compact convex control range $\Omega \subset \mathbb{R}^m$.
We will consider the associated control flow and controllability properties as well as exponential stability properties.
Crucial insight will be gained by analyzing the projection to (real) projective space $\mathbb{P}^{d-1}$.

Different approaches to bilinear control systems can be found e.g. in
D.L. Elliott, Bilinear Control Systems, 2009
San Martin/Seco, Erg.Th.Dyn.Syst.(2010) based on semigroups in Lie groups.

The linear oscillator

The linear oscillator with control/uncertainty in the restoring force:

$$\ddot{x} + b\dot{x} + [1 + u(t)]x = 0, \text{ with } u(t) \in [-\rho, \rho], b = 1.5 > 0.$$

or, in state space form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $u(t) \in [-\rho, \rho]$ and $b > 0$.

The linear control flow

As in the general case, a bilinear control system defines a control flow on $\mathcal{U} \times \mathbb{R}^d$, given by

$$\Phi(t, u, x) = (\theta_t u, \varphi(t, x, u)), t \in \mathbb{R}.$$

The special property of this control flow is its linearity with respect to x,

$$\Phi(t, u, \alpha x + \beta y) = \alpha\Phi(t, u, x) + \beta\Phi(t, u, y), \alpha, \beta \in \mathbb{R}.$$

The state space $\mathcal{U} \times \mathbb{R}^d$ has the structure of a (topologically trivial) vector bundle with compact metric base space $\mathcal{U}$.
Furthermore, we know that the periodic points are dense for the shift θ, hence the base space is chain transitive.

Projective space

Linearity of $\Phi(t,u,x)$ in x immediately implies that one gets an induced flow on $\mathcal{U} \times \mathbb{P}^{d-1}$.
$\mathbb{P}^{d-1}$ may be obtained by identifying opposite points on the unit sphere.
For a solution $x(t) = \varphi(t, x_0, u)$ of $\dot{x} = A(u)x$ one obtains with $s(t) = \frac{x(t)}{\|x(t)\|}$, where $\|x(t)\| = \sqrt{\langle x(t), x(t)\rangle}$,

$$\dot{s}(t) = \left[A(u) - s(t)^T A(u) s(t) \cdot I\right] s(t).$$

In fact,

$$\begin{aligned} \dot{s} &= \frac{\dot{x}\ \|x\| - x\langle \dot{x}, x\rangle / \|x\|}{\|x\|^2} = \frac{A(u)x\ \|x\| - x\langle A(u)x, x\rangle / \|x\|}{\|x\|^2} \\ &= \left[A(u) - s(t)^T A(u) s(t) \cdot I\right] s(t). \end{aligned}$$

Abbreviating $h(s,u) = \left[A(u) - s^T A(u) s \cdot I\right] s$ we can write this as

$$\dot{s}(t) = h(s(t), u(t)) \text{ on } \mathbb{S}^{d-1}.$$

The subtracted term $\left[s^T A(u) s\right] s$ is the radial component of $A(u)s$.

Exponential growth rates

The exponential growth rate or Lyapunov exponent of a solution for (u, x_0) is

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, x_0, u)\|.$$

Somewhat surprisingly, also the Lyapunov exponents are determined by the induced system on projective space,

$$\lambda(u, x_0) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(u(\tau), s(\tau))d\tau \text{ with } q(u, s) := s^\top A(u)s.$$

Selgrade's Theorem

Theorem. Let Φ be a continuous linear flow on on a vector bundle with compact chain transitive base space $\mathcal{U} \times \mathbb{R}^d$. Then the induced flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ has only finitely many chain recurrent components $\mathcal{M}_1, \dots, \mathcal{M}_\ell, 1 \leq \ell \leq d$.

Every $\mathcal{M}_i$ defines an invariant subbundle via

$$\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i) = \{(u, x) \in \mathcal{U} \times \mathbb{R}^d \,|(u, \mathbb{P}x) \in \mathcal{M}_i\}$$

and the following decomposition into a Whitney sum holds

$$\mathcal{U} \times \mathbb{R}^d = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\ell.$$

Example

Consider

$$\dot{x} = Ax.$$

For an eigenvector x corresponding to a real eigenvalue μ of A the point $\mathbb{P}x$ is an equilibrium in $\mathbb{P}^{d-1}$.

More generally, let $\lambda_1, \ldots, \lambda_\ell$ be the pairwise different real parts of the eigenvalues of A and denote by $V(\lambda_i)$ be the direct sum of all generalized eigenspaces for the eigenvalues with real part equal to λ_i. Then the $\mathcal{M}_i := \mathbb{P}V_i$ are the chain recurrent components and

$$\mathbb{R}^d = \bigoplus_{i=1}^{\ell} V(\lambda_i) = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{M}_i.$$

The chain control sets

Corollary. For a bilinear control system $\dot{x} = A(u)x$, $u(t) \in \Omega$, there are $1 \leq \ell \leq d$ chain control sets E_i for the induced system in projective space and there is a Whitney decomposition

$$\mathcal{U} \times \mathbb{R}^d = \bigoplus_{i=1}^{\ell} \mathbb{P}^{-1}\mathcal{E}_i,$$

where the $\mathcal{E}_i$ are the lifts of the chain control sets E_i in $\mathbb{P}^{d-1}$,

$$\mathcal{E}_i = \{(u,p) \in \mathcal{U} \times \mathbb{P}^{d-1} \,|s(t) \in E_i, t \in \mathbb{R}, \text{ for } \dot{s} = h(u,s), s(0) = p\}.$$

Questions:

- Proof of Selgrade's theorem?

