Subspace entropy and controlled invariant subspaces

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1 Introduction

This paper discusses the following problem: Given a controlled invariant subspace V of a linear control system, what is the minimal amount of information per unit time (measured via an entropy notion) that has to be transferred to a controller in order to keep the system in or near V? This problem connects the analysis of control under communication constraints to classical geometric control theory. It was motivated by earlier investigations on invariance entropy (Colonius and Kawan [4]) for a similar problem, concerning controlled invariance of compact subsets with nonvoid interior in the state space where geometric structures did not play a role. The joint paper Colonius and Helmke [3] presented an important insight-the associated entropy for controlled invariant subspaces coincides with the subspace entropy of the linear flow associated with the uncontrolled system. The latter entropy notion was introduced in this paper and several estimates were derived. The present paper extends this line of research by giving a closer analysis of the subspace entropy.

Since the notion of controlled invariant subspaces is a cornerstone of geometric control theory, it is hoped that this will contribute to a closer connection of the theory of control under communication constraints to the more classical parts of state space control theory.

The contents of this paper is as follows: Section 2 collects results on topological entropy of linear differential equations and defines subspace entropy. Section 3 defines entropy for controlled invariant subspaces and explains the equivalence to subspace entropy. Final Section 4 presents the main results of this paper by analyzing the subspace entropy. It is shown that the subspace entropy is bounded above by the topological entropy of an induced system; a sufficient condition for equality is given which leads to a characterization of the subspace entropy (and hence the invariance entropy) by certain positive eigenvalues of the uncontrolled system.

This problem grew out of a discussion with Uwe, when we returned from a meeting of the DFG Priority Research Program 1305 “Control of Digitally Connected Dynamical Systems”. The successful application for funding of this research initiative by Deutsche Forschungsgemeinschaft (DFG) owes a lot to Uwe’s broad knowledge, his many fruitful ideas, and his vigor.

Notation. The distance of a point x in a normed vector space to a closed subset M is defined by $\mathrm{dist}(x,M) := \inf_{y\in M} \|x-y\|$.

2 Topological entropy and subspace entropy

In this section we first recall results on topological entropy of the flow for a linear differential equations. Then the subspace entropy is defined which is a suitable modification of the topological entropy. Later we will use it for the uncontrolled system $\dot{x} = Ax$ and relate it to the entropy of controlled invariant subspaces. It is worth to emphasize that an open loop control system does not define a flow, since the control functions $u(\cdot)$ are time-dependent, and hence it is not covered by this definition.

For a linear map $A : \mathcal{X} \to \mathcal{X}$ on an n-dimensional normed vector space $\mathcal{X}$, let $\Phi : \mathbb{R} \times \mathcal{X} \to \mathcal{X}, \Phi(t,x) := e^{tA}x,\ t \in \mathbb{R},\ x \in \mathcal{X}$, be the induced flow (actually, throughout this paper, only the semiflow defined for $t \geq 0$ will be relevant.) A set R in $\mathcal{X}$ is called (T,ε)-spanning if for every $x \in K$ there is $y \in R$ such that for all $t \in [0,T]$ one has

$$\|\Phi(t,x) - \Phi(t,y)\| = \left\|e^{tA}(x-y\right\| < \varepsilon.$$

Denote by $r_{\text{top}}(T,\varepsilon,K)$ the minimal cardinality of such a (T,ε,K)-spanning set.

Definition 1. With the notation above, the topological entropy of Φ with respect to K is defined by

$$h_{\text{top}}(\varepsilon,K) := \limsup_{T\to\infty} \frac{1}{T} \log r_{\text{top}}(T,\varepsilon,K),$$
$$h_{\text{top}}(K) := \lim_{\varepsilon \searrow 0} h_{\text{top}}(\varepsilon,K).$$

and the topological entropy with respect to a subspace V of $\mathcal{X}$ is

$$h_{\text{top}}(V) = \sup_{K \subset V} h_{\text{top}}(K),$$

where the supremum is taken over all compact subsets $K \subset V$.

Where appropriate, we also write $h_{\text{top}}(V;\Phi)$, if the considered flow has to be specified. For the topological entropy of linear flows and $V = \mathcal{X}$, a classical result by R. Bowen [2] shows

$$h_{\text{top}}(\mathcal{X}) := \sup_K h_{\text{top}}(K) = \sum_{i=1}^{n} \max(0, \operatorname{Re}\lambda_i),$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A; see also Walters [11, Theorem 8.14] and Matveev and Savkin [8, Theorem 2.4.2] for proofs. The supremum is attained for any compact set K with nonvoid interior in $\mathcal{X}$.

