A Splitting Theorem for Affine Skew-Product Flows

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1 Introduction

This paper discusses topological conjugacy for skew product flows on topological group bundles and semidirect products. One one hand, our results are generalizations of contributions to topological conjugacy of linear and affine skew product flows, which were mainly motivated by bilinear control systems; see Ayala, Colonius and Kliemann [2] and Colonius and Santana [5], as well as Kawan, Rocio and Santana [7] for flows on Lie groups. On the other hand, there are close analogies to classical (and much more general) results by Lind [9] in the framework of ergodic skew product systems and splitting theorems. Our results may be viewed as topological versions of some of Lind's results.

The contents of this paper is as follows: In Section 2 some basic properties of topological groups and semidirect products are collected. Section 3 discusses flows on semidirect products, in particular, of Lie groups and topological conjugacy on theses groups. Section 4 generalizes the discussion of topological conjugacies to skew product flows on topological group bundles. Theorem 12 shows that topological skew conjugacy is equivalent to solvability of a certain functional equation. It is shown that conjugacy results for linear and affine skew product which are valid for hyperbolic systems are a special case. Furthermore the relation to Lind's [9] functional equation and his splitting theorem for measurable conjugacy is explained. Finally, Section 5 presents an application to skew-product transformation semigroups and fiber entropy.

2 Preliminaries

In this section we recall some notations (see, e.g., Kechris [8, Appendix I]) and facts on automorphisms of topological groups.

Let $(H, \otimes)$ and $(G, +)$ be topological groups. Consider the following continuous action given by homeomorphisms:

$$\xi : H \times G \to G \text{ such that } \xi(a, b) = a \star b.$$

Suppose that for all $h_1, h_2 \in H$ and $g \in G$

$$e_H \star g = g, (h_1 \otimes h_2) \star g = h_1 \star (h_2 \star g), \tag{1}$$

and for all $h \in H$ and $g_1, g_2 \in G$

$$h \star e_G = e_G, h \star (g_1 + g_2) = h \star g_1 + h \star g_2, \tag{2}$$

where e_H and e_G are the neutral elements of H and G, respectively. A simple consequence is that for all $h \in H, g \in G$

$$h \star g + h \star (-g) = h \star (g + (-g)) = h \star e_G = e_G, \text{ hence } h \star (-g) = -(h \star g). \tag{3}$$

Furthermore h induces an automorphism (i.e., an invertible topological group homomorphism) S_h defined by $S_h(g) := h \star g, g \in G$. The homomorphism property is (2) and invertibility follows using (1)

$$(S_{h^{-1}} \circ S_h)(g) = S_{h^{-1}}(h \star g) = h^{-1} \star (h \star g) = \left(h^{-1} \otimes h\right) \star g = e_H \star g = g, g \in G.$$

Then the semi-direct product $H \rtimes G$ of H with G is given by the group structure defined for $(h_1, g_1), (h_2, g_2) \in H \times G$ as

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \otimes h_2, h_1 \star g_2 + g_1).$$

Note that the neutral element is (e_H, e_G) and the inverse is given by $(h, g)^{-1} = (h^{-1}, h^{-1} \star -g)$.

Clearly, $H \rtimes G$ again is a topological group and there is a natural continuous action of $H \rtimes G$ on G given by

$$(h, g_1) \cdot g_2 = h \star g_2 + g_1 \text{ for } h \in H, g_1, g_2 \in G. \tag{4}$$

An affine automorphism of G is any map σ of the form $\sigma(g) = \zeta(g) + h$, where h is a fixed element of G and ζ is an automorphism of G. Any such σ can be identified with the pair (ζ, h), ζ being the automorphic part of σ and $h = \sigma(e_G)$, the translation part. If $\sigma_1 = (\zeta, h_1)$, $\sigma_2 = (\zeta, h_2)$ are affine automorphisms, then $\sigma_1\sigma_2 = (\zeta_1\zeta_2, \zeta_1(h_1) + h_2)$, thus the group of affine automorphisms of G can be identified with the semidirect product of the automorphism group of G with G, where the automorphism group of G acts by evaluation on G. If now H acts by automorphisms on G, then $H \rtimes G$ acts by affine automorphisms on G given by above equality (4). If H acts on G faithfully (i.e., if $h \star g = g$ for all g, then $h = e_H$), then H can be viewed as a group of automorphisms of G. In this case also $H \rtimes G$ acts faithfully on G, hence $H \rtimes G$ can be viewed as a group of affine automorphisms of G.

If H and G are Lie groups, the Lie algebra of $H \rtimes G$ is denoted by $\mathfrak{h} \rtimes \mathfrak{g}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{g}$ is the Lie algebra of G. Note that for simplicity we use the same product symbol for the group and algebra. As usual $\exp_{H \rtimes G} : \mathfrak{h} \rtimes \mathfrak{g} \to H \rtimes G$ denotes the known exponential map.

3 Flows on semidirect products of Lie groups

In this section we discuss topological conjugacy of flows on semidirect products of Lie groups.

Recall the following concepts. Let M be a smooth manifold and $\mathfrak{X}(M)$ the space of smooth vector fields on M. Let $X_1, X_2 \in \mathfrak{X}(M)$ be complete vector fields and Φ_1 and Φ_2 their associated flows (or dynamical systems). The flows Φ_1 and Φ_2, and the vector fields X_1, X_2, are called topologically conjugate if there exists a homeomorphism $h : M \to M$ such that $h(\Phi_1(t,x)) = \Phi_2(t, h(x))$ for all $x \in M$ and $t \in \mathbb{R}$, and h is called a topological conjugacy.

As in the classical case of linear autonomous differential equations here the existence of homeomorphic fundamental domains is essential. We define a type of fundamental domain in a general context: Consider a topological space M and a flow $\Phi : \mathbb{R} \times M \longrightarrow M$. A *fundamental domain* of Φ is a pair (Z, τ) where Z is a subset of M and $\tau : M \to \mathbb{R}$ is a continuous map such that for all $x \in M$ it holds that $\Phi(t,x) \in Z$ if and only if $t = \tau(x)$.