- How are the Lyapunov exponents related to the chain control sets?

- Do the chain control sets coincide with the control sets in projective space?

- What about the control sets in $\mathbb{R}^d$?

- Consequences for stability and stabilizability?

On the proof of Selgrade's theorem

This is based on the relation between chain recurrence, Morse decompositions and attractor-repeller pairs.

Recall:
A **Morse decomposition** of a flow is given by $\{\mathcal{M}_i \,|i = 1, \ldots, \ell\}$ with nonvoid, pairwise disjoint and compact isolated invariant sets s.t.
(i) $\forall x \in X : \omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$;
(ii) there are no cycles.

If the number of chain recurrent components is finite, this corresponds to the finest Morse decomposition. In particular, if the number of chain control sets in a compact invariant set is finite, this corresponds to the finest Morse decomposition of the control flow.

Definition. For a flow on a compact metric space X an attractor A is a compact invariant set with a nbhd N such that

$$A = \omega(N) := \{ y \in X \,|\, \exists (x_n) \in N, \exists t_n \to \infty : y = \lim x_n \cdot t_n \}.$$

A compact invariant set R is a repeller if there is a nbhd N^* such that

$$R = \alpha(N^*) := \{ y \in X \,|\, \exists (x_n) \in N^*, \exists t_n \to -\infty : y = \lim x_n \cdot t_n) \}.$$

Proposition. For every attractor,

$$R := \{ x \in X \,|\, \omega(x) \cap A = \varnothing \}$$

is a repeller, called the complementary repeller.

Morse decompositions and attractor-repeller pairs

Theorem. Let $\mathcal{M}_i$, $i = 1, \ldots, n$, be subsets of X. Equivalent are:

(i) $\{\mathcal{M}_i \,|\, i = 1, \ldots, \}$ form a Morse decomposition;

(ii) there is an increasing sequence of attractors

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that $\mathcal{M}_{n-i} = A_{i+1} \cap A_i^*$ for $0 \leq i \leq n-1$.

Example.

$$\dot{x} = x(x-1)(x-2)^2(x-3).$$

A Morse decomposition is given by

$$\mathcal{M}_1 = \{0\} \preceq \mathcal{M}_3 = \{1\} \succeq \mathcal{M}_2 = [2,3].$$

Here $n = 3, A_0 = \varnothing, A_0^* = [0,3], A_1 = \{1\}, A_1^* = \{0\} \cup [2,3]$, $A_2 = [1,3], A_2^* = \{0\}, A_3 = [0,3], A_3^* = \varnothing$ and

$$A_1 \cap A_0^* = \{1\} = \mathcal{M}_3, A_2 \cap A_1^* = [2,3] = \mathcal{M}_2, A_3 \cap A_2^* = \{0\} = \mathcal{M}_1.$$

Proof of Selgrade's theorem

Steps of the proof: Show that

- an attractor for the projectivized flow $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}^{d-1}$ defines a (linear!) subbundle of $\mathcal{U} \times \mathbb{P}^{d}$.

- an attractor-repeller pair defines an invariant subbundle decomposition for the linear flow Φ on $\mathcal{U} \times \mathbb{R}^{d}$.

- then one can use the dimension of the subbundles to show that there is a finest Morse decomposition into Morse sets $\mathcal{M}_i$, hence

- this are the chain recurrent components in $\mathcal{U} \times \mathbb{P}^{d}$

- defining a decomposition of $\mathcal{U} \times \mathbb{R}^{d}$ into invariant subbundles $\mathcal{V}_i := \mathbb{P}^{-1}(\mathcal{M}_i)$.

The Morse spectrum of the bilinear system I

Recall: For $\varepsilon, T > 0$ an (ε, T)-chain ζ in $\mathcal{U} \times \mathbb{P}^{d-1}$ is given by

$$n \in \mathbb{N}, T_0, T_1, \ldots, T_{n-1} > T, (u_0, p_0), \ldots, (u_n, p_n) \in \mathcal{U} \times \mathbb{P}^{d-1}$$

such that

$$d(\mathbb{P}\Phi(T_i, (u_i, p_i)), (u_{i+1}, p_{i+1})) < \varepsilon \text{ for all } i.$$

With $\mathbb{P}x_i = p_i$ the chain exponent of ζ is

$$\lambda(\zeta) = \left(\sum_{i=1}^{n-1} T_i \right)^{-1} \sum_{i=1}^{n-1} \left(\log \| \varphi(T_i, x_i, u_i) \| - \log \| x_i \| \right),$$

The **Morse spectrum** is

$$\textstyle\sum_{Mo} = \{ \lambda \in \mathbb{R} \,|\, \exists \varepsilon_n \to 0, \exists T_n \to \infty, (\varepsilon_n, T_n)\text{-chains } \zeta_n : \lim \lambda(\zeta_n) = \lambda \}.$$

The Morse spectrum of the bilinear system II

Results:

(i) $\sum_{Mo} = \bigcup_{i=1}^{\ell} \sum_{Mo}(\mathcal{M}_i)$

(ii) Each $\sum_{Mo}(\mathcal{M}_i)$ consists of a closed interval $[\kappa_i^*, \kappa_i]$.

(iii) For $i < j$ we have $\kappa_i^* < \kappa_j^*$ and $\kappa_i < \kappa_j$.

(iv) $\sum_{Ly} \subset \sum_{Mo}$ and the κ_i^*, κ_i are actually Lyapunov exponents.

(Un)stable subbundle

The upper spectral interval $\Sigma_{Mo}(\mathcal{M}_\ell) = [\kappa_\ell^*, \kappa_\ell]$ determines the robust stability of $\dot{x} = A(u(t))x$ (and stabilizability of the system if the set $\mathcal{U}$ is interpreted as a set of admissible control functions).

