DECAY RATES FOR STABILIZATION OF LINEAR CONTINUOUS-TIME SYSTEMS WITH RANDOM SWITCHING

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Abstract. For a class of linear switched systems in continuous time a controllability condition implies that state feedbacks allow to achieve almost sure stabilization with arbitrary exponential decay rates. This is based on the Multiplicative Ergodic Theorem applied to an associated system in discrete time.

Key words. Random switching, almost sure stabilization, arbitrary rate of convergence, Lyapunov exponents, persistent excitation, Multiplicative Ergodic Theorem

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1. Introduction.

Let N be a positive integer and consider the family of N control systems

$$\dot{x}_i(t) = A_i x_i(t) + \alpha_i(t) B_i u_i(t), \qquad i \in \{1, \dots, N\}, \tag{1.1}$$

where, for $i \in \{1, \dots, N\}$, $x_i(t) \in \mathbb{R}^{d_i}$ is the state of the subsystem i, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input of the subsystem i, d_i and m_i are non-negative integers, A_i and B_i are matrices with with real entries and appropriate dimensions, and $\alpha_i : \mathbb{R}_+ \to \{0, 1\}$ is a switching signal determining the activity of the control input on the i-th subsystem. We assume that at each time the control input is active in exactly one subsystem, i.e.,

$$\sum_{i=1}^{N} \alpha_i(t) = 1 \text{ for all } t \in \mathbb{R}_+. \tag{1.2}$$

This paper analyzes the stabilizability of all subsystems in (1.1) by linear feedback laws $u_i(t) = K_i x_i(t)$ under randomly generated switching signals $\alpha_1, \dots, \alpha_N$ satisfying (1.2), and the maximal almost sure exponential decay rates that can be achieved with such feedbacks.

System (1.1) is a switched control system, where the switching signals $\alpha_1, \dots, \alpha_N$ affect the activity of the control input. Switched systems have been extensively studied in the literature, both for deterministic switching signals, such as in the monographs Liberzon [23] and Sun and Ge [30] and the surveys Lin and Antsaklis [24], Margaliot [25], and Shorten, Wirth, Mason, Wulff, and King [28], and for random switching signals, such as in the monographs Costa, Fragoso, and Todorov [12] and Davis [14], and papers such as Benaïm, Le Borgne, Malrieu, and Zitt [3], Cloez and Hairer [11], and Guyon, Iovleff, and Yao [18]. Such systems are useful models in several applications, ranging from air traffic control, electronic circuits, and automotive engines to chemical processes and population models in biology.

An important motivation for our work comes from the theory of persistently excited control systems, in which one considers systems of the form

$$\dot{x}(t) = Ax(t) + \alpha(t) Bu(t), \tag{1.3}$$

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with $x(t) \in \mathbb{R}^d$, $u(t) \in \mathbb{R}^m$, A and B matrices with real entries and appropriate dimensions, and α a (T, μ)-persistently exciting (PE) signal for some positive constants $T \geq \mu$, i.e., a signal $\alpha \in L^\infty(\mathbb{R}_+, [0,1])$ satisfying, for every $t \geq 0$,

$$\int_t^{t+T} \alpha(s)\,\mathrm{d}s \geq \mu \tag{1.4}$$

(cf. Chaillet, Chitour, Loría, and Sigalotti [5], Chitour, Colonius, and Sigalotti [8], Chitour, Mazanti, and Sigalotti [9], Chitour and Sigalotti [10], Srikant and Akella [29]). Notice that, when α takes its values in $\{0,1\}$, (1.3) can be seen as a particular case of (1.1) by adding a trivial subsystem (cf. Remark 5.2). The stabilizability problem for (1.3) consists in investigating if, given A, B, T, and μ, one can find a linear feedback $u(t) = Kx(t)$ which stabilizes (1.3) exponentially for every (T, μ)-persistently exciting signal α. This problem has been considered in [10], where the authors provide sufficient conditions for stabilizability and prove that, in contrast to the situation for autonomous linear control systems, controllability does not imply stabilizability with arbitrary decay rates, even if one considers only persistently exciting signals taking values in $\{0,1\}$. The main result of our paper, Theorem 5.1, implies that, if one requires the feedback to stabilize (1.3) for *almost every* randomly generated signal α (with respect to the random model described in Section 2), then one can retrieve stabilizability with arbitrary decay rates, giving thus a positive answer to an open problem stated by Chitour and Sigalotti.

In this paper, in order to study the stabilizability by linear feedback laws of (1.1), we rewrite it as

$$\dot{x}(t) = \widehat{A}x(t) + \widehat{B}_{\alpha(t)}u_{\alpha(t)}(t), \tag{1.5}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_i(t) \\ \vdots \\ x_N(t) \end{pmatrix} \in \mathbb{R}^d, \quad \widehat{A} = \begin{pmatrix} A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_N \end{pmatrix}, \quad \widehat{B}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{pmatrix}, \tag{1.6}$$

$d = d_1 + \cdots + d_N$, and $\alpha : \mathbb{R}_+ \to \{1, \dots, N\}$ is defined from $\alpha_1, \dots, \alpha_N : \mathbb{R}_+ \to \{0,1\}$ by setting $\alpha(t)$ to be the unique index $i \in \{1, \dots, N\}$ such that $\alpha_i(t) = 1$. We then look for linear feedback laws of the form $u_i(t) = K_i P_i x$, where $P_i \in \mathcal{M}_{d_i,d}(\mathbb{R})$ is the matrix associated with the canonical projection onto the i-th factor of the product $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}$. With such feedback laws, (1.5) reads

$$\dot{x}(t) = \left(\widehat{A} + \widehat{B}_{\alpha(t)}K_{\alpha(t)}P_{\alpha(t)}\right)x(t).$$

Before considering the stabilizability of (1.5), we begin the paper by the stability analysis of the linear switched system with random switching

$$\dot{x}(t) = L_{\alpha(t)}x(t), \tag{1.7}$$

where $L_1, \dots, L_N \in \mathcal{M}_d(\mathbb{R})$ and $\alpha : \mathbb{R}_+ \to \{1, \dots, N\}$ is as before. We characterize its exponential behavior through its Lyapunov exponents, using the classical Multiplicative Ergodic Theorem due to Oseledets (cf. Arnold [1]). It turns out that a direct application of this theorem to systems in continuous time with random switching is not feasible, since in general they do not define random dynamical systems in the sense of [1] (cf. Example 2.5). Instead, we apply the Multiplicative Ergodic Theorem to an associated system in discrete time and then deduce results for the Lyapunov exponents of the continuous-time system. We remark that Lyapunov exponents for continuous-time systems with random switching are also considered by Li, Chen, Lam, and Mao in [22], but under assumptions on the random switching signal α guaranteeing that the corresponding switched system is a random dynamical system, which allows the direct use of the Multiplicative Ergodic Theorem in continuous time.

The considered linear equations with random switching (1.7) form Piecewise Deterministic Markov Processes (PDMP). These processes were introduced in Davis [13] and have since been extensively studied in the literature. For an analysis of their invariant measures, in particular, their supports, cf. Bakhtin and Hurth [2] and Benaïm, Le Borgne, Malrieu, and Zitt [3], also for further references. An important particular case which also attracts much research interest is that of Markovian jump linear systems (MJLS), in which one assumes that the random switching signal is generated by a continuous-time Markov chain. For more details, we refer to Bolzern, Colaneri, and De Nicolao [4], Fang and Loparo [15], and to the monograph Costa, Fragoso, and Todorov [12]. The case of nonlinear switched systems with random switching signals has also been considered in the literature, cf. e.g. Chatterjee and Liberzon [6], where multiple Lyapunov functions are used to derive a stability criterion under some slow switching condition that contains as a particular case switching signals coming from continuous-time Markov chains. We also remark that several different notions of stability for systems with random switching have been used in the literature; see, e.g., Feng, Loparo, Ji, and Chizeck [16] for a comparison between the usual notions in the context of MJLS. The one considered in this paper is that of almost sure stability.

The contents of this paper is as follows:

Section 2 constructs the random signals α in (1.5) and (1.7). Example 2.5 shows that, in general, (1.7) endowed with such random switching signals does not define a random dynamical system, and Remark 2.6 discusses the relation to previous works in the literature. Section 3 introduces an associated system in discrete time, which defines a random dynamical system in discrete time. We discuss relations between the Lyapunov exponents for continuous- and discrete-time systems and state the conclusions we obtain from the Multiplicative Ergodic Theorem. Section 4 derives a formula for the maximal Lyapunov exponent, which is the main ingredient in the stability analysis of (1.7). Finally, Section 5 presents the main result of this paper, namely that almost sure stabilization can be achieved for (1.1) with arbitrary decay rate under a controllability hypothesis.

Notation: The sets $\mathbb{N}^*$ and $\mathbb{N}$ are used to denote the positive and nonnegative integers, respectively. For $N \in \mathbb{N}^*$ we let $\underline{N} := \{1, ..., N\}$ and $\mathbb{R}_+ := [0, \infty), \mathbb{R}_+^* := (0, \infty)$.

2. Random model for the switching signal. Let $N, d \in \mathbb{N}^*$ and $L_1, \ldots, L_N \in \mathcal{M}_d(\mathbb{R})$ and consider System (1.7) with a switching signal α belonging to the set $\mathcal{P}$ defined by

$$\mathcal{P} := \{\alpha : \mathbb{R}_+ \to \underline{N} \text{ piecewise constant and right continuous}\}.$$

Recall that a piecewise constant function has only finitely many discontinuity points on any bounded interval. Given an initial condition $x_0 \in \mathbb{R}^d$ and $\alpha \in \mathcal{P}$, system (1.7) admits a unique solution defined on $\mathbb{R}_+$, which we denote by $\varphi_{\mathrm{c}}(\cdot; x_0, \alpha)$. In order to simplify the notation, for $i \in \underline{N}$, we denote by Φ^i the linear flow defined by the matrix L_i, i.e., $\Phi_t^i = e^{L_i t}$ for every $t \in \mathbb{R}$.