Since all norms on $\mathcal{X}$ are equivalent, we may assume that $\mathcal{X}$ is a Hilbert space and we endow $\mathcal{X}$ with the following inner product which is adapted to the decomposition into the Lyapunov spaces $L_j, 1 \leq j \leq l$. Recall that a Lyapunov space L_j is the sum of all generalized (real) eigenspaces corresponding to an eigenvalue of A with real part equal to λ_j. We order these Lyapunov exponents such that

$$\lambda_1 > \ldots > \lambda_l.$$

Take a basis corresponding to the Jordan normal form: for each j, one has a basis $e_1^j,\ldots,e_{n_j}^j$ of L_j which is orthonormal with respect to an inner product in L_j. Define

$$\left\langle e_{i_1}^{j_1}, e_{i_2}^{j_2}\right\rangle = \begin{cases} 0 & \text{for} \quad j_1 \neq j_2 \text{ or } i_1 \neq i_2 \\ 1 & \text{for} \quad j_1 = j_2 \text{ and } i_1 = i_2. \end{cases} \tag{1}$$

In order to simplify the notation a bit, we number the basis elements by $1,\ldots,n$ and denote them by x_j. They form an orthonormal basis for an inner product on $\mathcal{X}$. Recall that we can identify the Grassmannian manifold $\mathbb{G}_k\mathcal{X}$ of k-dimensional subspaces with the subset of projective space $\mathbb{P}(\bigwedge^k\mathcal{X})$ obtained from the indecomposable elements in the exterior product $\bigwedge^k\mathcal{X}$. We endow $\mathbb{G}_k\mathcal{X}$ with the corresponding metric.

Following Colonius, San Martin, da Silva [6] we first describe the chain recurrent components in the Grassmannian; see, e.g., Robinson [9] for a discussion of this notion for flows on compact metric spaces.

Theorem 2. *Let $A:\mathcal{X}\to\mathcal{X}$ be a linear map with flow $\Phi_t = e^{tA}$ on $\mathcal{X}$. Let L_i, $i = 1,\ldots,l$, be the Lyapunov spaces of A. For $k\in\{1,\ldots,n\}$ define the index set*

$$I(k) = \{(k_1,\ldots,k_l) \mid k_1+\ldots+k_l = k \text{ and } 0\le k_i\le n_i = \dim L_i\}. \tag{2}$$

Then the chain recurrent components (also called Morse sets) of the induced flow on the Grassmannian $\mathbb{G}_k\mathcal{X}$ are

$$\mathcal{M}^k_{k_1,\ldots,k_l} = \mathbb{G}_{k_1}L_1\oplus\ldots\oplus\mathbb{G}_{k_l}L_l,\ (k_1,\ldots,k_l)\in I(k). \tag{3}$$

Here the sum on the right-hand side denotes the set of all k-dimensional subspaces $V^k\subset\mathcal{X}$ with

$$\dim(V^k\cap L_i) = k_i,\ i = 1,\ldots,l.$$

In particular, every ω-limit set for the induced flow on $\mathbb{G}_k\mathcal{X}$ is contained in one of these chain recurrent components.

If A is hyperbolic, i.e., there are no eigenvalues of A on the imaginary axis, then one can decompose $\mathcal{X}$ into the stable and the unstable subspaces, $\mathcal{X} = \mathcal{X}^-\oplus\mathcal{X}^+$. We denote by $\pi^\pm$ the projection of $\mathcal{X}$ to $\mathcal{X}^\pm$ and let $\Phi^\pm$ be the associated restrictions of Φ.

Theorem 3. *Consider a linear flow $\Phi_t = e^{tA}$ and assume that A is hyperbolic. Let V be a k-dimensional subspace. Then the topological entropy with respect to a compact set $K\subset V$ satisfies*

$$h_{\text{top}}(K,\Phi) = h_{\text{top}}(\pi^+K,\Phi^+).$$

and the topological entropy of V is

$$h_{\text{top}}(V,\Phi) = \sum_{i=1}^{l} k_i\max(0,\lambda_i),$$

where $\mathcal{M}^k_{k_1,\ldots,k_l}\subset\mathbb{G}_k\mathcal{X}$ is the Morse set that contains the omega limit of the point V for the induced flow $\mathbb{G}_k\Phi$ on the k-Grassmannian $\mathbb{G}_k\mathcal{X}$. Furthermore, for every compact subset $K\subset V$ with nonvoid interior, $h_{\text{top}}(K;\Phi)$ equals the volume growth rate of π^+K under the flow Φ^+.