Let $\pi : H \rtimes G \to H$ be the canonical projection onto the Lie group H. If $\Upsilon_t(h,g) = \exp_{H \rtimes G}(tX)(h,g)$ is a flow on $H \rtimes G$, where $X = (A,B) \in \mathfrak{h} \rtimes \mathfrak{g}$ and $(h,g) \in H \rtimes G$, we denote by Ω the flow on H given by

$$\Omega_t(h) = \exp_H(tA)h \text{ with } h \in H. \tag{5}$$

The next result from Moskowitz and Sacksteder [10, Theorem 3.1] describes the flow on $H \rtimes G$. In this theorem, the exponential map $\exp_{H \rtimes G}$ for a semidirect product $H \rtimes G$ of arbitrary connected Lie groups H and G is given by

$$\exp_{H \rtimes G}(h,g) = (\exp_H(h), \mathcal{E}_G^D(g)),$$

where $\mathcal{E}_G^D$ is defined as follows: Let $g(t)$ be the solution of

$$g'(t) = dL_{g(t)}\gamma(t) \tag{6}$$

with $g(0) = e_G$, $\gamma(t) = \exp(tD)\gamma_0$ and $\gamma : \mathbb{R} \to \mathfrak{g}$. Define

$$\mathcal{E}_G^D : \mathfrak{g} \to G, \quad \mathcal{E}_G^D(\gamma_0) := g(1).$$

where h is identified with $(h,0)$ and hence induces a derivation $D = \mathrm{ad}_h$ on the Lie algebra of G.

Lemma 1 *For $X = (A,B) \in \mathfrak{h} \rtimes \mathfrak{g}$, the flow $\Upsilon_t(h,g) = \exp_{H \rtimes G}(tX) \cdot (h,g)$ on $H \rtimes G$ is given by*

$$\exp_{H \rtimes G}(tX) \cdot (h,g) = \left(\exp_H(tA)h, \mathcal{E}_G^D(tB)\right).$$

Let M and P be topological spaces and $\pi : M \times P \to M$ the canonical projection, then we have the following result proved in Kawan, Rocio and Santana [7, Corollary 4].

Lemma 2 *Let M and P be topological spaces and $\pi : M \times P \to M$ the canonical projection. Assume that $\widetilde{\Lambda}^1$ and $\widetilde{\Lambda}^2$ are flows on M, and Λ^1 and Λ^2 are flows on $M \times P$ such that the diagram*

$$\begin{tikzcd} M \times P \arrow[r, "\Lambda_t^i"] \arrow[d, "\pi"'] & M \times P \arrow[d, "\pi"] \\ M \arrow[r, "\widetilde{\Lambda}_t^i"'] & M \end{tikzcd}$$

commutes for every $t \in \mathbb{R}$. Suppose that there exist fundamental domains $(\widetilde{W}^i, \widetilde{\tau}^i)$ for the flows $\widetilde{\Lambda}^i$ such that $\widetilde{W}^1$ is homeomorphic to $\widetilde{W}^2$. Then Λ^1 is topologically conjugate to Λ^2.

The rest of this section is an adaptation of [7, Proposition 12 and Remark 13].

Proposition 3 *Consider the affine group $H \rtimes G$ for Lie groups G and H. Let $\pi : H \rtimes G \to H$ be the canonical projection. Take flows $\Upsilon_t^i(h,g) = \exp_{H \rtimes G}(tX_i)(h,g)$, $i = 1,2$, on $H \rtimes G$ and suppose that there exist homeomorphic fundamental domains for the flows Ω^i on H as in (5). Then Υ^1 is topologically conjugate to Υ^2.*

Proof. Let $X_i = (A_i, B_i) \in \mathfrak{h} \rtimes \mathfrak{g}$ for $i = 1,2$. By Lemma 1, we have that for every $t \in \mathbb{R}$

$$\pi(\Upsilon_t^i(h,g)) = \pi(\exp_{H \rtimes G}(t(A_i,B_i))(h,g)) = \exp_H(tA_i)h = \Omega_t^i(h) = \Omega_t^i(\pi(h,g)).$$

That is, the following diagram commutes:

$$\begin{tikzcd} H \rtimes G \arrow[r, "\Upsilon_t^i(\cdot)"] \arrow[d, "\pi"'] & H \rtimes G \arrow[d, "\pi"] \\ H \arrow[r, "\Omega_t^i(\cdot)"'] & H \end{tikzcd}$$

Then the proposition follows from Lemma 2. ■

As a consequence we obtain the following corollary.

Corollary 4 *If H is a nilpotent or abelian group, then any two flows on $H \rtimes G$ are topologically conjugate.*

Proof. In fact, using the same notation as above, take flows $\Upsilon_t^i(h,g) = \exp(tX_i)(h,g)$, $i = 1,2$, on $H \rtimes G$. Then $\Omega_t^i(h) = \exp_H(tA_i)h$ with $h \in H$ are flows on H. Hence, by Section 3 in [7] there exist fundamental domains for Ω^i, $i = 1,2$. Therefore, by Proposition 3 it follows that Υ^1 is topologically conjugate to Υ^2, if H is nilpotent or abelian. ■

Another consequence is the following result for a semisimple Lie group H. Consider the Iwasawa decomposition $H = ANK = NAK$. Take $X_i = (A_i, B_i) \in \mathfrak{h} \rtimes \mathfrak{g}$, but with $A_i \in \mathfrak{a}$ for $i = 1, 2$ or $A_i \in \mathfrak{n}$ for $i = 1, 2$, where $\mathfrak{a}$ and $\mathfrak{n}$ are the Lie algebras of A and N, respectively. Then again by Section 3 in [7] the flows on H, $\Omega_t^1(h) = \exp_H(tA_1)h$ and $\Omega_t^2(h) = \exp_H(tA_2)h$ are topologically conjugate. Hence, by the last proposition it follows that these flows on $H \rtimes G$ are topologically conjugate.