We suppose in this paper that the signal α is randomly generated according to a Markov process which we describe now. Let $M \in \mathcal{M}_N(\mathbb{R})$ be an irreducible stochastic matrix, i.e., the irreducible matrix M has nonnegative entries and $\sum_{j=1}^N M_{ij} = 1$ for every $i \in \underline{N}$. Let p be the unique probability vector in $\mathbb{R}^N$ invariant under M, i.e., $p \in [0,1]^N$ is regarded as a row vector $p = (p_1, \dots, p_N)$ with $\sum_{i=1}^N p_i = 1$ and $pM = p$. Finally, let $\mu_1, \dots, \mu_N$ be probability measures on $\mathbb{R}_+^*$ with the Borel σ-algebra $\mathfrak{B}$ with finite expectation and denote by τ_i the expected value of μ_i, i.e., $\tau_i = \int_{\mathbb{R}_+^*} t \,\mathrm{d}\mu_i(t) \in (0, \infty)$ for every $i \in \underline{N}$. Whenever necessary, we will use that $\mu_1, \dots, \mu_N$ define probability measures on $\mathbb{R}_+$ with its Borel σ-algebra, that we also denote by $\mathfrak{B}$ for simplicity.

The random model for the signal α can be described as follows. We choose a random initial state $i \in \underline{N}$ according to the probability law defined by p. Then, at every time the system switches to a state i, we choose a random positive time T according to the probability law μ_i and stay in i during the time T, before switching to the next state, which is chosen randomly according to the probability law corresponding to the i-th row $(M_{ij})_{j=1}^N$ of the matrix M. Let us perform this construction more precisely.

Definition 2.1. *Let $\Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*}$ and endow Ω with the product σ-algebra $\mathfrak{F} = (\mathfrak{P}(\underline{N}) \times \mathfrak{B})^{\mathbb{N}^*}$ (cf. Halmos [20, §38, §49]), where $\mathfrak{P}(\underline{N})$ is the σ-algebra containing all subsets of $\underline{N}$. We define the probability measure $\mathbb{P}$ in $(\Omega, \mathfrak{F})$, for $n \in \mathbb{N}^*$, $i_1, \dots, i_n \in \underline{N}$, and $U_1, \dots, U_n \in \mathfrak{B}$, by*

$$\begin{aligned}&\mathbb{P}\left((\{i_1\} \times U_1) \times (\{i_2\} \times U_2) \times \cdots \times (\{i_n\} \times U_n) \times (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^* \setminus \underline{n}}\right)\\ &= p_{i_1} \mu_{i_1}(U_1) M_{i_1 i_2} \mu_{i_2}(U_2) \cdots M_{i_{n-1} i_n} \mu_{i_n}(U_n).\end{aligned}$$

Remark 2.2. *The construction from Definition 2.1 is a Markov chain in the state space $\underline{N} \times \mathbb{R}_+$. More precisely, denoting by $\mathrm{Pr}(X)$ the set of all probability measures on a given measurable space X and by $x_n : \Omega = (\underline{N} \times \mathbb{R}_+)^{\mathbb{N}^*} \to \underline{N} \times \mathbb{R}_+$ the canonical projection onto the n-th coordinate for $n \in \mathbb{N}^*$, one has that $(x_n)_{n=1}^\infty$ is the unique Markov process in $\underline{N} \times \mathbb{R}_+$ with transition probability $P : \underline{N} \times \mathbb{R}_+ \to \mathrm{Pr}(\underline{N} \times \mathbb{R}_+)$ defined by*

$$P(i,t)(\{j\} \times U) = M_{ij} \mu_j(U), \qquad \forall i, j \in \underline{N},\ \forall t \in \mathbb{R}_+,\ \forall U \in \mathfrak{B}, \tag{2.1}$$

and with initial law ν_1 given by

$$\nu_1(\{j\} \times U) = p_j \mu_j(U), \qquad \forall j \in \underline{N},\ \forall U \in \mathfrak{B}.$$

The transition operator $T : \mathrm{Pr}(\underline{N} \times \mathbb{R}_+) \to \mathrm{Pr}(\underline{N} \times \mathbb{R}_+)$ of this process is given by

$$T\nu(\{j\} \times U) = \sum_{i=1}^N \nu(\{i\} \times \mathbb{R}_+) M_{ij} \mu_j(U), \qquad \forall j \in \underline{N},\ \forall U \in \mathfrak{B}. \tag{2.2}$$

(For the definition of a Markov process and its transition probability, initial law, and transition operator, we refer to Hairer [19].) This can be proved by a straightforward computation using the definition of $\mathbb{P}$ and [19, Definition 2.13 and Proposition 2.38]. Notice that the canonical projection of $\underline{N} \times \mathbb{R}_+$ onto $\underline{N}$ transforms the Markov process

$(x_n)_{n=1}^{\infty}$ *into a discrete Markov chain in the finite state space* $\underline{N}$ *with transition matrix* M *and initial distribution* p.

To construct a random switching signal α from a certain $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$, we regard $(i_n)_{n=1}^{\infty}$ as the sequence of states taken by α and t_n as the time spent in the state i_n.

Definition 2.3. *We define the map* $\boldsymbol{\alpha} : \Omega \to \mathcal{P}$ *as follows: for* $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$, *we set* $s_0 = 0$, $s_n = \sum_{k=1}^{n} t_k$ *for* $n \in \mathbb{N}^*$, *and* $\boldsymbol{\alpha}(\omega)(t) = i_n$ *for* $t \in [s_{n-1}, s_n)$, $n \in \mathbb{N}^*$.

Notice that $\boldsymbol{\alpha}(\omega)$ is not well-defined if $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$ is such that $\sum_{n=1}^{\infty} t_n < \infty$. However, a straightforward argument using the definition of $\mathbb{P}$ shows that $\sum_{n=1}^{\infty} t_n = \infty$ for almost every $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$, and hence $\boldsymbol{\alpha} : \Omega \to \mathcal{P}$ is well-defined almost everywhere on Ω. In the sequel, we denote by Ω_0 the set of $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega$ for which $\sum_{n=1}^{\infty} t_n = \infty$, so that $\mathbb{P}(\Omega_0) = 1$ and $\boldsymbol{\alpha}$ is well-defined on Ω_0.

In order to consider solutions of (1.7) for signals α chosen randomly according to the previous construction, we use the solution map φ_{c} of (1.7) to provide the following definition.

Definition 2.4. *We define the continuous-time map*

$$\varphi_{\mathrm{rc}} : \left\{ \begin{array}{rcl} \mathbb{R}_+ \times \mathbb{R}^d \times \Omega_0 & \to & \mathbb{R}^d \\ (t; x_0, \omega) & \mapsto & \varphi_{\mathrm{c}}(t; x_0, \boldsymbol{\alpha}(\omega)). \end{array} \right. \tag{2.3}$$

For $x_0 \in \mathbb{R}^d \setminus \{0\}$ *and almost every* $\omega \in \Omega$, *we define the* Lyapunov exponent *of the continuous-time system* (2.3) *by*

$$\lambda_{\mathrm{rc}}(x_0, \omega) = \limsup_{t \to \infty} \frac{1}{t} \log \left\| \varphi_{\mathrm{rc}}(t; x_0, \omega) \right\|. \tag{2.4}$$

The Lyapunov exponent λ_{rc} is used to characterize the asymptotic behavior of (2.3). A natural idea to obtain information on such Lyapunov exponents would be to apply the continuous-time Multiplicative Ergodic Theorem (see, e.g., Arnold [1, Theorem 3.4.1]). To do so, φ_{rc} should define a random dynamical system on $\mathbb{R}^d \times \Omega$, i.e., one would have to provide a metric dynamical system θ on Ω — a measurable dynamical system $\theta : \mathbb{R}_+ \times \Omega \to \Omega$ on $(\Omega, \mathfrak{F}, \mathbb{P})$ such that θ_t preserves $\mathbb{P}$ for every $t \geq 0$ — in such a way that φ_{rc} becomes a cocycle over θ. However, in general the natural choice for θ to obtain the cocycle property for φ_{rc}, namely the time shift, does not define such a measure preserving map, as shown in the following example.

Example 2.5. *For* $t \geq 0$, *let* $\theta_t : \Omega \to \Omega$ *be defined for almost every* $\omega \in \Omega$ *as follows. For* $\omega = (i_j, t_j)_{j=1}^{\infty} \in \Omega_0$, *set* $s_0 = 0$, $s_k = \sum_{j=1}^{k} t_j$ *for* $k \in \mathbb{N}^*$. *Let* $n \in \mathbb{N}^*$ *be the unique integer such that* $t \in [s_{n-1}, s_n)$. *We define* $\theta_t(\omega) = (i_j^*, t_j^*)_{j=1}^{\infty}$ *by* $i_j^* = i_{n+j-1}$ *for* $j \in \mathbb{N}^*$, $t_1^* = s_n - t$, $t_j^* = t_{n+j-1}$ *for* $j \geq 2$. *One immediately verifies that* θ_t *corresponds to the time shift in* $\mathcal{P}$, *i.e., for every* $t, s \geq 0$ *and* $\omega \in \Omega_0$, *one has*

$$\boldsymbol{\alpha}(\theta_t \omega)(s) = \boldsymbol{\alpha}(\omega)(t + s).$$

However, the map θ_t *in* $(\Omega, \mathfrak{F})$ *does not preserve the measure* $\mathbb{P}$ *in general. Indeed, suppose that* $\mu_i = \delta_1$ *for every* $i \in \underline{N}$, *where* δ_1 *denotes the Dirac measure at* 1. *In particular, a set* $E \in \mathfrak{F}$ *has nonzero measure only if* E *contains a point* $(i_j, t_j)_{j=1}^{\infty}$ *with* $t_j = 1$ *for every* $j \in \mathbb{N}^*$. *For* $r \in \mathbb{N}^*$ *and* $i_1, \dots, i_r \in \underline{N}$, *let*

$$E = (\{i_1\} \times \{1\}) \times \cdots \times (\{i_r\} \times \{1\}) \times \left(\underline{N} \times \mathbb{R}_+\right)^{\mathbb{N}^* \setminus \underline{r}}.$$