The **stable, center, and unstable subbundles** of $\mathcal{U} \times \mathbb{R}^d$ are defined as

$$L^- = \bigoplus_{j:\ \kappa_j < 0} \mathcal{V}_j,\ \ L^0 = \bigoplus_{j:\ 0 \in [\kappa_j^*, \kappa_j]} \mathcal{V}_j, L^+ = \bigoplus_{j:\ \kappa_j^* > 0} \mathcal{V}_j.$$

Corollary. The zero solution of $\dot{x} = A(u(t))x, u \in \mathcal{U}$, is exponentially stable for all $u \in \mathcal{U}$ iff $\kappa_\ell < 0$ iff $L^- = \mathcal{U} \times \mathbb{R}^d$.

Stability radius

Suppse that $0 \in \mathrm{int}\Omega$ and consider the control ranges $\Omega^\rho = \rho\Omega$.

The maximal spectral value $\kappa_\ell(\rho)$ is continuous in ρ and we define the (asymptotic-) stability radius of this family as

$$\begin{aligned} r &= \inf\{\rho \geq 0 \,|\, \exists u \in \mathcal{U}^\rho : \dot{x}^\rho = A(u(t))x^\rho \text{ is not exp.stable}\} \\ &= \inf\{\rho \geq 0 \,|\, \kappa_\ell(\rho) > 0\} \end{aligned}$$

The linear oscillator

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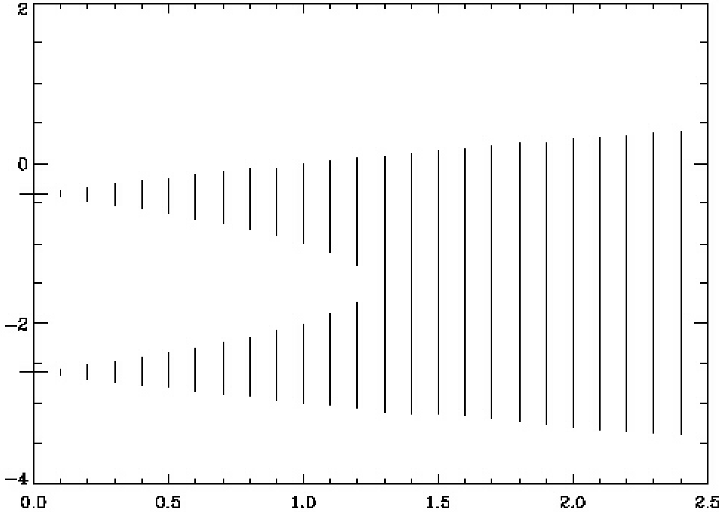
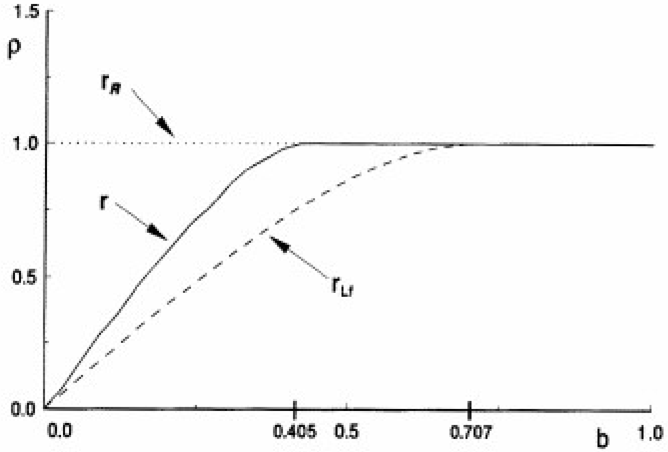
with $u(t) \in [-\rho, \rho]$ and $b > 0$. (For $b \leq 0$ the system is unstable even for constant perturbations.)

Spectral intervals for the linear oscillator





1.5
ρ
r_R
1.0
r
0.5
r_{Lf}
0.0
0.0
0.405
0.5
0.707
b
1.0

Control sets and chain control sets

Assume that the Lie algebra rank condition for the system on $\mathbb{P}^{d-1}$ holds.

Every chain control set contains a control set $\mathbb{P}D_j$, hence there are $0 < k \leq \ell \leq d$ control sets $\mathbb{P}D_j$ with nonvoid interior in $\mathbb{P}^{d-1}$.

Exactly one of them is an invariant control set.

Control sets and chain control sets

Suppose that for $\rho < \rho'$, i.e. for increasing control ranges $\rho\Omega \subset \rho'\Omega$, the reachable sets in $\mathbb{P}^{d-1}$ are strictly increasing.

Then for all up to at most $d-1$ ρ-values the closures of the control sets are the chain control sets and the spectral growth rates satisfy

$$\sum\nolimits_{Ly}(\mathbb{P}D_j) = \sum\nolimits_{Mo}(E_j).$$

The control sets of the bilinear system in $\mathbb{R}^d$ are exactly those cones over the control sets $\mathbb{P}D_j$ for which $0 \in \left(\kappa_j^*, \kappa_j\right)$.

The bilinear system is completely controllable in $\mathbb{R}^d \setminus \{0\}$ iff the projected system is completely controllable and $0 \in \left(\kappa^*, \kappa\right)$.