We note that the Morse set containing the omega limit of V can be determined in the following way. Let $v_1,...,v_k$ be a basis of V. Then we can express the v_i using the standard basis of $\mathcal{X}$ as introduced in (1) by

$$v_i = \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n = \sum_{\alpha_{ij}\neq 0} \alpha_{ij}x_j.$$

There is a minimal number of Lyapunov spaces such that

$$V \subset L_{i_1} \oplus \ldots \oplus L_{i_j},$$

and we number them such that $\lambda_{i_1} > \ldots > \lambda_{i_j}$. Note that generically $\sum \dim L_{i_j} > k$. Then $\omega(V)$ is contained in the Morse set $\mathcal{M}^k_{k_1,...,k_l}$, where the k_i are recursively obtained in the following way: k_1 is the maximal number of base vectors v_i which have a nontrivial component in L_1. Then eliminate these base vectors v_i and let k_2 be the maximal number of the remaining v_i which have a nontrivial component in L_2, etc.

Next we modify the definition of topological entropy in order to define the subspace entropy introduced in Colonius and Helmke [3]. Let V be a linear subspace of $\mathcal{X}$ and consider a linear map $A : \mathcal{X} \to \mathcal{X}$ with flow $\Phi_t = e^{tA}$. For any compact subset $K \subset V$ and for given $T,\varepsilon > 0$ we call $R \subset K$ a (T,ε)-spanning set, if for all $x \in K$ there exists $y \in R$ with

$$\max_{0\le t\le T} \operatorname{dist}(e^{tA}(x-y),V) < \varepsilon. \tag{4}$$

Let $r_{\text{sub}}(T,\varepsilon,K,V)$ denote the minimal cardinality of a such a (T,ε)-spanning set. If no finite (T,ε)-spanning set exists, we set $r_{\text{sub}}(T,\varepsilon,K,V;\Phi) = \infty$. If there exists some (T,ε)-spanning set, then one also finds a finite (T,ε)-spanning set using compactness of K and continuous dependence on the initial value. Note that the points y in R will, in general, not lead to solutions $e^{tA}y$ which remain for all $t \ge 0$ in the ε-neighborhood of V.

Definition 4. Let A be a linear map on $\mathcal{X}$ with associated flow $\Phi_t = e^{tA}$ and consider a subspace V of $\mathcal{X}$. For a compact subset $K \subset V$, we consider the exponential growth rate of $r_{\text{sub}}(T,\varepsilon,K,V)$ and set

$$\begin{aligned} h_{\text{sub}}(\varepsilon,K,V) &:= \limsup_{T\to\infty} \frac{1}{T} \log r_{\text{sub}}(T,\varepsilon,K,V), \\ h_{\text{sub}}(K,V) &:= \lim_{\varepsilon \searrow 0} h_{\text{sub}}(\varepsilon,K,V), \end{aligned}$$

and define the entropy of V with respect to Φ by

$$h_{\text{sub}}(V) := \sup_K h_{\text{sub}}(K,V),$$

where the supremum is taken over all compact subsets $K \subset V$.

Where appropriate, we write $h_{\text{sub}}(V;\Phi)$ in order to clarify which flow is considered. As usual in the context of topological entropy, one sees that, by monotonicity, the limit for $\varepsilon \searrow 0$ exists (it might be infinite.) Since all norms on a finite dimensional vector space are equivalent, the entropy does not depend on the norm used in (4). For simplicity, we require throughout that $\mathcal{X}$ is a Hilbert space. One easily sees that the subspace entropy $h(V;\Phi)$ is invariant under state space similarity, i.e., $h(SV;S\Phi S^{-1}) = h(V;\Phi)$ for S in the set $GL(\mathcal{X})$ of isomorphisms on $\mathcal{X}$; here $S\Phi_t S^{-1} = Se^{tA}S^{-1} = e^{SAS^{-1}t}, t \geq 0$.