Then $\mathbb{P}(E) = p_{i_1} M_{i_1 i_2} \cdots M_{i_{r-1} i_r}$, *and, for* $t \geq 0$, $\theta_t^{-1}(E)$ *is the set of points* $(i_j^*, t_j^*)_{j=1}^\infty$ *such that, setting* $s_0^* = 0$, $s_k^* = \sum_{j=1}^k t_j^*$ *for* $k \in \mathbb{N}^*$, *and* $n \in \mathbb{N}^*$ *the unique integer such that* $t \in \left[s_{n-1}^*, s_n^*\right)$, *one has* $s_n^* - t = 1$, $t_{n+j-1}^* = 1$ *for* $j = 2, \dots, r$, *and* $i_{n+j-1}^* = i_j$ *for* $j \in \underline{r}$. *If* $t \notin \mathbb{N}$, *then* $s_n^* = t + 1 \notin \mathbb{N}$, *and thus there exists* $j \in \underline{n}$ *such that* $t_j^* \neq 1$. *We have shown that, if* $t \notin \mathbb{N}$, *then, for every* $\omega = (i_j^*, t_j^*)_{j=1}^\infty \in \theta_t^{-1}(E)$, *there exists* $j \in \mathbb{N}^*$ *such that* $t_j^* \neq 1$, *and thus* $\mathbb{P}(\theta_t^{-1}(E)) = 0$, *hence* θ_t *does not preserve the measure* $\mathbb{P}$.

REMARK 2.6. *For some particular choices of* $\mu_1, \dots, \mu_N$, *the time-shift* θ_t *may preserve* $\mathbb{P}$, *in which case the continuous-time Multiplicative Ergodic Theorem can be applied directly to* (2.3). *This special case falls in the framework of Li, Chen, Lam, and Mao [22]. An important particular case where* θ_t *preserves* $\mathbb{P}$ *is when* $\mu_1, \dots, \mu_N$ *are chosen in such a way that* $\boldsymbol{\alpha}$ *becomes a homogeneous continuous-time Markov chain, which is the case treated, e.g., in Bolzern, Colaneri, and De Nicolao [4], and in Fang and Loparo [15]. The results we provide in Section 4 generalize the corresponding almost sure stability criteria from [4, 15, 22] to randomly switching signals constructed according to Definitions 2.1 and 2.3.*

3. Associated discrete-time system and Lyapunov exponents.

Example 2.5 shows that in general one cannot expect to obtain a random dynamical system from φ_{rc} in order to apply the continuous-time Multiplicative Ergodic Theorem. Our strategy to study the exponential behavior of φ_{rc} relies instead on defining a suitable discrete-time map φ_{rd} associated with φ_{rc}, in such a way that φ_{rd} does define a discrete-time random dynamical system — to which the discrete-time Multiplicative Ergodic Theorem can be applied — and that the exponential behavior of φ_{rc} and φ_{rd} can be compared.

DEFINITION 3.1. *For* $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$, *we set* $s_n(\omega) = \sum_{k=1}^n t_k$ *for* $n \in \mathbb{N}^*$ *and* $s_0(\omega) = 0$. *We define the discrete-time map* φ_{rd} *by*

$$\varphi_{\mathrm{rd}} : \begin{cases} \mathbb{N} \times \mathbb{R}^d \times \Omega_0 & \to \quad \mathbb{R}^d \\ (n; x_0, \omega) & \mapsto \quad \varphi_{\mathrm{rc}}(s_n(\omega); x_0, \omega). \end{cases} \tag{3.1}$$

For $x_0 \in \mathbb{R}^d \setminus \{0\}$ *and almost every* $\omega \in \Omega$, *we define the* Lyapunov exponent *of the discrete-time system* (3.1) *by*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = \limsup_{n \to \infty} \frac{1}{n} \log \left\| \varphi_{\mathrm{rd}}(n; x_0, \omega) \right\|. \tag{3.2}$$

The map φ_{rd} corresponds to regarding the continuous-time map φ_{rc} only at the switching times $s_n(\omega)$. It is the solution map of the random discrete-time equation

$$x_n = e^{L_{i_n} t_n} x_{n-1}. \tag{3.3}$$

System (3.3) is obtained from (1.7) by taking the values of a continuous-time solution at the discrete times $s_n(\omega)$. The sequence $(s_n(\omega))_{n=0}^\infty$ contains all the discontinuities of $\boldsymbol{\alpha}(\omega)$ and may also contain times with trivial jumps. The Lyapunov exponent λ_{rd} characterizes the asymptotic behavior of φ_{rd}.

Notice that the solution maps φ_{rc} and φ_{rd} satisfy, for every $x_0 \in \mathbb{R}^d$ and almost every $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$,

$$\begin{aligned} &\varphi_{\mathrm{rc}}(0; x_0, \omega) = x_0, \\ &\varphi_{\mathrm{rc}}(t; x_0, \omega) = \Phi_{t - s_n(\omega)}^{\boldsymbol{\alpha}(\omega)(s_n(\omega))} \varphi_{\mathrm{rc}}(s_n(\omega); x_0, \omega), \text{ for } n \in \mathbb{N} \text{ and } t \in (s_n(\omega), s_{n+1}(\omega)], \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\varphi_{\mathrm{rd}}(0;x_0,\omega)&=x_0,\\ \varphi_{\mathrm{rd}}(n+1;x_0,\omega)&=\Phi_{t_{n+1}}^{\boldsymbol{\alpha}(\omega)(s_n(\omega))}\varphi_{\mathrm{rd}}(n;x_0,\omega),\qquad \text{for } n\in\mathbb{N}.\end{aligned}\tag{3.5}$$

We now prove that φ_{rd} defines a discrete-time random dynamical system on $\mathbb{R}^d\times\Omega$. To do so, we must first provide a discrete-time metric dynamical system θ on $(\Omega,\mathfrak{F},\mathbb{P})$, which can be chosen simply as the usual shift operator. Let $\theta:\Omega\to\Omega$ be defined by

$$\theta((i_n,t_n)_{n=1}^{\infty})=(i_{n+1},t_{n+1})_{n=1}^{\infty}.\tag{3.6}$$

One can easily verify, using Definition 2.1 and the fact that $pM=p$, that the measure $\mathbb{P}$ is invariant under θ, and thus θ is a discrete-time metric dynamical system in $(\Omega,\mathfrak{F},\mathbb{P})$. Moreover, since $\theta(\Omega_0)=\Omega_0$, θ also defines a metric dynamical system in $(\Omega_0,\mathfrak{F},\mathbb{P})$ (where $\mathfrak{F}$ and $\mathbb{P}$ are understood to be restricted to Ω_0).

We now consider the ergodicity of θ. We start by providing the following definition.

Definition 3.2. *Let $(\Omega,\mathfrak{F})$ be the measurable space from Definition 2.1 and $\nu\in\Pr(\underline{N}\times\mathbb{R}_+)$. We define the probability measure $\mathbb{P}_\nu$ in $(\Omega,\mathfrak{F})$ by requiring that, for every $n\in\mathbb{N}^*$, $i_1,\dots,i_n\in\underline{N}$, and $U_1,\dots,U_n\in\mathfrak{B}$,*

$$\begin{aligned}&\mathbb{P}_\nu\left((\{i_1\}\times U_1)\times(\{i_2\}\times U_2)\times\cdots\times(\{i_n\}\times U_n)\times(\underline{N}\times\mathbb{R}_+)^{\mathbb{N}^*\setminus\underline{n}}\right)\\ &=\nu(\{i_1\}\times U_1)M_{i_1i_2}\mu_{i_2}(U_2)\cdots M_{i_{n-1}i_n}\mu_{i_n}(U_n).\end{aligned}\tag{3.7}$$

Remark 3.3. *If $\nu(\{i\}\times U)=p_i\mu_i(U)$ for every $i\in\underline{N}$ and $U\in\mathfrak{B}$, then $\mathbb{P}_\nu$ coincides with the measure $\mathbb{P}$ from Definition 2.1. Moreover, for every $\nu\in\Pr(\underline{N}\times\mathbb{R}_+)$, $\mathbb{P}_\nu$ is the probability measure associated with a Markov process in $\underline{N}\times\mathbb{R}_+$ with transition probability P given by* (2.1)*, transition operator T given by* (2.2)*, and with initial law ν.*

Lemma 3.4. *The measure $\mathbb{P}_\nu$ is invariant under the shift θ if and only if $\nu(\{i\}\times U)=p_i\mu_i(U)$ for every $i\in\underline{N}$ and $U\in\mathfrak{B}$.*

Proof. Notice that $\mathbb{P}_\nu$ is invariant under θ if and only if $T\nu=\nu$. Hence $\mathbb{P}_\nu$ is invariant under θ if and only if

$$\nu(\{j\}\times U)=\sum_{i=1}^{N}\nu(\{i\}\times\mathbb{R}_+)M_{ij}\mu_j(U),\qquad\forall j\in\underline{N},\ \forall U\in\mathfrak{B}.\tag{3.8}$$

If (3.8) holds, we apply it to $U=\mathbb{R}_+$ to get that $(\nu(\{i\}\times\mathbb{R}_+))_{i=1}^{N}$ is a left eigenvector of M associated with the eigenvalue 1. Since M is irreducible, we then obtain than $\nu(\{i\}\times\mathbb{R}_+)=p_i$ for $i\in\underline{N}$. It follows that $\nu(\{j\}\times U)=p_j\mu_j(U)$ for every $j\in\underline{N}$ and $U\in\mathfrak{B}$. The converse is immediate. □

Lemma 3.4 shows that the only invariant measure for θ under the form $\mathbb{P}_\nu$ is the measure $\mathbb{P}$ from Definition 2.1. In particular, we obtain immediately from Hairer [19, Theorem 5.7] the following result.