An example: control sets vs chain control sets

Consider the bilinear control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + u_1(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u_2(t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\Omega = [0, \frac{1}{2}] \times [1, 2]$. For $u_1 \equiv 0,\ u_2 \equiv 1$ we have a double eigenvalue $\lambda_{1,2} = 1$ of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with eigenspace $\mathbb{R}^2$, hence there is a single chain control set $E = \mathbb{P}^1$.

There are two control sets in $\mathbb{P}^1$ given by

$$D_1 = \left(-\frac{\pi}{4}, 0 \right) \text{ and } D_2 = \left[\frac{\pi}{4}, \frac{\pi}{2} \right].$$

Concluding remarks

Bilinear control systems may be viewed as linear flows on vector bundles.

Their topological analysis via chain transitivity, Morse decompositions and attractors leads to a spectral theory which allows us to find results on controllability and stability. There are further results on stabilizability by (time varying) feedbacks.

Imperial College, London
June 2018

Control under Communication Constraints and Invariance Entropy

Fritz Colonius
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Goal

Determine **fundamental limitations** in control

Here: Describe the "information" needed to make a subset invariant for a control system

A recent survey on various definitions and application areas of **entropy** is Amigó et al. DCDS B (2015).

Classically, entropy is used in dynamical systems theory in order to describe the information generated by the systems and to classify them.

Control systems:

> **Delchamps** (1990) (ergodic theory for quantized feedback)

Topological versions have been analyzed, in particular, by

> **Nair, Evans, Mareels and Moran** (2004)
>
> **Kawan**, Springer LNM Vol. 2089 (2013)

Control systems

We consider control system in discrete time given by

$$x_{n+1} = f(x_n, u_n), n \in \mathbb{N} = \{0, 1, ...\},$$

where $f : M \times \Omega \to M$ is continuous and M and Ω are metric spaces. The solution with $x_0 = x$ and $u = (u_n) \in \mathcal{U} := \Omega^N$ is denoted by $\varphi(n, x, u)$, $n \in \mathbb{N}$.

We assume that for every $x \in Q \subset M$ there is $u(x) \in \Omega$ with $f(x, u(x)) \in Q$.
What is the "**information**" necessary to keep the system in Q?

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What is the "**information**" necessary to keep the system in Q?

Motivation: Suppose that the present state x_n of the system is measured. If the controller has complete information about the present state, it can adjust a feedback control $u(x)$ appropriately. However, if the measurement is sent to the controller via a (noiseless) digital channel with bounded data rate it is of interest to determine the minimal data rate needed to make Q invariant. More abstractly: What is the minimal average information needed to make Q invariant?

Overview

This talk consists of three parts:

- Some motivation from classical entropy of dynamical systems

- Topological and measure-theoretic invariance entropy for control systems

- Relations to controllability properties

Topological entropy for dynamical systems

Let $T : X \to X$ be a continuous map on a compact metric space.
Suppose $\mathcal{B}$ is a finite open cover of X, i.e., the sets in $\mathcal{B}$ are open, their union is X.
For an **itinerary** $\alpha = (B_0, B_1, ..., B_{n-1}) \in \mathcal{B}^n$ let

$$\mathcal{B}_n(\alpha) = \{x \in X \,|\, T^j(x) \in B_j \text{ for } j = 0, .., n-1\} = B_0 \cap .. \cap T^{-(n-1)} B_{n-1}.$$

They again form an open cover of X,

$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$.

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$$\mathfrak{B}^{(n)} = \{\mathcal{B}_n(\alpha) \,|\alpha \in \mathcal{B}^n\}.$$

Denote the minimal number of elements of a subcover by $N(\mathfrak{B}^{(n)})$. Then the entropy of $\mathcal{B}$ is given by

$$h(\mathcal{B}, T) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)})$$

and the **topological entropy** of T is

$$h_{top}(T) = \sup_{\mathcal{B}} h(\mathcal{B}, T).$$

Adler, Konheim, McAndrew (1965)

A classical example

Consider the **logistic map** on the interval $X = [0,1]$ given by

$$F_4(x) = 4x(1-x), x \in [0,1].$$

The topological entropy of F_4 is

$$h_{top}(F_4) = \log_2 2 = 1 > 0.$$

Hence this is a **chaotic map**.

Metric entropy for dynamical systems

For a probability measure μ and a partition $\mathcal{P}$ of X the **Shannon entropy** is

$$H_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P).$$

Let μ be invariant for a map S on X, i.e., $\mu(S^{-1}B) = \mu(B)$ for all $B \subset X$.

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They yield a partition $\mathcal{P}^{(n)} = \{P_n(\alpha)\,|\alpha \in \mathcal{P}^n\}$ and

$$h_\mu(\mathcal{P}, S) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\mathcal{P}^{(n)}\right).$$

The **Kolmogorov-Sinai entropy** of S is

$$h_\mu(S) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}, S).$$

The logistic map again

Recall

$$F_4(x) = 4x(1-x) \text{ on } [0,1].$$

A (trivial) invariant measure is $\mu = \delta_0$ with entropy $h_{\delta_0}(F_4) = 0$.
A nontrivial invariant measure is given by its density (with respect to Lebesgue measure)

$$\frac{1}{\pi\sqrt{x(1-x)}}, x \in [0,1].$$

The corresponding **metric entropy** is

$$h_\mu(F_4) = \log_2 2 = 1$$

(hence equal to the topological entropy).

The **Variational Principle** states that

$$\sup_{\mu} h_{\mu}(T) = h_{top}(T)$$

and invariant measures μ with maximal entropy, i.e., $h_{\mu}(T) = h_{top}(T)$, are of special relevance.

Often, entropy can be characterized by (the positive) **Lyapunov exponents**.