Remark 5. If we choose $V = \{0\}$ condition (4) is trivial, since only $K = \{0\}$ is allowed; furthermore, if we choose $V = \mathcal{X}$, the distance in (4) is always equal to zero. In particular, the subspace entropy does not recover the usual definition of topological entropy for the linear flow $\Phi(t,x) = e^{tA}x$; see Definition 1.

3 Entropy for controlled invariant subspaces

This section briefly summarize some well-known definitions and facts concerning controlled invariant subspaces. Then invariance entropy for controlled invariant subspaces of linear control systems on $\mathcal{X}$ is defined and related to the subspace entropy of linear flows as defined in the previous section.

The notion of controlled invariant subspaces (also called (A,B)–invariant subspaces) was introduced by Basile and Marro [1]; see the monographs Wonham [12] and Trentelman, Stoorvogel and Hautus [10] for expositions of the theory.

Consider linear control systems in state space form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

with linear maps $A : \mathcal{X} \to \mathcal{X}$ and $B : \mathbb{R}^m \to \mathcal{X}$, where $\mathcal{X}$ is an n-dimensional normed vector space. The solutions $\varphi(t,x,u), t \geq 0$, of (5) with initial condition $\varphi(0,x,u) = x$ are given by the variation-of-constants formula

$$\varphi(t,x,u) = e^{tA}x + \int_0^t e^{A(t-s)}Bu(s)ds.$$

Recall that a subspace V is called controlled invariant, if for all $x \in V$ there is $u \in \mathbb{R}^m$ with $Ax + Bu \in V$, i.e., if $AV \subset V + \operatorname{Im} B$. Equivalently, there is a linear map $F : \mathcal{X} \to \mathbb{R}^m$, called a friend of V, such that

$$(A + BF)V \subset V.$$

This also shows that V is controlled invariant iff for every $x \in V$ there is an (open loop) continuous control function $u : [0,\infty) \to \mathbb{R}^m$ with $\varphi(t,x,u) \in V$ for all $t \geq 0$. In fact, differentiating the solution one finds

$$V \ni \frac{d}{dt}\varphi(0,x,u) = Ax + Bu(0).$$

For the converse, define for $x \in V$ a control by $u(t) = Fe^{(A+BF)t}x, t \geq 0$.

We now introduce the central notion of this paper, invariance entropy for controlled invariant subspaces of linear control system (5) and relate it to the subspace entropy defined in the previous section.

In the following, we consider a fixed controlled invariant subspace V of $\mathcal{X}$ with $\dim V = k$. Furthermore, we admit arbitrary controls in the space $C([0,\infty),\mathbb{R}^m)$ of continuous functions $u:[0,\infty)\to\mathbb{R}^m$.

Definition 6. For a compact subset $K\subset V$ and for given $T,\varepsilon>0$ we call a set $\mathcal{R}\subset C([0,\infty),\mathbb{R}^m)$ of control functions (T,ε)-*spanning* if for all $x_0\in K$ there is $u\in\mathcal{R}$ with

$$\operatorname{dist}(\varphi(t,x_0,u),V)<\varepsilon \text{ for all } t\in[0,T]. \tag{6}$$

By $r_{\text{inv}}(T,\varepsilon,K,V)$ we denote the minimal cardinality of such a (T,ε)-spanning set. If no finite (T,ε)-spanning set exists, we set $r_{\text{inv}}(T,\varepsilon,K,V)=\infty$.

In other words, we require for a (T,ε)-spanning set $\mathcal{R}$ that, for every initial value in K, there is a control in $\mathcal{R}$ such that up to time T the trajectory remains in the ε-neighborhood of V. Note that, in contrast to the definitions of topological entropy and subspace entropy for flows, Definition 4, here a number of control functions is counted, not a number of initial values. Hence this notion is intrinsic for control systems.

The following elementary observation shows that one cannot require that there are finitely many control functions u such that instead of (6) one has $\varphi(t,x_0,u)\in V$ for all $t\in[0,T]$. Hence the invariance condition has to be relaxed as indicated above using $\varepsilon>0$.