Corollary 3.5. *The metric dynamical system θ is ergodic in $(\Omega,\mathfrak{F},\mathbb{P})$.*

Now that we have defined the random discrete-time system (3.1) and provided the metric dynamical system θ, we can show that the pair $(\theta,\varphi_{\mathrm{rd}})$ defines a random dynamical system.

Proposition 3.6. *$(\theta, \varphi_{\mathrm{rd}})$ is a discrete-time random dynamical system over $(\Omega, \mathfrak{F}, \mathbb{P})$.*

Proof. Since θ is a discrete-time metric dynamical system over $(\Omega, \mathfrak{F}, \mathbb{P})$, one is only left to show that φ_{rd} satisfies the cocycle property

$$\varphi_{\mathrm{rd}}(n+m; x_0, \omega) = \varphi_{\mathrm{rd}}(n; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)), \quad \forall n, m \in \mathbb{N},\ \forall x_0 \in \mathbb{R}^d,\ \forall \omega \in \Omega_0. \tag{3.9}$$

Let $\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega_0$. Then it follows immediately from the definitions of $\boldsymbol{\alpha}$ and s_n that, for $n, m \in \mathbb{N}$,

$$s_n(\theta^m(\omega)) = \sum_{k=1}^{n} t_{k+m} = \sum_{k=m+1}^{m+n} t_k = s_{n+m}(\omega) - s_m(\omega),$$
$$\boldsymbol{\alpha}(\theta^m(\omega))(s_n(\theta^m(\omega))) = i_{n+m} = \boldsymbol{\alpha}(\omega)(s_{n+m}(\omega)).$$

We prove (3.9) by induction on n. When $n = 0$, (3.9) is clearly satisfied for every $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, and $\omega \in \Omega_0$. Suppose now that $n \in \mathbb{N}$ is such that (3.9) is satisfied for every $m \in \mathbb{N}$, $x_0 \in \mathbb{R}^d$, and $\omega \in \Omega_0$. Using (3.5), we obtain

$$\begin{aligned}
&\varphi_{\mathrm{rd}}(n+1; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)) \\
&= \Phi^{\boldsymbol{\alpha}(\theta^m(\omega))(s_n(\theta^m(\omega)))}_{s_{n+1}(\theta^m(\omega)) - s_n(\theta^m(\omega))} \varphi_{\mathrm{rd}}(n; \varphi_{\mathrm{rd}}(m; x_0, \omega), \theta^m(\omega)) \\
&= \Phi^{\boldsymbol{\alpha}(\omega)(s_{n+m}(\omega))}_{s_{n+m+1}(\omega) - s_{n+m}(\omega)} \varphi_{\mathrm{rd}}(n+m; x_0, \omega) = \varphi_{\mathrm{rd}}(n+m+1; x_0, \omega),
\end{aligned}$$

which concludes the proof of (3.9). □

We now compare the asymptotic behavior of (2.3) and (3.1) by considering the relation between the Lyapunov exponents $\lambda_{\mathrm{rc}}(x_0, \omega)$ and $\lambda_{\mathrm{rd}}(x_0, \omega)$ of the continuous- and discrete-time systems. We first establish the following result.

Proposition 3.7. *For almost every $\omega \in \Omega$, one has*

$$\lim_{n \to \infty} \frac{s_n(\omega)}{n} = \sum_{i=1}^{N} p_i \int_{\mathbb{R}_+} t \,\mathrm{d}\mu_i(t) = \sum_{i=1}^{N} p_i \tau_i =: m. \tag{3.10}$$

Proof. Consider the map $f : \Omega_0 \to \mathbb{R}_+^*$ given by $f((i_n, t_n)_{i=1}^{\infty}) = t_1$. For every $k \in \mathbb{N}$, $f \circ \theta^k((i_n, t_n)_{n=1}^{\infty}) = t_{k+1}$. Since θ is ergodic, Birkhoff's Ergodic Theorem shows that, for almost every $\omega \in \Omega$,

$$\lim_{n \to \infty} \frac{s_n(\omega)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \theta^k(\omega) = \int_{\Omega} f(\omega) \,\mathrm{d}\mathbb{P}(\omega) = \sum_{i=1}^{N} p_i \int_{\mathbb{R}_+} t \,\mathrm{d}\mu_i(t),$$

as required. □

The next result provides the relation between λ_{rc} and λ_{rd}.

Proposition 3.8. *For every $x_0 \in \mathbb{R}^d \backslash \{0\}$ and almost every $\omega \in \Omega$, the Lyapunov exponents of the continuous- and discrete-time systems* (2.3) *and* (3.1)*, given by* (2.4) *and* (3.2)*, are related by*

$$\lambda_{\mathrm{rd}}(x_0, \omega) = m \lambda_{\mathrm{rc}}(x_0, \omega).$$

Proof. Let us first show that $\lambda_{\mathrm{rd}}(x_0, \omega) \leq m \lambda_{\mathrm{rc}}(x_0, \omega)$. For every $n \in \mathbb{N}^*$, one has

$$\frac{1}{n} \log \|\varphi_{\mathrm{rd}}(n; x_0, \omega)\| = \frac{s_n(\omega)}{n} \frac{1}{s_n(\omega)} \log \|\varphi_{\mathrm{rc}}(s_n(\omega); x_0, \omega)\|.$$

Moreover

$$\limsup_{n\to\infty} \frac{1}{s_n(\omega)} \log \|\varphi_{\mathrm{rc}}(s_n(\omega); x_0, \omega)\| \leq \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi_{\mathrm{rc}}(t; x_0, \omega)\|,$$

and then the conclusion follows since $\frac{s_n(\omega)}{n} \to m$ as $n \to \infty$ for almost every $\omega \in \Omega$.

We now turn to the proof of the inequality $\lambda_{\mathrm{rd}}(x_0, \omega) \geq m\lambda_{\mathrm{rc}}(x_0, \omega)$. Let $C, \gamma > 0$ be such that $\left\|\Phi^i_t x\right\| \leq Ce^{\gamma t} \|x\|$ for every $i \in \underline{N}$, $x \in \mathbb{R}^d$, and $t \geq 0$. For $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$, let $n_t \in \mathbb{N}$ be the unique integer such that $t \in (s_{n_t}(\omega), s_{n_t+1}(\omega)]$, which is well-defined for almost every $\omega \in \Omega$. Then

$$\begin{aligned} &\frac{1}{t} \log \|\varphi_{\mathrm{rc}}(t; x_0, \omega)\| = \frac{1}{t} \log \left\|\Phi^{\boldsymbol{\alpha}(\omega)(s_{n_t}(\omega))}_{t - s_{n_t}(\omega)} \varphi_{\mathrm{rc}}(s_{n_t}(\omega); x_0, \omega)\right\| \\ &= \frac{1}{t} \log \left\|\Phi^{\boldsymbol{\alpha}(\omega)(s_{n_t}(\omega))}_{t - s_{n_t}(\omega)} \varphi_{\mathrm{rd}}(n_t; x_0, \omega)\right\| \leq \frac{\log C}{t} + \gamma \frac{t - s_{n_t}(\omega)}{t} + \frac{1}{t} \log \|\varphi_{\mathrm{rd}}(n_t; x_0, \omega)\|. \end{aligned} \tag{3.11}$$

Since $t \in (s_{n_t}(\omega), s_{n_t+1}(\omega)]$, one has, for almost every $\omega \in \Omega$,

$$0 \leq \frac{t - s_{n_t}(\omega)}{t} \leq \frac{s_{n_t+1}(\omega)}{s_{n_t}(\omega)} - 1 \xrightarrow[t\to\infty]{} 0, \tag{3.12}$$

where we use (3.10) to obtain that $\frac{s_{n_t+1}(\omega)}{s_{n_t}(\omega)} \to 1$ as $t \to \infty$. We write $\frac{1}{t} = \frac{n_t}{t} \frac{1}{n_t}$. Since $t \in (s_{n_t}(\omega), s_{n_t+1}(\omega)]$, one has $\frac{n_t}{t} \in \left[\frac{n_t}{s_{n_t+1}(\omega)}, \frac{n_t}{s_{n_t}(\omega)}\right)$. Now

$$\lim_{t\to\infty} \frac{n_t}{s_{n_t}(\omega)} = \frac{1}{m} \quad \text{and} \quad \lim_{t\to\infty} \frac{n_t}{s_{n_t+1}(\omega)} = \lim_{t\to\infty} \left(\frac{n_t + 1}{s_{n_t+1}(\omega)} - \frac{1}{s_{n_t+1}(\omega)}\right) = \frac{1}{m},$$

and thus $\frac{n_t}{t} \to \frac{1}{m}$ as $t \to \infty$. Using this fact and inserting (3.12) into (3.11), one obtains the conclusion of the theorem by letting $t \to \infty$. □

We also find it useful to prove the following proposition, which evaluates the average time spent in a certain state k.

PROPOSITION 3.9. *Let $i \in \underline{N}$. For almost every $\omega \in \Omega$, one has*

$$\lim_{T\to\infty} \frac{\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = i\}}{T} = \frac{p_i \tau_i}{m},$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}$.