Invariance entropy for control systems

Describe the **minimal information** to make a compact $Q \subset M$ invariant for

$$x_{n+1} = f(x_n, u_n), \ \ u_n \in \Omega,$$

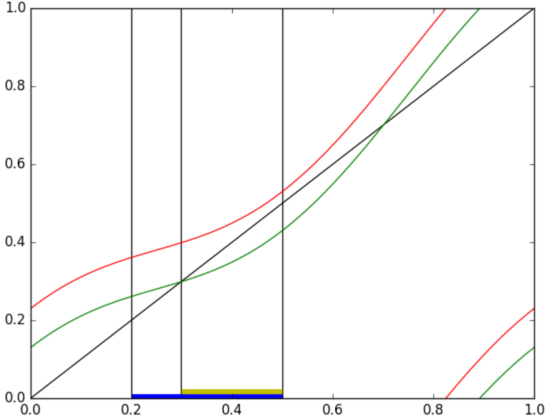
with solutions $\varphi(n, x_0, u), n \in \mathbb{N}$, in M.

This can be done in a topological or in a measure-theoretic framework. Topological invariance entropy is based on **itineraries in Q** corresponding to **invariant open covers** of Q. They are constructed by feedbacks keeping the system in Q and replace the open covers.

Observe: This is not directly related to the entropy of the uncontrolled system which may behave very wildly in Q, while Q itself may be invariant. Hence the entropy of the dynamical system may be positive while the invariance problem is trivial.

$$f_\alpha(x,\omega) = x + \sigma\cos(2\pi x) + A\omega + \alpha \mod 1,\ \omega \in \Omega = [-1,1].$$

With $A = 0.05, \sigma = 0.1, \alpha = 0.08$ consider the set $Q = [0.2, 0.5]$.



An **invariant open cover** $\mathcal{C}_\tau = (\mathcal{B}, F)$ is given by $\tau \in \mathbb{N}$, an open cover $\mathcal{B}$ of Q and $F : \mathcal{B} \to \Omega^\tau$ with

$$\varphi(j, B, F(B)) \subset \mathrm{int}Q \text{ for } j = 1, ..., \tau \text{ and } B \in \mathcal{B}.$$

Topological invariance entropy for control systems

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For a **$\mathcal{C}_\tau$-itinerary** $\alpha = (B_0, ..., B_{n-1}) \in \mathcal{B}^n$ define $u_\alpha = (F(B_0), F(B_1), ...)$ and

$$B_n(\alpha) = \{x \in Q \,|\, \varphi(i\tau, x, u_\alpha) \in B_i \text{ for } i = 0, ..., n-1\}.$$

These sets again form an open cover of Q,

$$\mathfrak{B}^{(n)} = \{B_n(\alpha) \,|\, \alpha \in \mathcal{B}^n\}.$$

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These sets again form an open cover of Q,

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The invariance entropy of $\mathcal{C}_\tau$ is

$$h(\mathcal{C}_\tau, Q) := \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{B}^{(n)} \,|\, Q)$$

and the **topological invariance entropy** of Q is

$$h_{top}^{inv}(Q) := \inf_{\mathcal{C}_\tau(\mathcal{B},F)} h(\mathcal{C}_\tau, Q).$$

Relations to data rates and coder-controllers

A **coder-controller** has the form $\mathcal{H} = (S, \gamma, \delta, \tau)$ where
- $S = (S_k)_{k \in \mathbb{N}}$ denotes finite coding alphabets
- the coder mapping $\gamma_k : M^{k+1} \to S_k$ associates to the present and past states the symbol $s_k \in S_k$
- at time $k\tau$ the controller mapping is $\delta_k : S_0 \times \cdots \times S_k \to \Omega^\tau$.

The **transmission data rate** is

$$R(\mathcal{H}) = \liminf_{k \to \infty} \frac{1}{k\tau} \sum_{j=0}^{k-1} \log \# S_j .$$

$\mathcal{H}$ renders Q invariant if for every $x_0 \in Q$ the sequence

$$x_{k+1} := \varphi(\tau, x_k, u_k), k \in \mathbb{N}, \tag{1}$$

with

$$u_k = \delta_k(\gamma_0(x_0), \gamma_1(x_0, x_1), \ldots, \gamma_k(x_0, x_1, \ldots, x_k)) \in \Omega^\tau \tag{2}$$

satisfies

$$\varphi(i, x_k, u_k) \in Q \text{ for all } i \in \{1, \ldots, \tau\} \text{ and all } k \in \mathbb{N}. \tag{3}$$

The data rate theorem

Theorem. For a compact and controlled invariant set Q it holds that

$$h_{inv}^{top}(Q) = \inf R(\mathcal{H}),$$

where the infimum is taken over all coder-controllers $\mathcal{H}$ that render Q invariant.

Comments and some further results

- Let $K \subset Q$ be compact. Then one can define (using spanning sets of controls) the invariance entropy $h_{inv}(K, Q)$ of K with respect to Q.