4 Skew product flows and semidirect products

In this section we generalize our discussion of topological conjugacy to skew product flows on topological group bundles.

Consider with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$ a continuous dynamical system $\vartheta_t : B \to B, t \in \mathbb{T}$, on a base space B, which is a metric space, and consider the topological group bundle $H \times B$. Then let $\Phi : \mathbb{T} \times H \times B \to H \times B$ be a continuous map of the form

$$\Phi_t(h, b) := (\phi_t(h, b), \vartheta_t b), \tag{7}$$

where the first component satisfies for all $s, t \in \mathbb{T}$ and $b \in B, h \in H$

$$\phi_0(h, b) = h, \phi_{t+s}(h, b) = \phi_t(e_H, \vartheta_s b) \otimes \phi_s(h, b). \tag{8}$$

Then one finds for $s = 0$

$$\phi_t(h, b) = \phi_t(e_H, b) \otimes \phi_0(h, b) = \phi_t(e_H, b) \otimes h,$$

and hence ϕ satisfies the cocycle property

$$\phi_{t+s}(h, b) = \phi_t(e_H, \vartheta_s b) \otimes \phi_s(h, b) = \phi_t(\phi_s(h, b), \vartheta_s b) \text{ for all } t, s \in \mathbb{R}.$$

Thus Φ is a skew product system, and hence this construction provides us with a class of continuous skew product systems on $H \times B$, where by (8) the associated cocycle ϕ_t is compatible with the group structure on H.

Remark 5 *Here and in the following we consider group bundles which are topologically trivial. The results are also valid, when B is a compact metric space and the considered group bundle is only locally trivial.*

Now we will construct "affine" skew product systems Ψ on $(H \rtimes G) \times B$ which are compatible with the group structures and with the natural actions of H on G and of $H \rtimes G$ on G.

Definition 6 *A skewing function $a : \mathbb{T} \times B \to G$ is a continuous map satisfying $a(0, b) = e_G$ and*

$$a(t+s, b) = \phi_t(e_H, \vartheta_s b) \star a(s, b) + a(t, \vartheta_s b) = (\phi_t(e_H, \vartheta_s b), a(t, \vartheta_s b)) \cdot a(s, b). \tag{9}$$

We think of a as an inhomogeneous term. We use it to define a skew product system Ψ on $(H \rtimes G) \times B$. Consider, for $t \in \mathbb{T}$, the map Ψ_t defined by

$$\Psi_t : (H \rtimes G) \times B \rightarrow (H \rtimes G) \times B,\ \Psi_t(h, g, b) := (\psi_t(h, g, b), \vartheta_t b)$$

with $\psi_t : (H \rtimes G) \times B \rightarrow H \rtimes G$ given by

$$\psi_t(h, g, b) := (\phi_t(h, b), \phi_t(e_H, b) \star g + a_t(b)) = (\phi_t(h, b), (\phi_t(e_H, b), a_t(b)) \cdot g) \quad (10)$$

and ϕ_t given in (7) and (8). Since $a : \mathbb{T} \times B \to G$ is continuous, also $\psi : \mathbb{T} \times (H \rtimes G) \times B \rightarrow H \rtimes G$ and $\Psi : \mathbb{T} \times (H \rtimes G) \times B \rightarrow (H \rtimes G) \times B$ are continuous. By condition (9) the dynamical system property for Ψ or, equivalently, the cocycle property for the first component,

$$\psi_{t+s}(h, g, b) = \psi_t(\psi_s(h, g, b), \vartheta_s b) \in H \rtimes G$$

holds: The left hand side is

$$\psi_{t+s}(h, g, b) = (\phi_{t+s}(h, b), \phi_{t+s}(e_H, b) \star g + a_{t+s}(b)) \quad (11)$$

and the right hand side is

$$\begin{aligned} &\psi_t(\psi_s(h, g, b), \vartheta_s b) \\ &= \psi_t((\phi_s(h, b), \phi_s(e_H, b) \star g + a_s(b)), \vartheta_s b) \\ &= (\phi_t(\phi_s(h, b), \vartheta_s b), \phi_t(e_H, \vartheta_s b) \star [\phi_s(e_H, b) \star g + a_s(b)] + a_t(\vartheta_s b)). \end{aligned} \quad (12)$$

The first components of (11) and (12) (i.e., the components in G) coincide. For the second component (in H) we use the distributive law (2), the cocycle property of ϕ and the defining property (9) of a. It follows that

$$\begin{aligned} &\phi_t(e_H, \vartheta_s b) \star [\phi_s(e_H, b) \star g + a_s(b)] + a_t(\vartheta_s b) \quad (13) \\ &= \phi_t(e_H, \vartheta_s b) \star \phi_s(e_H, b) \star g + \phi_t(e_H, \vartheta_s b) \star a_s(b) + a_t(\vartheta_s b) \\ &= \phi_{t+s}(e_H, b) \star g + a_{t+s}(b). \end{aligned}$$

Thus Ψ_t is given by the skew product flow Φ on $G \times B$ (with base flow ϑ on B and cocycle ϕ) and the associated skewing map (the affine term)

$$a : \mathbb{R} \times B \to G : (t, b) \mapsto a_t(b)$$

satisfying (9).