Proof. Fix $i \in \underline{N}$. Let $\varphi_i : \Omega \to \mathbb{R}_+$ be given by

$$\varphi_i((i_n, t_n)_{n=1}^\infty) = \begin{cases} t_1, & \text{if } i_1 = i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Birkhoff's Ergodic Theorem, one has, for almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_i(\theta^j \omega) = \int_\Omega \varphi_i(\omega) \,\mathrm{d}\mathbb{P}(\omega) = p_i \tau_i. \tag{3.13}$$

On the other hand, by definition of $\boldsymbol{\alpha}$, for almost every $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega$,

$$\sum_{j=0}^{n-1} \varphi_i(\theta^j \omega) = \sum_{\substack{j=1 \\ i_j = i}}^{n} t_j = \mathcal{L}\{t \in [0, s_n(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = i\}.$$

Hence it follows from Proposition 3.7 and (3.13) that, for almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{\mathcal{L}\{t \in [0, s_n(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = i\}}{s_n(\omega)} = \lim_{n\to\infty} \frac{n}{s_n(\omega)} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_i(\theta^j \omega) = \frac{p_i \tau_i}{m}. \tag{3.14}$$

Let $\omega \in \Omega$ be such that (3.14) holds and take $T \in \mathbb{R}_+$. Choose $n_T \in \mathbb{N}$ such that $s_{n_T}(\omega) \leq T < s_{n_T+1}(\omega)$. Then

$$\frac{1}{T}\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = i\} \leq \frac{1}{s_{n_T}(\omega)}\mathcal{L}\{t \in [0, s_{n_T+1}(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = i\}$$

and

$$\frac{1}{T}\mathcal{L}\{t \in [0, T] \mid \boldsymbol{\alpha}(\omega)(t) = i\} \geq \frac{1}{s_{n_T+1}(\omega)}\mathcal{L}\{t \in [0, s_{n_T}(\omega)] \mid \boldsymbol{\alpha}(\omega)(t) = i\}.$$

The conclusion of the proposition then follows since, by Proposition 3.7, $\frac{s_{n+1}(\omega)}{s_n(\omega)} \to 1$ as $n \to \infty$ for almost every $\omega \in \Omega$. □

Remark 3.10. *The choice of s_n in Definition 3.1 is not unique, and one might be interested in other possible choices. The times $s_n(\omega)$ correspond to the transitions of the Markov chain from Remark 2.2. However, if some of the diagonal elements of M are non-zero, then the discrete part of the Markov chain, i.e., its component in $\underline{N}$, may switch from a certain state to itself. In practical situations, it may be possible to observe only switches between different states, and another possible choice for $s_n(\omega)$ that may be of practical interest is to consider only the times corresponding to such non-trivial switches. Defining θ as the shift to the next different state, θ defines a metric dynamical system if we suppose that, instead of having $pM = p$, we have $p\tilde{M} = p$, where $\tilde{M}_{ij} = \frac{M_{ij}}{1-M_{ii}}$ for $i, j \in \underline{N}$ with $i \neq j$ and $\tilde{M}_{ii} = 0$ for $i \in \underline{N}$. (Notice that $M_{ii} \neq 1$ for every $i \in \underline{N}$ since M is irreducible.) The counterparts of the previous results can be proved in this framework with no extra difficulty.*

Remark 3.11. *The fact that systems (1.7) and (3.3) are linear has been used only in the proof of Proposition 3.8, where one uses an exponential bound on the growth of the flows $\Phi^i_t = e^{L_i t}$, namely that there exist constants $C, \gamma > 0$ such that $\left\|e^{L_i t}\right\| \leq C e^{\gamma t}$ for every $t \geq 0$ and $i \in \underline{N}$. If we consider, instead of system (1.7), the nonlinear switched system*

$$\dot{x}(t) = f_{\alpha(t)}(x(t)),$$

where $f_1, \ldots, f_N$ are complete vector fields generating flows $\Phi^1, \ldots, \Phi^N$, and modify the discrete-time system (3.3) accordingly, all the previous results remain true, with the same proofs, under the additional assumption that there exist constants $C, \gamma > 0$ such that $\left\|\Phi^i_t x\right\| \leq C e^{\gamma t} \|x\|$ for every $t \geq 0$, $i \in \underline{N}$, and $x \in \mathbb{R}^d$. However, the next results do not generalize to the nonlinear framework.

In order to conclude this section, we apply the discrete-time Multiplicative Ergodic Theorem (see, e.g., Arnold [1, Theorem 3.4.1]) in the one-sided invertible case to system (3.1) and we use Proposition 3.8 to obtain that several of its conclusions also hold for the continuous-time system (2.3).

Let $L : \Omega \to \mathcal{M}_d(\mathbb{R})$ be the function defined for $\omega = (i_n, t_n)_{n=1}^\infty$ by $L(\omega) = e^{L_{i_1} t_1}$, so that $\varphi_{\mathrm{rd}}(n; x_0, \omega) = L(\theta^{n-1}\omega)\varphi_{\mathrm{rd}}(n-1; x_0, \omega)$ for every $x_0 \in \mathbb{R}^d$, $n \in \mathbb{N}^*$, and almost every $\omega \in \Omega$. For $n \in \mathbb{N}$ and almost every $\omega \in \Omega$, we denote $\Phi(n, \omega)$ the linear operator defined by $\Phi(n, \omega)x = \varphi_{\mathrm{rd}}(n; x, \omega)$ for every $x \in \mathbb{R}^d$, which is thus given by $\Phi(n, \omega) = e^{L_{i_n} t_n} \cdots e^{L_{i_1} t_1}$ for $\omega = (i_j, t_j)_{j=1}^\infty \in \Omega$ and $n \in \mathbb{N}^*$.

Proposition 3.12. *There exists a measurable subset $\widehat{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure and invariant under θ such that*

(i) for every $\omega \in \widehat{\Omega}$, the limit $\Psi(\omega) = \lim_{n\to\infty} \left(\Phi(n,\omega)^{\mathrm{T}} \Phi(n,\omega)\right)^{1/2n}$ exists and is a positive definite matrix;

(ii) there exist an integer $q \in \underline{d}$ and q integers $d_1 > \cdots > d_q$ such that, for every $\omega \in \widehat{\Omega}$, there exist q vector subspaces $V_1(\omega), \dots, V_q(\omega)$ with respective dimensions $d_1 > \cdots > d_q$ such that

$$V_q(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d,$$

and $L(\omega)V_i(\omega) = V_i(\theta(\omega))$ for every $i \in \underline{q}$;

(iii) for every $x_0 \in \mathbb{R}^d \setminus \{0\}$ and $\omega \in \widehat{\Omega}$, the Lyapunov exponents $\lambda_{\mathrm{rd}}(x_0,\omega)$ and $\lambda_{\mathrm{rc}}(x_0,\omega)$ exist as limits, i.e.,

$$\lambda_{\mathrm{rd}}(x_0,\omega) = \lim_{n\to\infty} \frac{1}{n} \log \left\|\varphi_{\mathrm{rd}}(n;x_0,\omega)\right\|,$$
$$\lambda_{\mathrm{rc}}(x_0,\omega) = \lim_{t\to\infty} \frac{1}{t} \log \left\|\varphi_{\mathrm{rc}}(t;x_0,\omega)\right\|;$$

(iv) there exist real numbers $\lambda_1^{\mathrm{d}} > \cdots > \lambda_q^{\mathrm{d}}$ and $\lambda_1^{\mathrm{c}} > \cdots > \lambda_q^{\mathrm{c}}$ such that, for every $i \in \underline{q}$ and $\omega \in \widehat{\Omega}$,

$$\lambda_{\mathrm{rd}}(x_0,\omega) = \lambda_i^{\mathrm{d}} \iff \lambda_{\mathrm{rc}}(x_0,\omega) = \lambda_i^{\mathrm{c}} \iff x_0 \in V_i(\omega) \setminus V_{i+1}(\omega),$$

where $V_{q+1}(\omega) = \{0\}$;

(v) for every $\omega \in \widehat{\Omega}$, the eigenvalues of $\Psi(\omega)$ are $e^{\lambda_1^{\mathrm{d}}} > \cdots > e^{\lambda_q^{\mathrm{d}}}$, and their respective algebraic multiplicities are $m_i = d_i - d_{i+1}$, with $d_{q+1} = 0$.

Proof. Let us show that Multiplicative Ergodic Theorem can be applied to the random dynamical system $(\theta, \varphi_{\mathrm{rd}})$. Recall that there are $C \geq 1$, $\gamma > 0$ such that, for every $i \in \underline{N}$ and $t \in \mathbb{R}$, $\left\|e^{L_i t}\right\| \leq C e^{\gamma\left|t\right|}$. Then, for $\omega = (i_n, t_n)_{n=1}^\infty \in \Omega_0$, $\log^+ \left\|L(\omega)^{\pm 1}\right\| \leq \log C + \gamma t_1$, so that

$$\int_\Omega \log^+ \left\|L(\omega)^{\pm 1}\right\| \mathrm{d}\mathbb{P}(\omega) \leq \log C + \gamma \sum_{i=1}^N p_i \tau_i < \infty.$$

Then the Multiplicative Ergodic Theorem can be applied to $(\theta, \varphi_{\mathrm{rd}})$, yielding all the conclusions for Ψ, q, d_i, V_i, $\lambda_{\mathrm{rd}}(x_0,\omega)$, and λ_i^{d}. The conclusions concerning $\lambda_{\mathrm{rc}}(x_0,\omega)$ and λ_i^{c} in (iv) follow from Proposition 3.8, with $\lambda_i^{\mathrm{c}} = \frac{1}{m}\lambda_i^{\mathrm{d}}$. One is now left to show that the Lyapunov exponent $\lambda_{\mathrm{rc}}(x_0,\omega)$ exists as a limit.

Notice that $\left\|e^{-L_i t} x\right\| \leq C e^{\gamma t} \left\|x\right\|$ for every $i \in \underline{N}$, $x \in \mathbb{R}^d$ and $t \geq 0$, and hence $\left\|e^{L_i t} x\right\| \geq C^{-1} e^{-\gamma t} \left\|x\right\|$. Let $t > 0$ and choose $n_t \in \mathbb{N}$ such that $t \in (s_{n_t}(\omega), s_{n_t+1}(\omega)]$. Then, proceeding as in (3.11), one gets

$$\frac{1}{t} \log \left\|\varphi_{\mathrm{rc}}(t;x_0,\omega)\right\| \geq -\frac{\log C}{t} - \gamma \frac{t - s_{n_t}}{t} + \frac{1}{t} \log \left\|\varphi_{\mathrm{rd}}(n_t;x_0,\omega)\right\|.$$

Using (3.12), we thus obtain that

$$\liminf_{t\to\infty} \frac{1}{t} \log \left\|\varphi_{\mathrm{rc}}(t;x_0,\omega)\right\| \geq \frac{1}{m} \lambda_{\mathrm{rd}}(x_0,\omega) = \lambda_{\mathrm{rc}}(x_0,\omega),$$

which yields the existence of the limit. □

4. The maximal Lyapunov exponent. We are interested in this section in the maximal Lyapunov exponents for systems (2.3) and (3.1), i.e., the real numbers λ_1^{c} and λ_1^{d} from Proposition 3.12(iv). We denote these numbers by $\lambda_{\max}^{\mathrm{c}}$ and $\lambda_{\max}^{\mathrm{d}}$, respectively. Before proving the main results of this section, we state the following lemma, which shows that the Gelfand formula for the spectral radius ρ holds uniformly over compact sets of matrices. This follows from the estimates derived in Green [17, Section 3.3]. For the reader's convenience, we provide a proof.