- For **linear control systems** in $\mathbb{R}^d$

$$x_{n+1} = Ax_n + Bu_n, u_n \in \Omega \subset \mathbb{R}^m,$$

with $\mathrm{int}K \neq \emptyset$ and (A, B) controllable, A hyperbolic and Ω a compact nbhd of 0, one has for K contained in the unique control set D

$$h_{top}^{inv}(K, D) = \sum\nolimits_{\lambda \in \sigma(A)} \max(0, \log |\lambda|).$$

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- hyperbolicity of the control flow on $\mathcal{U} \times Q$ gives a formula in terms of Lyapunov exponents for periodic solutions

Kawan (2014),

- also for linear control systems on Lie groups

da Silva (2014)

Comments and some further results

DA SILVA AND KAWAN, DISC.CONT.DYNAM.SYST. (2016):

Theorem. Consider a uniformly hyperbolic chain control set E with nonempty interior of a control-affine continuous-time system. Assume that
(i) the Lie Algebra Rank Condition holds on $\mathrm{int}E$ and
(ii) for each $u \in \mathcal{U}$ there exists a unique $x \in E$ with $(u, x) \in \mathcal{E}$, i.e., $\mathcal{E}$ is a graph over $\mathcal{U}$.
Then E is the closure of a control set D and for every compact set $K \subset D$ with positive volume,

$$h_{inv}(K, D) = \inf_{(u,x)\in\mathcal{E}} \limsup_{t\to\infty} \log J^{+}\varphi_{t,u}(x)$$

where $J^{+}\varphi_{t,u}(x)$ is the unstable determinant of $d\varphi_{t,u}(x)$.

Invariance pressure

Introduce a potential $f \in C(\Omega, \mathbb{R})$ for the control values.
Let $K \subset Q$ be compact s.t. $\forall x \in K\ \exists u \in \mathcal{U} : \varphi(\mathbb{R}_+, x, u) \subset Q$.
A set $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)-spanning set if

$$\forall x \in K\ \exists u \in \mathcal{S} : \varphi([0,\tau], x, , u) \subset Q.$$

With $(S_\tau f)(u) := \int_0^\tau f(u(t))dt$ let

$$a_\tau(f, K, Q) := \inf\{\sum_{u \in \mathcal{S}} e^{(S_\tau f)(u)};\ \mathcal{S} \text{ is } (\tau, K, Q)\text{-spanning}\}.$$

The **invariance pressure** is

$$P_{inv}(f, K, Q) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q).$$

If $f \equiv 0$, $\sum_{u \in \mathcal{S}} e^{(S_\tau f)(u)} = \#\mathcal{S}$. Then this reduces to a known characterization of the invariance entropy.

Invariance pressure for linear control systems

Consider a **linear control systems** in $\mathbb{R}^d$

$$\dot{x} = Ax + Bu, u(t) \in \Omega \subset \mathbb{R}^m,$$

with a compact neighborhood Ω of 0 and assume (A, B) controllable, A hyperbolic.
For $K \subset D$, the unique control set with $\mathrm{int} D \neq \varnothing$, one has:

$$P_{inv}(f, K, D) \leq \sum_{\lambda \in \sigma(A)} \max(0, \mathrm{Re}\, \lambda) + \inf_{T, u(\cdot)} \frac{1}{T} \int_0^T f(u(s)) ds,$$

where the infimum is taken over all $T > 0$ and all T-periodic controls $u(\cdot)$ with values in a compact subset of $\mathrm{int}\Omega$ and a T-periodic $x(\cdot) \subset \mathrm{int} D$.

Measure-theoretic invariance entropy for control systems

Describe the **minimal information** to make a compact $Q \subset M$ invariant for

$$x_{n+1} = f(x_n, u_n), \;\; u_n \in \Omega,$$

with solutions $\varphi(n, x_0, u)$, $n \in \mathbb{N}$, in M.

We will need **itineraries in Q** corresponding to **invariant partitions** of Q. They will be constructed by feedbacks keeping the system in Q and replace the partitions.

Measure-theoretic invariance entropy: the ingredients

We need

- partitions and itineraries in Q for a map S

- a probability measure (quasi-stationary)

- a notion of entropy

Let η be a probability measure on Q.
An **invariant partition** $\mathcal{C}_\tau = (\mathcal{P}, F)$ is given by $\tau \in \mathbb{N}$, a partition $\mathcal{P}$ of Q and $F : \mathcal{P} \to \Omega^\tau$ such that for $P \in \mathcal{P}$

$$\varphi(j, x, F(P)) \in Q \text{ for } j = 1, ..., \tau \text{ and } \eta\text{-a.a. } x \in P.$$

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$$\varphi(j, x, F(P)) \in Q \text{ for } j = 1, ..., \tau \text{ and } \eta\text{-a.a. } x \in P.$$

Define

$$A(P) = \{u \in \mathcal{U} \,|\, \varphi(j, x, u) \in Q,\, j = 1, ..., \tau, \eta\text{-a.a. } x \in P\} \times P \subset \mathcal{U} \times Q$$

and

$$\mathfrak{A} = \mathfrak{A}(\mathcal{C}_\tau) = \{A(P) \,|\, P \in \mathcal{P}\}.$$

Then $\mathfrak{A}$ consists of pairwise disjoint subsets in $\mathcal{U} \times Q$.

Let the shift θ on $\mathcal{U} = \Omega^{\mathbb{N}_0}$ be $(\theta u)_n := u_{n+1}, n \in \mathbb{N}$. The control system

$$x_{n+1} = f(x_n, u_n), \; u_n \in \Omega,$$

is described by the **control flow** given by S on $\mathcal{U} \times M$ and its iterations,

$$S(u, x) := (\theta u, f(x, u_0)) \text{ for } u = (u_n) \in \mathcal{U} \text{ and } x \in M.$$

Then

$$S^n(u, x) = (\theta^n u, \varphi(n, x, u)), n \in \mathbb{N}.$$

We are interested in the restriction

$$S_Q : \mathcal{U} \times Q \to \mathcal{U} \times M.$$

A sequence $\alpha = (A(P_0), ..., A(P_{n-1}))$ is called an **itinerary** if for $u_\alpha := (F(P_0), F(P_1), ..., F(P_{n-1}))$

$$\eta\{x \in Q \,|\, \varphi(j\tau, x, u_\alpha) \in P_j, j = 0, 1, ..., n-1\} > 0.$$