Proposition 7. *Let V be a controlled invariant subspace. Furthermore, consider a neighborhood K of the origin in V, let $T>0$, and suppose that there is $v\in V$ with $e^{AT}v\notin V$. Then there is no finite set $\mathcal{R}$ of controls such that for every $x_0\in K$ there is $u\in\mathcal{R}$ with $\varphi(t,x_0,u)\in V$ for all $t\in[0,T]$.*

Proof. We may assume that $\gamma v\in K$ for all $\gamma\in(0,1)$. The proof is by contradiction. Suppose that $\mathcal{R}=\{u_1,\ldots,u_r\}$ is a finite set of controls such that for every $x_0\in V$ there is a control u_j in $\mathcal{R}$ with $\varphi(T,x_0,u_j)\in V$. There is a control in $\mathcal{R}$, say u_1, with

$$\varphi(T,v,u_1)=e^{TA}v+\int_0^T e^{(T-s)A}Bu_1(s)ds\in V.$$

Since $e^{TA}v\notin V$, it follows that

$$\varphi(T,0,u_1)=\int_0^T e^{(T-s)A}Bu_1(s)ds\notin V.$$

We find for $\gamma\in(0,1)$

$$\begin{aligned}\varphi(T,\gamma v,u_1)&=\gamma\left[e^{TA}v+\int_0^T e^{(T-s)A}Bu_1(s)ds\right]+(1-\gamma)\int_0^T e^{(T-s)A}Bu_1(s)ds\\&=\gamma\varphi(T,v,u_1)+(1-\gamma)\varphi(T,0,u_1).\end{aligned}$$

This implies $\varphi(T,\gamma v,u_1)\notin V$ for all $\gamma\in(0,1)$. Choose $\gamma_1\in(0,1)$ and let $v_1:=\gamma_1 v$. There is a control in $\mathcal{R}$, say $u_2\neq u_1$, such that $\varphi(T,v_1,u_2)\in V$. Iterating the arguments above one arrives at a contradiction. $\square$

On the other hand, there are always finite (T,ε)-spanning sets of controls as shown by the following remark.

Remark 8. Let $K \subset V$ be compact and $\varepsilon, T > 0$. By controlled invariance of V there is for every $x \in K \subset V$ a control function u with $\varphi(t,x,u) \in V$ for all $t \geq 0$. Hence, using continuous dependence on initial values and compactness of K, one finds finitely many controls $u_1, \ldots, u_r$ such that for every $x \in K$ there is u_j with $\mathrm{dist}(\varphi(t,x,u_j), V) < \varepsilon$ for all $t \in [0,T]$. Hence $r_{\mathrm{inv}}(T,\varepsilon,K,V) < \infty$.

Now we consider the exponential growth rate of $r_{\mathrm{inv}}(T,\varepsilon,K,V)$ as in Definition 6 for $T \to \infty$ and let $\varepsilon \to 0$. The resulting invariance entropy is the main subject of the present paper.

Definition 9. Let V be a controlled invariant subspace for a control system of the form (5). Then, for a compact subset $K \subset V$, the *invariance entropy* $h_{\mathrm{inv}}(K,V)$ is defined by

$$h_{\mathrm{inv}}(\varepsilon,K,V) := \limsup_{T\to\infty} \frac{1}{T} \log r_{\mathrm{inv}}(T,\varepsilon,K,V),$$
$$h_{\mathrm{inv}}(K,V) := \lim_{\varepsilon \searrow 0} h_{\mathrm{inv}}(\varepsilon,K,V).$$

Finally, the invariance entropy of V is defined by

$$h_{\mathrm{inv}}(V;A,B) := \sup\nolimits_K h_{\mathrm{inv}}(K,V),$$

where the supremum is taken over all compact subsets $K \subset V$.

In the sequel, we will use the shorthand notation $h_{\mathrm{inv}}(V)$ for $h_{\mathrm{inv}}(V;A,B)$, when it is clear which control system is considered. Note that $h_{\mathrm{inv}}(\varepsilon_1,K,V) \leq h_{\mathrm{inv}}(\varepsilon_2,K,V)$ for $\varepsilon_2 \leq \varepsilon_1$. Hence the limit for $\varepsilon \to 0$ exists (it might be infinite.) Since all norms on finite dimensional vector spaces are equivalent, the invariance entropy of V is independent of the chosen norm. We will show later that every controlled invariant subspace has finite invariance entropy. It is clear by inspection, that, as the subspace entropy $h_{\mathrm{sub}}(V)$, also the invariance entropy $h_{\mathrm{inv}}(V)$ is invariant under state space similarity; i.e. $h_{\mathrm{inv}}(SV;SAS^{-1},SB) = h_{\mathrm{inv}}(V;A,B)$ for $S \in GL(\mathcal{X})$.