LEMMA 4.1. *Let $\mathcal{A} \subset \mathcal{M}_d(\mathbb{R})$ be a compact set of matrices. Then the limit*

$$\lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A)$$

is uniform over $\mathcal{A}$.

Proof. Let $\varepsilon > 0$ and define a continuous function $F : \mathcal{A} \to \mathcal{M}_d(\mathbb{R})$ by

$$F(A) = \frac{1}{\rho(A) + \varepsilon} A.$$

Then $F(\mathcal{A})$ is compact and for every $F(A) \in F(\mathcal{A})$ its spectral radius is $\rho(F(A)) = \frac{\rho(A)}{\rho(A)+\varepsilon} < 1$. Fix $A \in \mathcal{A}$. Then (see, e.g., Horn and Johnson [21, Lemma 5.6.10]) there is a norm $\|\cdot\|_A$ in $\mathbb{R}^d$ with $\|F(A)\|_A < \frac{1+\rho(F(A))}{2}$. Then for all B in a neighborhood U of A

$$\|F(B)\|_A < \frac{1+\rho(F(A))}{2}.$$

Since all norms on $\mathcal{M}_d(\mathbb{R})$ are equivalent, there is $\beta_A > 0$ such that for all $B \in U$

$$\|F(B)^n\| \leq \beta_A \|F(B)^n\|_A \leq \beta_A \|F(B)\|_A^n \leq \beta_A \left(\frac{1+\rho(F(A))}{2}\right)^n.$$

Then there is $N \in \mathbb{N}^*$, depending only on A and ε, such that for all $n \geq N$ and all $B \in U$,

$$\frac{1}{\rho(B)+\varepsilon} \|B^n\|^{1/n} = \|F(B)^n\|^{1/n} < 1,$$

implying $\|B^n\|^{1/n} < \rho(B) + \varepsilon$. Since this holds for every B in a neighborhood U of A and $\|B^n\|^{1/n} \geq \rho(B)$ for every $n \in \mathbb{N}^*$, one obtains that the convergence in U is uniform, and the assertion follows by compactness of $\mathcal{A}$. ☐

We can now prove our first result regarding the characterization of $\lambda_{\max}^{\mathrm{c}}$ and $\lambda_{\max}^{\mathrm{d}}$.

PROPOSITION 4.2. *For almost every $\omega \in \Omega$, we have*

$$\lambda_{\max}^{\mathrm{d}} = \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|. \tag{4.1}$$

Moreover,

$$\lambda_{\max}^{\mathrm{d}} \leq \inf_{n\in\mathbb{N}^*} \frac{1}{n}\int_\Omega \log\|\Phi(n,\omega)\| \,\mathrm{d}\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n}\int_\Omega \log\|\Phi(n,\omega)\| \,\mathrm{d}\mathbb{P}(\omega). \tag{4.2}$$

Proof. Notice that (4.1) and (4.2) do not depend on the norm in $\mathcal{M}_d(\mathbb{R})$. We fix in this proof the norm induced by the Euclidean norm in $\mathbb{R}^d$, given by $\|A\| = \sqrt{\rho(A^{\mathrm{T}}A)}$. Notice that, in this case, $\left\|A^{\mathrm{T}}A\right\| = \sqrt{\rho((A^{\mathrm{T}}A)^2)} = \rho(A^{\mathrm{T}}A) = \|A\|^2$.

By Proposition 3.12(v), $e^{\lambda^{\mathrm{d}}_{\max}}$ is the spectral radius $\rho(\Psi(\omega))$ of $\Psi(\omega)$ for almost every $\omega \in \Omega$. By continuity of the spectral radius and Proposition 3.12(i), one then gets that

$$e^{\lambda^{\mathrm{d}}_{\max}} = \lim_{n\to\infty} \rho\left[\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{1/2n}\right] = \lim_{n\to\infty}\lim_{k\to\infty}\left\|\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{k/2n}\right\|^{1/k}, \tag{4.3}$$

using also Gelfand's Formula for the spectral radius. The sequence of matrices $\left(\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{1/2n}\right)_{n=1}^{\infty}$ converges to $\Psi(\omega)$, hence this sequence is bounded in $\mathcal{M}_d(\mathbb{R})$. By Lemma 4.1, the limit in Gelfand's Formula is uniform, which shows that one can take the limit in (4.3) along the line $k = 2n$ to obtain

$$e^{\lambda^{\mathrm{d}}_{\max}} = \lim_{n\to\infty}\left\|\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right\|^{1/2n} = \lim_{n\to\infty}\left\|\Phi(n,\omega)\right\|^{1/n}.$$

Hence (4.1) follows by taking the logarithm.

In order to prove (4.2), fix $m \in \mathbb{N}^*$. By (4.1), for almost every $\omega \in \Omega$,

$$\lambda^{\mathrm{d}}_{\max} = \lim_{n\to\infty}\frac{1}{nm}\log\left\|\Phi(nm,\omega)\right\|. \tag{4.4}$$

One has $\Phi(nm,\omega) = \Phi(m,\theta^{(n-1)m}\omega)\cdots\Phi(m,\theta^m\omega)\,\Phi(m,\omega)$, and thus

$$\frac{1}{nm}\log\left\|\Phi(nm,\omega)\right\| \leq \frac{1}{nm}\sum_{k=0}^{n-1}\log\left\|\Phi(m,\theta^{mk}\omega)\right\|. \tag{4.5}$$

Since θ^m preserves $\mathbb{P}$ and $\log\left\|\Phi(m,\cdot)\right\| \in L^1(\Omega,\mathbb{R})$, Birkhoff's Ergodic Theorem shows that

$$\lim_{n\to\infty}\frac{1}{nm}\sum_{k=0}^{n-1}\log\left\|\Phi(m,\theta^{mk}\omega)\right\| = \frac{1}{m}\int_\Omega \log\left\|\Phi(m,\omega)\right\|\mathrm{d}\mathbb{P}(\omega). \tag{4.6}$$

Combining (4.4), (4.5), and (4.6), one obtains the inequality in (4.2). The sequence $\left(\int_\Omega \log\left\|\Phi(n,\omega)\right\|\mathrm{d}\mathbb{P}(\omega)\right)_n$ is subadditive, since $\Phi(n+m,\omega) = \Phi(m,\theta^n\omega)\,\Phi(n,\omega)$ for $n, m \in \mathbb{N}$ and θ preserves $\mathbb{P}$. This subadditivity implies that the equality in (4.2) holds. □

Under some extra assumptions on the probability measures μ_i, $i \in \underline{N}$, one obtains that the inequality in (4.2) is actually an equality.

PROPOSITION 4.3. *Suppose there exists $r > 1$ such that, for every $i \in \underline{N}$, one has $\int_{(0,\infty)} t^r\,\mathrm{d}\mu_i(t) < \infty$. Then $\lambda^{\mathrm{d}}_{\max}$ is given by*

$$\lambda^{\mathrm{d}}_{\max} = \inf_{n\in\mathbb{N}^*}\frac{1}{n}\int_\Omega \log\left\|\Phi(n,\omega)\right\|\mathrm{d}\mathbb{P}(\omega) = \lim_{n\to\infty}\frac{1}{n}\int_\Omega \log\left\|\Phi(n,\omega)\right\|\mathrm{d}\mathbb{P}(\omega).$$

Proof. One clearly has, using (4.1), that

$$\lambda^{\mathrm{d}}_{\max} = \int_\Omega \lambda^{\mathrm{d}}_{\max}\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \lim_{n\to\infty}\frac{1}{n}\log\left\|\Phi(n,\omega)\right\|\mathrm{d}\mathbb{P}(\omega).$$

The theorem is proved if we show one can exchange the limit and the integral in the above expression, which we do by applying Vitali's convergence theorem (see, e.g., Rudin [27, Chapter 6]). We are thus left to show that the sequence of functions

$\left(\frac{1}{n}\log\|\Phi(n,\cdot)\|\right)_{n=1}^{\infty}$ is uniformly integrable, i.e., for every $\varepsilon>0$, there exists $\delta>0$ such that, for every $E\in\mathfrak{F}$ with $\mathbb{P}(E)<\delta$, one has $\frac{1}{n}\left|\int_E \log\|\Phi(n,\omega)\|\,\mathrm{d}\mathbb{P}(\omega)\right|<\varepsilon$.

For $\omega=(i_n,t_n)_{n=1}^{\infty}\in\Omega_0$ and $n\in\mathbb{N}^*$, one has $\Phi(n,\omega)=e^{L_{i_n}t_n}\cdots e^{L_{i_1}t_1}$. Let $C,\gamma>0$ be such that $\left\|e^{L_i t}\right\|\leq Ce^{\gamma t}$ for every $i\in\underline{N}$ and $t\geq 0$. Then

$$\log\|\Phi(n,\omega)\|\leq n\log C+\gamma\sum_{i=1}^{n}t_i=n\log C+\gamma s_n(\omega),$$

where $s_n(\omega)=\sum_{i=1}^{n}t_i$. Hence, it suffices to show that the sequence $\left(\frac{s_n}{n}\right)_{n=1}^{\infty}$ is uniformly integrable.