We illustrate the constructions above by discussing the relations to linear and affine flows. A linear skew product flow $\Phi_t(h, b) = (\phi_t(h, b), \vartheta_s b)$ on the group bundle $\mathrm{Gl}(d, \mathbb{R}) \times B \to B$ with $H = \mathrm{Gl}(d, \mathbb{R})$ is defined as follows. Here $\otimes$ is the matrix multiplication in $\mathrm{Gl}(d, \mathbb{R})$ and $e_H = I$ and for $b \in B, h \in \mathrm{Gl}(d, \mathbb{R})$ one has $\phi_t(e_H, b) = \phi_t(e_H, b) \otimes h = \phi_t(I, b)h$ and

$$\phi_0(h, b) = h \text{ and } \phi_{t+s}(h, b) = \phi_t(I, \vartheta_s b)\phi_s(h, b), \quad (14)$$

hence

$$\Phi_{t+s}(h,b) = (\phi_{t+s}(h,b), \vartheta_{t+s}b) = (\phi_t(I, \vartheta_s b)\phi_s(h,b), \vartheta_t\vartheta_s b)$$
$$= \Phi_t(\phi_s(h,b), \vartheta_s b) = \Phi_t(\Phi_s(h,b)).$$

More specifically, this flow may be generated by a homogeneous bilinear control system in $\mathrm{Gl}(d,\mathbb{R})$ of the form

$$\dot{X} = \left[A_0 + \sum_{i=1}^{m} u_i(t)A_i\right] X, u \in \mathcal{U}; \tag{15}$$

here the controls $u = (u_i)$ are taken in

$$\mathcal{U} := \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m)\, | u(t) \in U \text{ for almost all } t \in \mathbb{R}\},$$

where U is a compact convex subset of $\mathbb{R}^m$. Then $B = \mathcal{U}$ is a metrizable weak* compact set and the shift $\vartheta : \mathbb{Z} \times \mathcal{U} \times \mathcal{U}, (t,u) \mapsto u(t+\cdot)$ is continuous. Let $X_u(t)$ be the solution of the ordinary differential equation in (15) with initial condition $X_u(0) = I$. Then one has for $(h,u) \in \mathrm{Gl}(d,\mathbb{R}) \times \mathcal{U}$

$$\phi_t(h,u) = X_u(t)h = \phi_t(I,u)h. \tag{16}$$

Then the corresponding skew product flow $\Phi = (\vartheta, \phi)$ on $\mathcal{U} \times \mathrm{Gl}(d,\mathbb{R})$ is continuous (see, e.g., Colonius and Kliemann [3, Theorem 9.5.5]).

We note that homogeneous bilinear control systems of special type induce linear skew product flows for other Lie groups, e.g., on $\mathcal{U} \times \mathrm{Sl}(d,\mathbb{R})$.

Remark 7 *It is well known, that the homogeneous bilinear control system (15) also induces a semigroup in the Lie group* $\mathrm{Gl}(d,\mathbb{R})$. *This is an entirely different construction.*

For linear flows one takes G as the additive group $\mathbb{R}^d$ and defines a skewing map a by

$$a_t(b) = \int_0^t \Phi_{t-\tau}(f(b,\tau), \theta_\tau b)d\tau = \int_0^t \varphi_{t-\tau}(f(b,\tau), \theta_\tau b)d\tau,$$

where integration is only performed in $\mathbb{R}^d$ (see Colonius and Santana [5, Definition 2.1 and Remark1]) and

$$f(b, t+s) = f(\theta_s(b), t) \text{ for all } b \in B \text{ and almost all } t, s \in \mathbb{R}.$$

Here $e_H = I_d \in \mathrm{Gl}(d,\mathbb{R})$ and $a(t,b)$ satisfies (9) (cf. [5, Proposition 1]), hence it is indeed a skewing function:

$$\begin{aligned}
a_{t+s}(b) &= \int_0^{t+s} \varphi_{t+s-\tau}(f(b,\tau), \theta_\tau b) d\tau \\
&= \int_0^{s} \varphi_{t+s-\tau}(f(b,\tau), \theta_\tau b) d\tau + \int_s^{t+s} \varphi_{t+s-\tau}(f(b,\tau), \theta_\tau b) d\tau \\
&= \int_0^{s} \phi_t(I, \vartheta_{s-\tau+\tau} b) \phi_{s-\tau}(f(b,\tau), \theta_\tau b) d\tau + \int_0^{t} \varphi_{t-\tau}(f(b,\tau+s), \theta_{\tau+s} b) d\tau \\
&= \phi_t(I, \vartheta_s b) \int_0^{s} \phi_{s-\tau}(f(b,\tau), \theta_\tau b) d\tau + \int_0^{t} \varphi_{t-\tau}(f(b,\tau+s), \theta_{\tau+s} b) d\tau \\
&= \phi_t(I, \vartheta_s b) \star \int_0^{s} \Phi_{t-\tau}(\theta_\tau b, f(b,\tau)) d\tau + \int_0^{t} \varphi_{t-\tau}(f(\theta_s b,\tau), \theta_\tau \theta_s b) d\tau \\
&= \phi_t(e_H, \vartheta_s b) \star a_s(b) + a_t(\vartheta_s b).
\end{aligned}$$

Returning to the general situation, the beautiful thing is that each of the skew product systems Φ on $H \times B$ and Ψ on $(H \rtimes G) \times B$ induces a skew product system on $G \times B$. They are defined in the following way, beginning with the "linear" flow Φ.

Definition 8 *Define the following skew product flow $\tilde{\Phi}_t$ in $G \times B$*

$$\tilde{\Phi}_t : G \times B \to G \times B, \tilde{\Phi}_t(g,b) = (\tilde{\phi}_t(g,b), \vartheta_t b)$$

with

$$\tilde{\phi}_t : G \times B \to G, \tilde{\phi}_t(g,b) = \phi_t(e_H, b) \star g.$$

This is a system of the form (10), where we take the trivial affine term $a(b) = e_G, b \in B$, and omit the first component in H. One also sees that the first component of $\tilde{\Phi}_t$ is a cocycle, since

$$\tilde{\phi}_{t+s}(g,b) = \phi_{t+s}(e_H, b) \star g = [\phi_t(e_H, \vartheta_s b) \otimes \phi_s(h,b)] \star g$$

and, using the associated law (1),

$$\begin{aligned}
\tilde{\phi}_t(\tilde{\phi}_s(g,b), \vartheta_s b)) &= \phi_t(e_H, \vartheta_s b) \star \tilde{\phi}_s(g,b) = \phi_t(e_H, \vartheta_s b) \star [\phi_t(e_H, b) \star g] \\
&= [\phi_t(e_H, \vartheta_s b) \otimes \phi_t(e_H, b)] \star g.
\end{aligned}$$

Hence $\tilde{\Phi}$ is a skew product flow.