Construction of metric invariance entropy: itineraries

A sequence $\alpha = (A(P_0), \ldots, A(P_{n-1}))$ is called an **itinerary** if for $u_\alpha := (F(P_0), F(P_1), \ldots, F(P_{n-1}))$

$$\eta\{x \in Q \,|\, \varphi(j\tau, x, u_\alpha) \in P_j, j = 0, 1, \ldots, n-1\} > 0.$$

Let

$$A(\alpha) = A(P_0) \cap S^{-\tau} A(P_1) \cap \cdots \cap S^{-(n-1)\tau} A(P_{n-1}) \subset S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$$

be the set of all (u, x) following this itinerary and

$$\mathfrak{A}^{(n)} = \{A(\alpha) \,|\, \alpha \text{ an itinerary of length } n\}.$$

Then $\mathfrak{A}^{(n)}$ consists of pairwise disjoint sets in $S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$.

Choice of the probability measure

For

$$x_{n+1} = f(x_n, u_n), u_n \in \Omega, \;\; n \in \mathbb{N} = \{0, 1, ...\},$$

let ν be a probability measure on Ω and define Markov transition probabilities by

$$p(x, B) := \nu\{\omega \in \Omega \,|\, f(x, \omega) \in B\}, B \subset M.$$

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$$p(x, B) := \nu\{\omega \in \Omega \,|\, f(x, \omega) \in B\}, B \subset M.$$

Let η be a **quasi-stationary measure**, i.e. a probability measure on $Q \subset M$ with

$$\rho \cdot \eta(B) = \int_Q p(x, B)\eta(dx) \text{ for } B \subset Q,$$

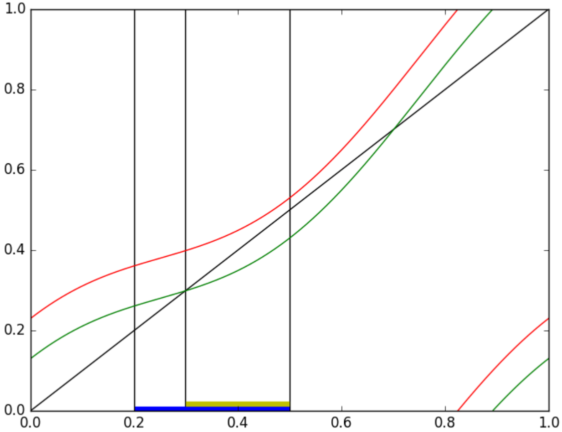
with $\rho := \int_Q p(x, Q)\eta(dx) \in (0, 1)$. η is stationary iff $\rho = 1$.

The measure $\mu := \nu^{\mathbb{N}} \times \eta$ on $\mathcal{U} \times Q$ is a **conditionally invariant measure** for the control flow S.

Collett, Martinez, San Martin (2013), Méléard, Villemonais (2012)

Pianigiani and Yorke (1979), Demers and Young (2006)

$$f(x,\omega) = x + \sigma\cos(2\pi x) + A\omega + \alpha \bmod 1, \omega \in \Omega = [-1,1], Q = [0.2, 0.5].$$



For the uniform distribution ν on $\Omega = [-1,1]$ one can prove that there is a quasi-stationary measure η for Q.

The entropy notion

Recall that $\mathfrak{A}^{(n)}$ is the collection of all sets

$$A_n(\alpha) = A(P_0) \cap S^{-\tau} A(P_1) \cap \cdots \cap S^{(n-1)\tau} A(P_{n-1}) \subset S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$$

consisting of the pairs (u, x) following an **itinerary** $\alpha = (P_0, ..., P_{n-1}) \in \mathcal{P}^n$.

It *does not work* to use the Shannon entropy of $\mathfrak{A}^{(n)}$ w.r.t. μ

$$H_\mu(\mathfrak{A}^{(n)}) = -\sum_\alpha \mu(A_n(\alpha)) \log \mu(A_n(\alpha)),$$

since η is only quasi-stationary with constant $\rho \in (0, 1)$ and $\mu = \nu^{\mathbb{N}} \times \eta$.

Construction of metric invariance entropy

Since $\rho^{-1} \cdot \mu$ is a probability measure on $S_Q^{-1}(\mathcal{U} \times Q)$ consider

$$H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau))$$

for the partition $\mathfrak{A}^{(n)}(\mathcal{C}_\tau)$ in $S_Q^{-(n-1)\tau}(\mathcal{U} \times Q)$ and then take the average of the required information as time tends to ∞ to get

$$h(\mathcal{C}_\tau, Q) = \limsup_{n\to\infty} \frac{1}{n\tau} H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau)).$$

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$$h(\mathcal{C}_\tau, Q) = \limsup_{n\to\infty}\frac{1}{n\tau}H_{\rho^{-(n-1)\tau}\mu}(\mathfrak{A}^{(n)}(\mathcal{C}_\tau)).$$

Define the **metric invariance entropy** for the control system as

$$\boxed{h_\eta^{inv}(Q) := \limsup_{\tau\to\infty}\inf_{\mathcal{C}_\tau} h(\mathcal{C}_\tau, Q),}$$

where the infimum is taken over all invariant partitions $\mathcal{C}_\tau(\mathcal{P}, F)$.

Relation to topological invariance entropy

Theorem. For every quasi-stationary measure η on Q the η-invariance entropy is bounded by the topological invariance entropy,

$$\boxed{h_{\eta}^{inv}(Q) \leq h_{top}^{inv}(Q).}$$

Note that metric entropy is **invariant** under appropriate **conjugacies**.