For $n\in\mathbb{N}^*$ and $E\in\mathfrak{F}$, we have, by Hölder's inequality,

$$\int_E\frac{s_n(\omega)}{n}\,\mathrm{d}\mathbb{P}(\omega)=\frac{1}{n}\sum_{i=1}^{n}\int_E t_i\,\mathrm{d}\mathbb{P}(\omega)\leq\frac{1}{n}\sum_{i=1}^{n}\left(\int_\Omega t_i^r\,\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{r}}\mathbb{P}(E)^{\frac{1}{r'}}\leq K^{\frac{1}{r}}\mathbb{P}(E)^{\frac{1}{r'}},\tag{4.7}$$

where $r'\in(1,\infty)$ is such that $\frac{1}{r}+\frac{1}{r'}=1$ and $K=\max_{i\in\underline{N}}\int_{(0,\infty)}t^r\,\mathrm{d}\mu_i(t)<\infty$. Equation (4.7) establishes the uniform integrability of $\left(\frac{s_n}{n}\right)_{n=1}^{\infty}$, which yields the result. ☐

As an immediate consequence of Proposition 3.7, Proposition 3.8, Proposition 4.2, and Proposition 4.3, we obtain the following result.

Corollary 4.4. *The maximal Lyapunov exponents $\lambda_{\max}^{\mathrm{c}}$ and $\lambda_{\max}^{\mathrm{d}}$ satisfy*

$$m\lambda_{\max}^{\mathrm{c}}=\lambda_{\max}^{\mathrm{d}}\leq\inf_{n\in\mathbb{N}^*}\frac{1}{n}\int_\Omega\log\|\Phi(n,\omega)\|\,\mathrm{d}\mathbb{P}(\omega).\tag{4.8}$$

In particular, if

$$\text{there exists } n\in\mathbb{N}^* \text{ such that } \int_\Omega\log\|\Phi(n,\omega)\|\,\mathrm{d}\mathbb{P}(\omega)<0,\tag{4.9}$$

then systems (2.3) *and* (3.1) *are almost surely exponentially stable.*

If we have further that there exists $r>1$ such that $\int_{\mathbb{R}_+}t^r\,\mathrm{d}\mu_i(t)<\infty$ for every $i\in\underline{N}$, then the inequality in (4.8) *is an equality and* (4.9) *is equivalent to the almost sure exponential stability of* (2.3) *and to the almost sure exponential stability of* (3.1)*.*

We conclude this section with the following characterization of a weighted sum of the Lyapunov exponents λ_i^{d}, $i\in\underline{N}$.

Proposition 4.5. *Suppose there exists $r>1$ such that, for every $i\in\underline{N}$, one has $\int_{(0,+\infty)}t^r\,\mathrm{d}\mu_i(t)<\infty$. Then*

$$\sum_{i=1}^{q}m_i\lambda_i^{\mathrm{d}}=\sum_{i=1}^{N}p_i\tau_i\operatorname{Tr}(L_i),\tag{4.10}$$

where m_i is as in Proposition 3.12(v).

Proof. Thanks to Proposition 3.12(v), one obtains that, for almost every $\omega=(i_n,t_n)_{n=1}^{\infty}\in\Omega$,

$$\det\Psi(\omega)=\prod_{i=1}^{q}e^{m_i\lambda_i^{\mathrm{d}}},$$

which yields

$$\sum_{i=1}^{q} m_i \lambda_i^{\mathrm{d}} = \log\det\Psi(\omega) = \lim_{n\to\infty} \log\det\left(\Phi(n,\omega)^{\mathrm{T}}\Phi(n,\omega)\right)^{1/2n}$$
$$= \lim_{n\to\infty} \log\left(\prod_{k=1}^{n} \det e^{L_{i_k} t_k}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k \operatorname{Tr}(L_{i_k}).$$

Then

$$\sum_{i=1}^{q} m_i \lambda_i^{\mathrm{d}} = \int_\Omega \sum_{i=1}^{q} m_i \lambda_i^{\mathrm{d}} \,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k \operatorname{Tr}(L_{i_k}) \,\mathrm{d}\mathbb{P}(\omega)$$
$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \int_\Omega t_k \operatorname{Tr}(L_{i_k}) \,\mathrm{d}\mathbb{P}(\omega) = \sum_{i=1}^{N} p_i \tau_i \operatorname{Tr}(L_i),$$

where we exchange limit and integral thanks to Vitali's convergence theorem and to the fact that $\left(\frac{s_n(\omega)}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n}\sum_{k=1}^{n} t_k\right)_{n=1}^{\infty}$ is uniformly integrable, as shown in the proof of Proposition 4.3. ▯

5. Main result. In this section, we use the stability criterion from Corollary 4.4 to study the stabilization by linear feedback laws of (1.1). As stated in the Introduction, we write (1.1) under the form (1.5), which is a switched control system with dynamics given by the N equations $\dot{x} = \widehat{A}x + \widehat{B}_i u_i$, $i \in \underline{N}$.

We consider system (1.5) in a probabilistic setting by taking random signals $\boldsymbol{\alpha}(\omega)$ as in Definition 2.3, i.e., the random control system $\dot{x}(t) = \widehat{A}x(t) + \widehat{B}_{\boldsymbol{\alpha}(\omega)(t)} u_{\boldsymbol{\alpha}(\omega)(t)}(t)$. The problem treated in this section is the arbitrary rate stabilizability of this system by linear feedback laws $u_i = K_i P_i x$, $i \in \underline{N}$, where we recall that $P_i \in \mathcal{M}_{d_i,d}(\mathbb{R})$ is the projection onto the i-th factor of $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}$. More precisely, we consider the closed-loop random switched system

$$\dot{x}(t) = \left(\widehat{A} + \widehat{B}_{\boldsymbol{\alpha}(\omega)(t)} K_{\boldsymbol{\alpha}(\omega)(t)} P_{\boldsymbol{\alpha}(\omega)(t)}\right) x(t). \tag{5.1}$$

We wish to know if, given $\lambda \in \mathbb{R}$, there exist matrices $K_i \in \mathcal{M}_{m_i,d_i}(\mathbb{R})$, $i \in \underline{N}$, such that the maximal Lyapunov exponent $\lambda_{\max}^{\mathrm{c}}$ of the continuous-time system (5.1), defined as in Section 4, satisfies $\lambda_{\max}^{\mathrm{c}} \leq \lambda$. Our main result is the following, which states that this is true under the controllability of (A_i, B_i) for every $i \in \underline{N}$.

THEOREM 5.1. *Let $N \in \mathbb{N}^*$, $d_1, \dots, d_N, m_1, \dots, m_N \in \mathbb{N}$, $A_i \in \mathcal{M}_{d_i}(\mathbb{R})$, $B_i \in \mathcal{M}_{d_i,m_i}(\mathbb{R})$ for $i \in \underline{N}$, and assume that (A_i, B_i) is controllable for every $i \in \underline{N}$. Define $\widehat{A}$ and $\widehat{B}_i$ as in* (1.6)*. Then, for every $\lambda \in \mathbb{R}$, there exist matrices $K_i \in \mathcal{M}_{m_i,d_i}(\mathbb{R})$, $i \in \underline{N}$, such that the maximal Lyapunov exponent $\lambda_{\max}^{\mathrm{c}}$ of the closed-loop random switched system* (5.1) *satisfies $\lambda_{\max}^{\mathrm{c}} \leq \lambda$.*

Proof. Let $C \geq 1$, $\beta > 0$ be such that, for every $i \in \underline{N}$ and every $t \geq 0$, $\left\|e^{A_i t}\right\| \leq C e^{\beta t}$. Thanks to Cheng, Guo, Lin, and Wang [7, Proposition 2.1], we may assume that C is chosen large enough such that the following property holds: there exists $D \in \mathbb{N}^*$ such that, for every $\gamma \geq 1$ and $i \in \underline{N}$, there exists a matrix $K_i \in \mathcal{M}_{m_i,d_i}(\mathbb{R})$ with

$$\left\|e^{(A_i + B_i K_i)t}\right\| \leq C\gamma^D e^{-\gamma t}, \qquad \forall t \in \mathbb{R}_+. \tag{5.2}$$

Let $\widehat{K}_i = K_i P_i \in \mathcal{M}_{m_i,d}(\mathbb{R})$. Then

$$\widehat{A} + \widehat{B}_i \widehat{K}_i = \begin{pmatrix} A_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_i + B_i K_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & A_N \end{pmatrix},$$

and thus, for every $t \in \mathbb{R}$,

$$e^{(\widehat{A} + \widehat{B}_i \widehat{K}_i)t} = \begin{pmatrix} e^{A_1 t} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & e^{A_2 t} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{(A_i + B_i K_i)t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & e^{A_N t} \end{pmatrix}.$$

Since M is irreducible and p is invariant under M, we have $p_i > 0$ for every $i \in \underline{N}$. The irreducibility of M also provides the existence of $r \geq N$ and $(i_1^*, \dots, i_r^*) \in \underline{N}^r$ such that $\{i_1^*, \dots, i_r^*\} = \underline{N}$ and $M_{i_1^* i_2^*} \cdots M_{i_{r-1}^* i_r^*} > 0$. In order to apply Corollary 4.4, consider

$$\begin{aligned} \int_\Omega \log \|\Phi(r, \omega)\| \,\mathrm{d}\mathbb{P}(\omega) = & \sum_{(i_1, \dots, i_r) \in \underline{N}^r} p_{i_1} M_{i_1 i_2} \cdots M_{i_{r-1} i_r} \\ & \cdot \int_{(0,\infty)^r} \log \left\| e^{(\widehat{A} + \widehat{B}_{i_r} \widehat{K}_{i_r}) t_r} \cdots e^{(\widehat{A} + \widehat{B}_{i_1} \widehat{K}_{i_1}) t_1} \right\| \mathrm{d}\mu_{i_1}(t_1) \cdots \mathrm{d}\mu_{i_r}(t_r). \end{aligned} \tag{5.3}$$