Also for the affine system Ψ there is an induced skew product system in $G \times B$.

Definition 9 *Define the following skew product system $\tilde{\Psi}_t, t \in \mathbb{T}$, in $G \times B$*

$$\tilde{\Psi}_t : G \times B \to G \times B, \tilde{\Psi}_t(g,b) := (\tilde{\psi}_t(g,b), \vartheta_t b)$$

given by

$$\tilde{\psi}_t(g,b) = \phi_t(e_H, b) \star g + a_t(b) = (\phi_t(e_H, b), a_t(b)) \cdot g.$$

So this is just the G-component of ψ from (10) and a cocycle by equality (13).

In the remainder of this section we will be concerned with topological conjugations of the defined flows. We begin with the definition of topological conjugation of skew products flows.

Definition 10 *Let $\Gamma^1 = (\gamma^1, \vartheta^1)$ and $\Gamma^2 = (\gamma^2, \vartheta^2)$ be skew product flows on topological group bundles $H^1 \times B^1$ and $H^2 \times B^2$, respectively. We say that Γ^1 and Γ^2 are topologically skew conjugate, or just topologically conjugate, if there exists a skew homeomorphism*

$$(h_H, h_B) : H^1 \times B^1 \to H^2 \times B^2$$

such that $(h_H, h_B)(\Gamma^1_t(x,b)) = \Gamma^2_t((h_H, h_B)(x,b))$; i.e., $h_B : B^1 \to B^2, h_H : H^1 \times B^1 \to H^2$ are maps with

$$h_B(\vartheta^1_t b)) = \vartheta^2_t(h_B(b)) \text{ for all } t \in \mathbb{T} \text{ and } b \in B^1, \tag{17}$$

$$h_H(\gamma^1(t,x,b), \vartheta^1_t b) = \gamma^2(t, h_H(x,b), h_B(b)) \text{ for all } t \in \mathbb{T}, b \in B^1, \text{ and } x \in H^1. \tag{18}$$

Thus topological skew conjugacy requires that the base flows are topologically conjugate via the homeomorphism h_B and (18) holds.

Our first result, concerned with topological conjugations of these flows, is a consequence of Lemma 2.

Proposition 11 *Consider flows Φ^i on $H \times B$ and Ψ^i on $G \times B$ with $i = 1, 2$, as defined above. Assume that there exist homeomorphic fundamental domains for the flows Φ^1 and Φ^2. Then Ψ^1 is topologically conjugate to Ψ^2.*

Proof. Consider the following diagram

$$\begin{array}{ccc} (H \rtimes G) \times B & \xrightarrow{\Psi^i_t} & (H \rtimes G) \times B \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ H \times B & \xrightarrow[\Phi^i_t]{} & H \times B \end{array}$$

where $\pi : (H \rtimes G) \times B \to H \times B$ is defined as $\pi(h, g, b) = (h, b)$. Noting that

$$\begin{aligned} \pi(\Psi^i_t(h,g,b)) &= \pi(\phi^i_t(h,b), \phi^i_t(e_H, b) \star g + a_t(b), \vartheta_t b) \\ &= (\phi^i_t(h,b), \vartheta_t b) = \Phi^i_t(\pi(h,g,b)), \end{aligned}$$

one sees that the diagram above commutes. Hence by Lemma 2 the proposition is proved. ■

Next we consider the following question in the setting above. Consider flows Φ^1 and Φ^2 on $H \times B$ with homeomorphic fundamental domains. When does it follow that the flows $\tilde{\Phi}^1$ and $\tilde{\Phi}^2$ on $G \times B$ are topologically conjugate?

We give an affirmative answer in the case where H a Lie subgroup of a linear Lie group G and the influence on the cocycle is trivial. This may be expressed by taking $Y : B \to H$ such that $Y(\vartheta_t b) = Y(b)$ for all t and b. Define $\phi_t : H \times B \to H$ as

$$\phi_t(h, b) = \int_0^t Y(\vartheta_\tau b) d\tau \cdot h.$$

We can see that $\phi_t(h, b)$ is a cocycle and hence $\Phi_t(h, b) = (\phi_t(h, b), \vartheta_t b)$ is a skew product. Moreover, note that

$$\phi_t(h, b) = \int_0^t Y(\vartheta_\tau b) d\tau \cdot h = \int_0^t Y(b) d\tau + h = tY(b) \cdot h.$$

In this case, the action of H in G is given by the matrix product. Hence,

$$\tilde{\Phi}_t : G \times B \to G \times B,$$

is given by

$$\begin{aligned} \tilde{\Phi}_t(g, b) &:= (\tilde{\phi}_t(g, b), \vartheta_t b) \\ &= (\phi_t(e_H, b) \star g, \vartheta_t b) = (tY(b) \cdot e_H \cdot g, \vartheta_t b) \\ &= (tY(b) \cdot g, \vartheta_t b) = (\phi_t(g, b), \vartheta_t b), \end{aligned}$$

because $H \subset G$ is a Lie subgroup. So, $\Phi = \tilde{\Phi}|_{B \times H}$. Therefore, if we have homeomorphic fundamental domains for Φ^1 and Φ^2, then $\tilde{\Phi}^1$ and $\tilde{\Phi}^2$ are topologically conjugate. In general, the answer to the question posed in this remark depends on the products of H and G and the action of H on G.