We are interested in the problem to keep the system in the subspace V for all $t \geq 0$. Then the exponential growth rate of the required number of control functions will give information on the difficulty of this task. A motivation to consider open-loop controls in this context comes, in particular, from model predictive control (see, e.g., Grüne and Pannek [7]), where optimal open-loop controls are computed and applied on short time intervals.

The following theorem (taken from Colonius and Helmke [3]) shows that the entropy of a controlled invariant subspace V can be characterized by the entropy of V for the corresponding uncontrolled system $\dot{x} = Ax$. This result will be useful in order to compute entropy bounds.

Theorem 10. *Let V be a controlled invariant subspace for system (5) and consider the invariance entropy $h_{\mathrm{inv}}(V)$ of control system (5) and the subspace entropy $h_{\mathrm{sub}}(V)$ of V of the uncontrolled system $\Phi_t = e^{tA}$. Then*

$$h_{\mathrm{inv}}(V) = h_{\mathrm{sub}}(V;\Phi).$$

Proof. (i) Let $K \subset V$ be compact, and fix $T, \varepsilon > 0$. Consider a (T, ε, K, V)-spanning set $\mathcal{R} = \{u_1, ..., u_r\}$ of controls with minimal cardinality $r = r_{\text{inv}}(T, \varepsilon, K, V)$. This means that for every $x \in K$ there is u_j with

$$\text{dist}(\varphi(t, x, u_j), V) < \varepsilon \text{ for all } t \in [0, T].$$

By minimality, we can for every u_j pick $x_j \in K$ with $\text{dist}(\varphi(t, x_j, u_j), V) < \varepsilon$ for all $t \in [0, T]$. Then, using linearity, one finds for all $x \in K$ a control u_j and a point $x_j \in K$ such that for all $t \in [0, T]$

$$\text{dist}(e^{tA}x - e^{tA}x_j, V) = \text{dist}(\varphi(t, x, u_j) - \varphi(t, x_j, u_j), V) < 2\varepsilon.$$

This shows that the points x_j form a $(T, 2\varepsilon)$-spanning set for the subspace entropy, and hence

$$r_{\text{inv}}(T, \varepsilon, K, V) \geq r_{\text{sub}}(T, 2\varepsilon, K, V).$$

Letting T tend to infinity, then $\varepsilon \to 0$ and, finally, taking the supremum over all compact subsets $K \subset V$, one obtains $h_{\text{inv}}(V) \geq h_{\text{sub}}(V)$.

(ii) For the converse inequality, let K be a compact subset of V and $T, \varepsilon > 0$. Let $E = \{x_1, \ldots, x_r\} \subset K$ be a minimal (T, ε)-spanning set for the subspace entropy which means that for all $x \in K$ there is $j \in \{1, \ldots, r\}, r = r_{\text{sub}}(T, \varepsilon, K, V)$, such that for all $t \in [0, T]$

$$\text{dist}(e^{tA}x - e^{tA}x_j, V) = \inf_{z \in V} \left\| e^{tA}x - e^{tA}x_j - z \right\| < \varepsilon.$$

Since V is controlled invariant, we can assign to each $x_j, j \in \{1, \ldots, r\}$, a control function $u_j \in C([0, \infty), \mathbb{R}^m)$ such that $\varphi(t, x_j, u_j) \in V$ for all $t \geq 0$. Let $\mathcal{R} := \{u_1, \ldots, u_r\}$. Using linearity we obtain that for every $x \in K$ there is j such that for all $t \in [0, T]$

$$\text{dist}(\varphi(t, x, u_j) - \varphi(t, x_j, u_j), V) = \text{dist}(e^{tA}x - e^{tA}x_j, V) < \varepsilon.$$

Since $\varphi(t, x_j, u_j) \in V$ for $t \in [0, T]$, it follows that

$$\begin{aligned} \text{dist}(\varphi(t, x, u_j), V) &= \inf_{z \in V} \left\| \varphi(t, x, u_j) - z \right\| \\ &\leq \left\| \varphi(t, x, u_j) - \varphi(t, x_j, u_j) \right\| < \varepsilon. \end{aligned}$$

Thus for every $x \in K$ there is $u_j \in \mathcal{R}$ such that for all $t \in [0, T]$ one has $\text{dist}(\varphi(t, x, u_j), V) < \varepsilon$. Hence $\mathcal{R}$ is (T, ε)-spanning for the invariance entropy and it follows that

$$r_{\text{inv}}(T, \varepsilon, K, V) \leq r_{\text{sub}}(T, \varepsilon, K, V) \text{ for all } T, \varepsilon > 0,$$

and consequently $h_{\text{inv}}(K, V) \leq h_{\text{sub}}(V; \Phi)$. □

In view of this theorem, we will look more closely at the subspace entropy.