Since $\sum_{i=1}^N P_i^{\mathrm{T}} P_i = \mathrm{Id}_d$ and $P_i e^{(\widehat{A} + \widehat{B}_i \widehat{K}_i)t} P_j^{\mathrm{T}} = 0$ if $i \neq j$, we have, for every $(i_1, \dots, i_r) \in \underline{N}^r$ and $(t_1, \dots, t_r) \in \mathbb{R}_+^r$,

$$\begin{aligned} & e^{(\widehat{A} + \widehat{B}_{i_r} \widehat{K}_{i_r}) t_r} \cdots e^{(\widehat{A} + \widehat{B}_{i_1} \widehat{K}_{i_1}) t_1} \\ = & \left(\sum_{j_r=1}^N P_{j_r}^{\mathrm{T}} P_{j_r} \right) e^{(\widehat{A} + \widehat{B}_{i_r} \widehat{K}_{i_r}) t_r} \cdots \left(\sum_{j_1=1}^N P_{j_1}^{\mathrm{T}} P_{j_1} \right) e^{(\widehat{A} + \widehat{B}_{i_1} \widehat{K}_{i_1}) t_1} \left(\sum_{j_0=1}^N P_{j_0}^{\mathrm{T}} P_{j_0} \right) \\ = & \sum_{i=1}^N P_i^{\mathrm{T}} P_i e^{(\widehat{A} + \widehat{B}_{i_r} \widehat{K}_{i_r}) t_r} \cdots P_i^{\mathrm{T}} P_i e^{(\widehat{A} + \widehat{B}_{i_1} \widehat{K}_{i_1}) t_1} P_i^{\mathrm{T}} P_i \\ = & \sum_{i=1}^N P_i^{\mathrm{T}} e^{(A_i + \delta_{i i_r} B_i K_i) t_r} \cdots e^{(A_i + \delta_{i i_1} B_i K_i) t_1} P_i. \end{aligned} \tag{5.4}$$

Since, for every $i \in \underline{N}$ and $t \geq 0$, we have $\left\| e^{A_i t} \right\| \leq C e^{\beta t}$ and $\left\| e^{(A_i + B_i K_i)t} \right\| \leq C \gamma^D e^{-\gamma t}$, we get, for every $(i_1, \dots, i_r) \in \underline{N}^r$ and $(t_1, \dots, t_r) \in \mathbb{R}_+^r$,

$$\left\| e^{(\widehat{A} + \widehat{B}_{i_r} \widehat{K}_{i_r}) t_r} \cdots e^{(\widehat{A} + \widehat{B}_{i_1} \widehat{K}_{i_1}) t_1} \right\| \leq N C^r \gamma^{rD} e^{\beta \sum_{i=1}^r t_i}. \tag{5.5}$$

When $(i_1, \dots, i_r) = (i_1^*, \dots, i_r^*)$, we can obtain a sharper bound than (5.5). For $i \in \underline{N}$, denote by $N(i)$ the nonempty set of all indices $k \in \underline{r}$ such that $i_k^* = i$, and denote by $n(i) \in \mathbb{N}^*$ the number of elements in $N(i)$. Then

$$\left\| P_i^{\mathrm{T}} e^{(A_i + \delta_{i i_r^*} B_i K_i) t_r} \cdots e^{(A_i + \delta_{i i_1^*} B_i K_i) t_1} P_i \right\| \leq C^r \gamma^{n(i) D} e^{-\gamma \sum_{k \in N(i)} t_k} e^{\beta \sum_{k \in \underline{r} \setminus N(i)} t_k},$$

which shows, using (5.4), that

$$\begin{aligned} \left\| e^{(\widehat{A} + \widehat{B}_{i_r^*} \widehat{K}_{i_r^*}) t_r} \cdots e^{(\widehat{A} + \widehat{B}_{i_1^*} \widehat{K}_{i_1^*}) t_1} \right\| &\leq \sum_{i=1}^{N} C^r \gamma^{n(i) D} e^{-\gamma \sum_{k \in N(i)} t_k} e^{\beta \sum_{k \in \underline{r} \setminus N(i)} t_k} \\ &\leq N C^r \gamma^{r D} e^{-\gamma \min_{k \in \underline{r}} t_k} e^{r \beta \max_{k \in \underline{r}} t_k}. \end{aligned} \tag{5.6}$$

Let

$$\begin{aligned} E_0 &= \max_{i \in \underline{N}} \tau_i, \\ E_{\min} &= \int_{(0,\infty)^r} \min_{k \in \underline{r}} t_k \, \mathrm{d}\mu_{i_1^*}(t_1) \cdots \mathrm{d}\mu_{i_r^*}(t_r) > 0, \\ E_{\max} &= \int_{(0,\infty)^r} \max_{k \in \underline{r}} t_k \, \mathrm{d}\mu_{i_1^*}(t_1) \cdots \mathrm{d}\mu_{i_r^*}(t_r) < \infty. \end{aligned}$$

Then, combining (5.5) and (5.6), we obtain from (5.3) that

$$\begin{aligned} \int_\Omega \log \left\| \mathit{\Phi}(r, \omega) \right\| \mathrm{d}\mathbb{P}(\omega) &\leq N^r \left(\log(N C^r) + r D \log \gamma + r \beta E_0 \right) \\ &\quad + p_{i_1^*} M_{i_1^* i_2^*} \cdots M_{i_{r-1}^* i_r^*} \left(\log(N C^r) + r D \log \gamma - \gamma E_{\min} + r \beta E_{\max} \right). \end{aligned} \tag{5.7}$$

The right-hand side of (5.7) tends to $-\infty$ as $\gamma \to \infty$, which can be achieved by (5.2). Hence it follows from Corollary 4.4 that the maximal Lyapunov exponent of (5.1) can be made arbitrarily small. ◻

Recall that the main motivation for Theorem 5.1 comes from the stabilizability of persistently excited systems (1.3) under linear feedback laws. It was proved in [10, Proposition 4.5] that there are (two dimensional) controllable systems for which the achievable decay rates under persistently exciting signals through linear feedback laws are bounded below, even when we consider only persistently exciting signals α taking values in $\{0, 1\}$ instead of $[0, 1]$. Theorem 5.1 implies that, in the probabilistic setting defined above, one can get arbitrarily large (almost sure) decay rates for the generalization (1.5) of system (1.3), which is in contrast to the situation for persistently excited systems. An explanation for this fact is that the probability of having a signal α with very fast switching for an infinitely long time, such as the signals used in the proof of [10, Proposition 4.5], is zero, and hence such signals do not interfere with the behavior of the (random) maximal Lyapunov exponent.

REMARK 5.2. *In order to establish a more precise link between Theorem 5.1 and the case of deterministic persistently excited systems treated in [5, 8, 9, 10, 26], consider the case of* (1.3) *with* $\alpha(t) \in \{0, 1\}$ *and* (A, B) *controllable. This is a special case of (1.1) if we let* $A_1 = A, B_1 = B$ *and add a trivial second subsystem with* $d_2 = m_2 = 0$. *In this remark,* α *denotes the signal* α_1 *from* (1.1), *instead of the signal* α *from* (1.5). *In order to simplify, we assume that, in the probabilistic model of* $\boldsymbol{\alpha}$, *trivial switches do not occur, which amounts to choosing*

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with the unique invariant probability vector $p = \left(\frac{1}{2}, \frac{1}{2}\right)$. Notice that the assumptions of Theorem 5.1 are satisfied in this framework.

In general, such signals $\boldsymbol{\alpha}(\omega)$ cannot be persistently exciting. In fact, suppose that the measure μ_2 satisfies $\mu_2((0,T]) < 1$ for every $T > 0$ (recall that μ_i determines how long the system remains in the state i, and that $i = 2$ corresponds to the activation of the trivial subsystem). Then

$$\mathbb{P}\{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \boldsymbol{\alpha}(\omega) \text{ is a PE } (T,\mu)\text{-signal}\} = 0. \tag{5.8}$$

Indeed, since a (T,μ)-signal is also a (T',μ')-signal for every $T' \geq T$ and $0 < \mu' \leq \mu$, we have

$$\begin{aligned} &\{\omega \in \Omega \mid \exists T \geq \mu > 0 \text{ such that } \boldsymbol{\alpha}(\omega) \text{ is a PE } (T,\mu)\text{-signal}\} \\ = &\bigcup_{T>0} \bigcup_{\mu \in (0,T]} \{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \text{ is a PE } (T,\mu)\text{-signal}\} \\ = &\bigcup_{T \in \mathbb{N}^*} \bigcup_{\frac{1}{\mu} \in \mathbb{N}^*} \{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \text{ is a PE } (T,\mu)\text{-signal}\}. \end{aligned}$$

If α is a PE (T,μ)-signal, the PE condition implies that α cannot remain zero during time intervals longer than $T - \mu$, and thus

$$\begin{aligned} &\{\omega \in \Omega \mid \boldsymbol{\alpha}(\omega) \text{ is a PE } (T,\mu)\text{-signal}\} \\ \subset &\{\omega = (i_n, t_n)_{n=1}^{\infty} \in \Omega \mid \forall n \in \mathbb{N}^*,\ i_n = 2 \implies t_n \leq T - \mu\}. \end{aligned} \tag{5.9}$$

Since i_n takes the value 2 infinitely many times for almost every $\omega \in \Omega$ and $\mu_2((0, T - \mu]) < 1$, the right-hand side of (5.9) has measure zero, and thus (5.8) holds.

However, one can link the random signals $\boldsymbol{\alpha}(\omega)$ with a weaker, asymptotic notion of persistence of excitation. A (deterministic) measurable signal $\alpha : \mathbb{R}_+ \to [0,1]$ is said to be asymptotically persistently exciting *with constant $\rho > 0$ if*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t \alpha(s)\, \mathrm{d}s \geq \rho.$$

It follows easily from (1.4) that every persistently exciting (T,μ)-signal is also asymptotically persistently exciting with constant $\rho = \frac{\mu}{T}$. Proposition 3.9 implies that, for almost every $\omega \in \Omega$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \boldsymbol{\alpha}(\omega)(s)\, \mathrm{d}s = \frac{\tau_1}{\tau_1 + \tau_2},$$

and thus, in particular, almost every signal $\boldsymbol{\alpha}(\omega)$ is asymptotically persistently exciting with constant $\rho = \frac{\tau_1}{\tau_1 + \tau_2} > 0$.

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