Now we will give conditions implying that the induced flows $\tilde{\Psi}_t$ and $\tilde{\Phi}_t$ are topologically conjugate. So we need to find

$$(h_G, h_B) : G \times B \to G \times B,$$

conjugating $\tilde{\Psi}_t$ and $\tilde{\Phi}_t$ with $h_B = id_B$. Clearly, in this case, the only difficulty is in the construction of $h_G : G \times B \to G$ which must satisfy

$$h_G\left(\tilde{\psi}_t(g, b), \vartheta_t b\right) = \tilde{\phi}_t(h_G(g, b), b) \tag{19}$$

Theorem 12 *The flows $\tilde{\Psi}_t$ and $\tilde{\Phi}_t$ are conjugate if and only if there is a continuous map $b \mapsto e(b) : B \to G$ solving the functional equation*

$$e(\vartheta_t b) = \phi_t(e_H, b) \star e(b) + a_t(b) = (\phi_t(e_H, b), a_t(b)) \cdot e(b). \tag{20}$$

Proof. Suppose that equation (20) has a continuous solution $e(\cdot)$. It follows that (note that G is not necessarily Abelian)

$$-e(\vartheta_t b) = -a_t(b) - \phi_t(e_H, b) \star e(b). \tag{21}$$

Define a map $h_G : G \times B \to G$ as $h_G(g,b) := g - e(b)$. Then, using (3), one computes

$$\begin{aligned} h_G\left(\tilde{\psi}_t(g,b), \vartheta_t b\right) &= h_G(\phi_t(e_H,b) \star g + a_t(b)), \vartheta_t b) \\ &= \phi_t(e_H,b) \star g + a_t(b) - e(\vartheta_t b) \\ &= \phi_t(e_H,b) \star g + a_t(b) - a_t(b) - \phi_t(e_H,b) \star e(b) \\ &= \phi_t(e_H,b) \star g - \phi_t(e_H,b) \star e(b) \\ &= \phi_t(e_H,b) \star g + \phi_t(e_H,b) \star (-e(b)) \\ &= \phi_t(e_H,b) \star (g - e(b)) \\ &= \tilde{\phi}_t(g - e(b), b) \\ &= \tilde{\phi}_t(h_G(b,g), b). \end{aligned}$$

Therefore $\tilde{\Psi}_t$ is conjugate to $\tilde{\Phi}_t$. By continuity of $b \mapsto e(b)$ the map h_G is continuous. The continuous inverse of (h_G, id_B) is $(h_G, id_B)^{-1}(g,b) = (g + e(b), b)$, thus this is a homeomorphism.

Conversely, suppose that $\tilde{\Phi}_t$ and $\tilde{\Psi}_t$ are topologically conjugate with $h_B = id_B$ and $h_G : G \times B \to G$. Then h_G satisfies (19) and hence for all $b \in B, g \in G, t \in \mathbb{T}$

$$\begin{aligned} h_G(\phi_t(e_H,b) \star g + a_t(b)), \vartheta_t b) = h_G\left(\tilde{\psi}_t(g,b), \vartheta_t b\right) &= \tilde{\phi}_t(h_G(g,b), b) \\ &= \phi_t(e_H,b) \star h_G(g,b). \end{aligned}$$

Clearly, $e : B \to G, e(b) := h_G(b,g) - g$ is continuous. Then the same computation as above shows that e is a solution of the functional equation (21) which is equivalent to (20). ■

In the case of linear flows (14) additional assumptions are necessary in order to ensure topological conjugacy of $\tilde{\Phi}_t$ and $\tilde{\Psi}_t$.

Remark 13 ***(Linear flows)*** *The conditions above on $e(b) \in G$ coincide with $e(b,0) \in \mathbb{R}^d$ in the case of affine-linear flows on vector bundles, cf. [5, Theorem 2.5, in particular, assumption (iii)] (observe that here $I_d \in \mathrm{Gl}(d,\mathbb{R}) = H$ is the neutral element e_H): In the proof of [5, Lemma 2.3] it is shown that*

$$e(b,t) = \int_0^t \varphi(t-s, \theta_s b, f(b,s))ds + \varphi(t,b,x).$$

Note that $x = e(b,0)$ and $a(t,b) = \int_0^t \varphi(t-s, \theta_s b, f(b,s))ds$ and $e_H = I \in H = \mathrm{Gl}(d,\mathbb{R})$. Now by (16) $\phi_t(h,u) = X_u(t)h$, and hence

$$\phi_t(e_H,b) \star e(b) = X_u(t)Ie(b,0) = X_u(t)x = \varphi(t,b,x),$$

implying, as claimed,

$$e(b,t) = a(t,b) + \phi_t(e_H,b) \star e(b,0).$$

Now note that conjugacy of the flows $\tilde{\Phi}_t$ and $\tilde{\Psi}_t$ also implies conjugacy results for affine flows: In fact, denote the topological conjugacy by $\sim$, consider two flows $\tilde{\Psi}_t^1$ and $\tilde{\Psi}_t^2$ of the form

$$\begin{aligned}\tilde{\Psi}_t^1(g,b) &:= (\tilde{\psi}_t^1 g, b), \vartheta_t b) = (\phi_t(h,b), \phi_t(e_H,b)\star g + a_t^1(b)),\\ \tilde{\Psi}_t^2(v,b) &:= (\tilde{\psi}_t^2(g,b), \vartheta_t b) = (\phi_t(h,b), \phi_t(e_H,b)\star g + a_t^2(b))\end{aligned}$$

and take the corresponding linear flows $\tilde{\Phi}_t^1$ and $\tilde{\Phi}_t^2$. For linear flows (14) on vector bundles, Ayala, Colonius and Kliemann [2, Corollary 3.4] shows that a hyperbolicity condition implies topological conjugacy of two linear flows (with topologically conjugate base flows). In this case $\tilde{\Phi}_t^1 \sim \tilde{\Phi}_t^2$. Now if the hypothesis of Theorem 12 is valid, it also follows that $\tilde{\Psi}_t^i \sim \tilde{\Phi}_t^i$ and hence $\tilde{\Psi}_t^1 \sim \tilde{\Psi}_t^2$.