4 Analysis of the subspace entropy

This section presents an analysis of the subspace entropy. The main result is Theorem 20 which shows that the subspace entropy is bounded above by the topological entropy of an induced system; a sufficient condition for equality is given which leads to a characterization of the subspace entropy (and hence the invariance entropy) by certain positive eigenvalues of the uncontrolled system.

First we describe the behavior of the subspace entropy under a semiconjugacy to the induced flow on a quotient space.

Proposition 11. *Let W be an A-invariant subspace for a linear map A on $\mathcal{X}$. Then, for a subspace V of $\mathcal{X}$ the subspace entropies of the flow $\Phi_t = e^{tA}$ on $\mathcal{X}$ and the induced flow $\bar{\Phi}_t$ on the quotient space $\mathcal{X}/W$, respectively, satisfy*

$$h_{\text{sub}}(V,\Phi) \geq h_{\text{sub}}(V/W,\bar{\Phi}).$$

Proof. Let $K \subset V$ be compact and for $T,\varepsilon > 0$ consider a $(T,\varepsilon,K,V;\Phi)$-spanning set $R \subset K$. Denote the projection of $\mathcal{X}$ to $\mathcal{X}/W$ by π, hence $\pi V = V/W$. Then the set πR is (T,ε)-spanning for $\pi K \subset \pi V$ with respect to the flow $\bar{\Phi}$. In fact, let $R = \{x_1,\ldots,x_r\}$ and consider $\pi x \in \pi K$ for some element $x \in K$. Then there exists $x_j \in R$ with

$$\max_{0\leq t\leq T} \text{dist}(e^{tA}(x-x_j),V) < \varepsilon.$$

Denoting the map induced by A on $\mathcal{X}/W$ by $\bar{A}$ one finds for all $t \in [0,T]$

$$\begin{aligned}
\text{dist}(e^{t\bar{A}}(\pi x-\pi x_j),\pi V) &= \inf_{z\in V}\left\|e^{t\bar{A}}(\pi x-\pi x_j)-\pi z\right\| \\
&= \inf_{z\in V,w\in W}\left\|e^{tA}(x-x_j)-z-w\right\| \\
&\leq \text{dist}(e^{tA}(x-x_j),V) < \varepsilon.
\end{aligned}$$

It follows that the minimal cardinality $r_{\text{sub}}(T,\varepsilon,K,V)$ for Φ is greater than or equal to the minimal cardinality $r_{\text{sub}}(T,\varepsilon,\pi K,\pi V)$ for $\bar{\Phi}$. Then take the limit superior for $T \to \infty$ and let ε tend to 0. Finally, observe that for every compact set $K_1 \subset V/W$ there is a compact set $K \subset V$ with $\pi K = K_1$. Hence taking the supremum over all compact $K_1 \subset V/W$ one obtains the assertion. □

Note that the map A does not induce a map on the quotient space $\mathcal{X}/V$, since we are interested in the case where V is not invariant. Nevertheless, condition (4) determines a distance in $\mathcal{X}/V$.

Next we show that we may assume that all eigenvalues of A have positive real part. Decompose $\mathcal{X}$ into the center-stable and the unstable subspaces, $\mathcal{X} = \mathcal{X}^{-,0} \oplus \mathcal{X}^{+}$. Thus $\mathcal{X}^{-,0}$ is the sum of all real generalized eigenspaces corresponding to eigenvalues with nonpositive real part and $\mathcal{X}^{+}$ is the sum of all real generalized eigenspaces corresponding to eigenvalues with positive real part. We denote the corresponding projections of $\mathcal{X}$ by $\pi^{-,0}$ and π^{+}, respectively, and let $\Phi^{0,-}$ and Φ^{+} be the associated restrictions of Φ.