Conjugacies

The metric entropy is **invariant** under appropriate **conjugacies** preserving the measure: Consider

$$x_{n+1} = f_1(x_n, u_n) \text{ and } y_{n+1} = f_2(y_n, u_n) \text{ with } (u_n) \in \mathcal{U}.$$

Let μ_1 and μ_2 be conditionally invariant measures for Q_1 and Q_2, resp. A bimeasurable bijection $\pi : Q_1 \to Q_2$ is a **conjugacy**, if

$$\pi\varphi_1(n, x_0, u) = \varphi_2(n, \pi x_0, u) \text{ for all } n \geq 0$$

and $\mathrm{id}_{\mathcal{U}} \times \pi : \mathcal{U} \times Q_1 \to \mathcal{U} \times Q_2$ maps μ_1 onto μ_2, i.e.,

$$\mu_1\left((\mathrm{id}_{\mathcal{U}} \times \pi)^{-1}(B)\right) = \mu_2(B) \text{ for all } B \in \mathcal{B}(\mathcal{U} \times Q_2).$$

Then

$$h^{inv}_{\mu_1}(Q_1, S_1) = h^{inv}_{\mu_2}(Q_2, S_2).$$

Invariance Entropy and Controllability Properties

For **dynamical systems** it is well known that the metric and the topological entropy are already determined on the recurrent set.

What about invariance entropy?

For **control systems** recurrence properties are replaced by controllability properties.

Here subsets of complete approximate controllability (in Q) are of relevance, called **control sets**. They are analogous to communicating classes.

W-control sets

For an **open** subset W of the state space let $\varphi_W(n,x,u)$ be the trajectories within W and define the **reachable and controllable set within W** by

$$\begin{aligned}
\mathcal{R}_W(x) &= \{\varphi_W(n,x,u) \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\} \\
\mathcal{C}_W(x) &= \{y \in W \,|\, \varphi_W(n,y,u) = x \text{ for some } n \in \mathbb{N} \text{ and } u \in \mathcal{U}\}.
\end{aligned}$$

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Definition. A set D is called an **invariant W-control set** if
(i)

$$\overline{D}^W = \overline{\mathcal{R}_W(x)}^W \text{ for all } x \in D,$$

where the closure is taken with respect to W and
(ii) there is $x \in D$ with $x \in \mathrm{int}\mathcal{C}_W(x)$.

Remark. Condition (ii) is crucial for discrete-time systems.

Existence of invariant W-control sets

Theorem. Assume
- the state space M is a connected analytic Riemannian manifold
- $W \subset M$ is connected open and relatively compact
- the control range $\Omega \subset \overline{\mathrm{int}\Omega} \subset \mathbb{R}^m$ and $f : M \times \Omega \to M$ is analytic
- $\Omega_{sub} := \{\omega \in \Omega \,|\, f(\cdot, \omega)$ is submersive$\}$ is the complement of a proper analytic subset.

Then the following are equivalent:

(i) There are at least one and at most finitely many **invariant W-control sets** D and for every $x \in W$ there is D with

$$\mathcal{R}_W(x) \cap D \neq \emptyset.$$

(ii) There is a compact set $F \subset W$ with

$$F \cap \overline{\mathcal{R}_W(x)} \neq \emptyset \text{ for all } x \in W.$$

Albertini and Sontag (1993), Wirth (1998), Patrão and San Martin (2007)

Invariance entropy and W-control sets

Theorem. Under the assumptions of (i) in the previous theorem let $Q := \overline{W} \subset M$. Assume

(i) for the finitely many invariant W-control sets D_i

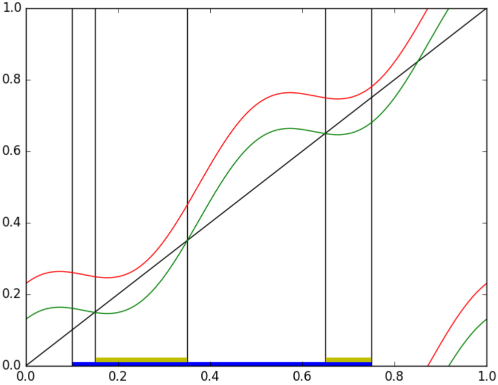
$$f(\bigcup_i \overline{D_i}, \Omega) \cap (\partial Q \setminus \bigcup_i \overline{D_i}) = \emptyset.$$

(ii) the maps $f(\cdot, \omega) : M \to M, \omega \in \Omega$, are nonsingular for a quasi-stationary measure η (i.e. preimages of null sets are null sets). Then

$$\boxed{h_\eta^{inv}(Q) = h_\eta^{inv}(\bigcup_i \overline{D_i}).}$$

Remark. In the continuous-time case a similar result for the topological invariance entropy has been shown in FC/Lettau (2016).

$$f_\alpha(x,\omega) = x + \sigma\cos(4\pi x) + A\omega + \alpha \mod 1.$$



Two W-control sets D_1 and D_2 (to the right) in $W = (0.1, 0.7)$. The invariance entropies for η on $Q = [0.1, 0.7]$ and on $\overline{D_2}$ coincide.

Final remarks

Classical entropy of dynamical systems describes the **total information** generated by the system topologically or with respect to an **invariant measure**.

In contrast, entropy for control systems describes the **minimal information** for invariance either topologically or with respect to a **quasi-stationary measure**.

The data rate theorem relates the topological invariance entropy to the minimal bit rate needed for invariance.

There are further results for other control problems, e.g. for stabilization or state estimation.

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