Next we discuss the relation of our constructions and Theorem 12 to the work of Lind [9] on a Splitting Theorem.

Lind considers the following situation. Let U be an invertible measure-preserving transformation acting on a Lebesgue measure space (X, μ). Let G be a separable compact abelian group equipped with the Borel σ-algebra and Haar measure, and S be a (continuous, algebraic) automorphism of G. Let $\alpha : X \to G$ be measurable, called a skewing map. Since both S and translations preserve Haar measure, the map $U \times_\alpha S$ called the skew product of U with S on the Lebesgue space $X \times G$ defined by

$$(U \times_\alpha S)(x,g) := (Ux, Sg + \alpha(x))$$

is measurable and preserve the product of μ with Haar measure.

The skew product $U \times_\alpha S$ algebraically splits if there is an isomorphism W on $X \times G$ of the form $W(x,g) = (x, g + \beta(x))$, where $\beta : X \to G$ is measurable such that the conjugation relation

$$(U \times_\alpha S)\, W = W\, (U \times S)$$

holds. This conjugation relation is equivalent to the functional equation

$$\alpha(x) = \beta(Ux) - S\beta(x). \tag{22}$$

Thus the algebraic splitting of $U \times_\alpha S$ is equivalent to solving this equation for a measurable function β, where α, U and S are given. Lind's main result ([9, p. 237]) shows that this is always possible.

Lind's Splitting Theorem. Equation (22) has a measurable solution β and hence skew products with ergodic automorphisms of compact abelian groups algebraically split.

The relation to our results is the following. In the discrete-time situation $\mathbb{T} = \mathbb{Z}$, one has

$$\vartheta_n = (\vartheta_1)^n \text{ for } n \in \mathbb{Z}.$$

Here $U := \vartheta_1$ is homeomorphism on B. Thus we replace the invertible measure preserving transformation U on the Lebesgue space X in Lind [9] by the homeomorphism ϑ_1 on the metric space B. Condition (9) for the affine term $a : \mathbb{Z} \times G$ takes the form: For every b one has $a_0(b) = e_G$ and

$$a_{n+m}(b) = \phi_n(e_H, \vartheta_m b) \star a_m(b) + a_n(\vartheta_m b) \text{ for all } n, m \in \mathbb{Z}.$$

and condition (10) has the form

$$\psi_n(h, g, b) := (\phi_n(h, b), \phi_n(e_H, b) \star g + a_n(b)) \text{ for } n \in \mathbb{Z}.$$

As observed at the end of Section 2, the element $\phi_1(e_H, b) \in H$ induces an automorphism ζ on G via $\zeta(g) = \phi_1(e_H, b) \star g, g \in G$. In Lind's framework [9], where $\vartheta_1 = U$, instead of $(H \ltimes G) \times B$ the space $B \ltimes G$ is considered and instead of $\phi_1(e_H, b) \star g, g \in G$, an automorphism S of the group $G = H$ is taken. If we also write $\alpha = a_1$ the formula

$$\phi_1(e_H, b) \star a_1(b) + a_1(\vartheta_1 b)$$

reduces to

$$S\alpha(b) + \alpha(Ub).$$

Here the flows $\tilde{\Phi}_t$ and $\tilde{\Psi}_t$ coincide with Φ and Ψ, respectively. Alluding to Lind's terminology, we consider the skew product of the flow ϑ on B with the (skew product flow) Φ on $B \times H$ with skewing function $\alpha : B \to G$. The result is the skew product system $\tilde{\Psi} = \Psi$ on $B \times G$.

In Theorem 12, we assume that equation (20) has a continuous solution e, i.e., there is a continuous map $b \mapsto e(b) : B \to G$ with

$$e(\vartheta_t b) = \phi_t(e_H, b) \star e(b) + a_t(b) = (\phi_t(e_H, b), a_t(b)) \cdot e(b).$$

We get, with $t = 1, \vartheta_1 = U, a_1(b) = \alpha(b)$, and the automorphism S instead of $\zeta(g) = \phi_t(e_H, b) \star g, g \in G$ a map $e : B \to G = H$ satisfying

$$e(Ub) = Se(b) + \alpha(b).$$

Thus we look for a solution $\beta = e$ of Lind's functional equation

$$\alpha(b) = \beta(Ub) - S\beta(b).$$

In the measure theoretic framework, the functional equation can always be solved by a measurable map β. Lind already noted that, in general, the topological analogue of his Splitting Theorem is not valid (cf. [9, p. 238]). In the topological framework additional assumptions are necessary, in order to get a solution (a continuous map β or $e : B \to G$) of Lind's functional equation. In the case of linear flows, a hyperbolicity assumption is needed, cf. Remark 13.

Remark 14 *When we want to emphasize the dependence of $\tilde{\Psi}$ on the skewing function a we write it, in analogy to Lind's notation, as $\tilde{\Psi} = \vartheta \times_a \phi$. Thus Theorem 12 shows that the flow $\vartheta \times_a \phi$ is topologically conjugate to the flow $\vartheta \times \phi$, i.e., it splits algebraically and topologically.*

Remark 15 *The discussion of linear and affine flows shows that here a hyperbolicity assumption is required in order to obtain topological conjugacy of the flows $\tilde{\Phi}_t^1$ and $\tilde{\Psi}_t^1$. A similar result for more general skew product flows would require the development of corresponding spectral theory and hyperbolicity. Of special interest would be the case of principal fibre bundles.*

5 Skew-product transformation semigroups and fiber entropy

Let S and T be topological semigroups acting on a topological spaces X and on a topological group $(H, \otimes)$ respectively. Suppose that S contains the identity e_S, represent the actions by $\sigma_1 : S \times X \to X$ and $\sigma_2 : T \times H \to H$ with $\sigma_1(s, x) = sx$ and $\sigma_2(t, h) = th$ and assume that both actions are surjective. Now we recall the definition of skew-product transformation semigroups given in Souza [11]. This notion is classical and there are several directions of research, for example in control theory the study of control flows (see e.g. [4]) and in topological dynamics the study of dynamics of group actions (see e.g. [6]).

In the context of the beginning of the previous section, the skew product transformation semigroup on the product space $H \times X$ is given by

$$\Phi : S \times H \times X \to H \times X \text{ with } \Phi(s, h, x) = (\phi(s, x)h, sx), \tag{23}$$

where $\phi : S \times X \to T$ is a cocycle, i.e., it is continuous and satisfies

$$\phi_{e_s}(x)h = h \text{ and } \phi_{s_1 s_2}(x) = \phi_{s_1}(s_2 x)\phi_{s_2}(x) \text{ for all } s_1, s_2 \in S, x \in X.$$

We can also write

$$\phi : S \times H \times X \to T \text{ with } \phi_s(h, x) = \phi_s(x)h.$$

Next define the following subsemigroup of T:

$$\mathcal{S} = \{\phi_{s_n}(x_n)\phi_{s_{n-1}}(x_{n-1}) \cdots \phi_{s_0}(x_0) : s_j \in S, x_j \in X, n \in \mathbb{N}\}.$$

As in Section 2, let $(G, +)$ be a topological group, suppose that H acts on G and take the correspondent actions defined in Section 2:

$$\star : H \times G \to G \text{ and } \cdot : (H \rtimes G) \times G \to G.$$

Hence we define the skew product flow

$$\Psi : S \times (H \rtimes G) \times X \to (H \rtimes G) \times X,\ \Psi_s(h, g, x) := (\psi_s(h, g, x), sx) \tag{24}$$

with $\psi_s : (H \rtimes G) \times X \to H \rtimes G$ defined as

$$\psi_s(h, g, x) := (\phi_s(x)h, \phi_s(x)e_H \star g + a_s(x)) = (\phi_s(x)h, (\phi_s(x)e_H, a_s(x)) \cdot g),$$

where $a : S \times X \to G$ is a continuous map (a skewing map) satisfying

$$a(e_S, x) = e_G \text{ and } a_{s_1 \cdot s_2}(x) = \phi_{s_1}(s_2 x)e_H \star a_{s_2}(x) + a_{s_1}(s_2 x).$$

Now we define fundamental domains in this context.

Definition 16 *Denote by R the topological group generated by S, that is, take R as the smallest group containing S. A fundamental domain for a skew product transformation semigroup $\Phi : S \times H \times X \to H \times X$ is a pair (Z, τ) where Z is a subset of $H \times X$ and $\tau : H \times X \to R$ is a continuous map such that for all $(h, x) \in H \times X$, it holds*

$$\Phi(s, (h, x)) \in Z \text{ if and only if } s = \tau(x).$$

Then it is not difficult to prove the following generalization of Proposition 11.

Proposition 17 *Take the skew product transformation semigroups Φ^i and Ψ^i, with $i = 1, 2$, as defined in (23) and (24). If there exist homeomorphic fundamental domains for Φ^i, $i = 1, 2$, then Ψ^1 is topologically conjugate to Ψ^2.*

Next we briefly discuss topological entropy in this context.

Consider a continuous skew product flow $\Phi : \mathbb{T} \times X \times B \to X \times B$ of the form (7) denoted by

$$\Phi_t(x, b) := (\phi_t(x, b), \vartheta_t b).$$

Suppose that X and B are metric spaces and B is compact. Furthermore, denote by $Q(b), b \in B$, a uniformly bounded family of compact subsets of X.

Remark 18 *Such a family may be viewed as an analogue to random sets in the theory of random dynamical systems; cf. Arnold [1].*

Fix $b \in B$ and let $T, \varepsilon > 0$. Define a (T, ε)-spanning set $R(T, \varepsilon)$ by the following: for every $x \in Q(\vartheta_{-T} b)$ there is $y \in R$ with

$$d(\phi_t(x, \vartheta_{-T} b), \phi_t(y, \vartheta_{-T} b) < \varepsilon \text{ for } t \in [0, T].$$

Let $r(T, \varepsilon)$ the minimal cardinality of such a set and define the entropy in the fiber over b by

$$h(\varepsilon, b, Q(\cdot)) := \limsup_{T \to \infty} \frac{1}{T} \ln r(T, \varepsilon, b, Q(\cdot)), h(b, Q(\cdot)) := \lim_{\varepsilon \searrow 0} h(\varepsilon, b, Q(\cdot)),$$

and, finally,

$$h := \sup_{b \in B} \sup_{Q(\cdot)} h(b, Q(\cdot))$$

We claim that this fiber entropy is invariant under skew conjugacy (for simplicity, we assume that the skew conjugacy is the identity on B).

Proposition 19 *Suppose that Φ^1 and Φ^2 are skew conjugate and let $Q^1(b), b \in B$ be as above. Define $Q^2(b) := h(Q^1(b), b), b \in B$. Then for every $b \in B$*

$$h_{inv}(b, Q^2(\cdot)) = h_{inv}(b, Q^1(\cdot)).$$

Proof. We know that

$$h(\phi^1(t,x,b),\vartheta_t b) = \phi^2(t,h(x,b),b).$$

It suffices to show that every $\varepsilon_2 > 0$ there is $\varepsilon_1 > 0$ such that for every $T > 0$ every (T,ε_1)-spanning set R for $Q^1(\cdot)$ is mapped to a (T,ε_2)-spanning set for $Q^2(\cdot)$. In fact, this implies $h_{inv}(\varepsilon_2,b,Q^2(\cdot)) \leq h_{inv}(\varepsilon_1,b,Q^1(\cdot))$ and hence $h_{inv}(b,Q^2(\cdot)) \leq h_{inv}(b,Q^1(\cdot))$; the assertion follows by interchanging the roles of $Q^1(\cdot)$ and $Q^2(\cdot)$. The rest is straightforward. ■

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