Proposition 12. *Let V be a subspace of $\mathcal{X}$. Then the subspace entropy Φ with respect to V and of Φ^+ with respect to π^+V coincide.*

Proof. Decompose Φ into $\Phi^{0,-}$ and Φ^+. The restriction $\Phi^{0,-}$ to the center-stable subspace has the property, that for a polynomial $p(t)$

$$\left\|\Phi_t^{0,-}(x-y)\right\| \leq p(t)\,\|x-y\|,$$

hence the subspace entropy here vanishes. Furthermore, the product of spanning sets for the stable and the unstable part yields spanning sets for the total system, hence

$$h_{\text{sub}}(V,\Phi) \leq h_{\text{sub}}(V,\Phi^+) + h_{\text{sub}}(V,\Phi^{0,-}) = h_{\text{sub}}(V,\Phi^+).$$

and clearly, $h_{\text{sub}}(V,\Phi^+) \leq h_{\text{sub}}(V,\Phi)$. □

Next we show that the subspace entropy is bounded above by the topological entropy of V.

Proposition 13. *Let V be a subspace of $\mathcal{X}$. Then the topological entropy of V and the subspace entropy of V satisfy $h_{\text{sub}}(V) \leq h_{\text{top}}(V)$.*

Proof. Let $K \subset V$ be compact and for $T,\varepsilon > 0$ consider a (T,ε)-spanning set $R = \{x_1,\ldots,x_r\} \subset K$ with minimal cardinality $r = r_{\text{top}}(T,\varepsilon,K)$. For every $x \in K$ there exists $x_j \in R$ such that for all $t \in [0,T]$

$$\left\|e^{tA}(x-x_j)\right\| < \varepsilon.$$

Then one finds for all $t \in [0,T]$

$$\text{dist}(e^{tA}(x-x_j),V) = \inf_{v\in V}\left\|e^{tA}(x-x_j)-v\right\| \leq \left\|e^{tA}(x-x_j)\right\| < \varepsilon.$$

It follows that the minimal cardinality $r_{\text{top}}(T,\varepsilon,K)$ for the topological entropy is greater than or equal to the minimal cardinality $r_{\text{sub}}(T,\varepsilon,K,V)$ for the subspace entropy. Then take the limit superior for $T \to \infty$ and let ε tend to 0. Finally, take the supremum over all compact sets $K \subset V$. □

The next proposition shows that only part of the state space $\mathcal{X}$ is relevant for the subspace entropy.

Proposition 14. *Let $V \subset \mathcal{X}$ be a subspace. Then the subspace entropies of V as a subspace of $\mathcal{X}$ and of the smallest A-invariant subspace $\langle A|V\rangle$ containing V coincide.*

Proof. Let $K \subset V$ be compact and consider $T,\varepsilon > 0$. Let $R \subset K$ in $\mathcal{X}$ be a (T,ε)-spanning set for the system in $\mathcal{X}$. Thus for every $x \in K$ there exists $y \in R$ with

$$\max_{0\leq t\leq T}\text{dist}(e^{tA}(x-y),V) < \varepsilon.$$

Since $e^{tA}(x-y) \in \langle A|V\rangle$, it follows that R is also a (T,ε)-spanning set for the system in $\langle A|V\rangle$. The converse is obvious. □

Hence it suffices to consider the system in $\langle A|V\rangle$. Next we show that we can also neglect the largest A-invariant subspace of V, denoted by $\ker(A;V)$. We denote the projection by

$$\pi : \langle A|V\rangle \to \langle A|V\rangle/\ker(A;V)$$

and hence $V/\ker(A;V) = \pi V$. The linear map A induces a linear map $\bar{A}$ on the quotient space $\langle A|V\rangle/\ker(A;V)$ and we let $\pi\Phi(t,\bar{x}) := e^{\bar{A}t}\bar{x}, t \in \mathbb{R}, \bar{x} \in \pi V$. Note that for all subspaces $W \subset V$ with $AW \subset W$ it follows that $W \subset \ker(A;V)$. Hence for a subspace $\pi W \subset \pi V$ with $\bar{A}(\pi W) = AW + \ker(A;V) \subset \pi W = W + \ker(A;V)$ the subspace $W + \ker(A;V) \subset V + \ker(A;V) \subset V$ is an A-invariant subspace of V and hence contained in $\ker(A;V)$.

Proposition 15. *The subspace entropies of Φ with respect to V and of $\pi\Phi$ with respect to $V/\ker(A;V)$ coincide,*

$$h_{\text{sub}}(V;\Phi) = h_{\text{sub}}(\pi V;\pi\Phi).$$

Proof. By Proposition 11, the inequality $h_{\text{sub}}(V;\Phi) \geq h_{\text{sub}}(\pi V;\pi\Phi)$ follows. For the converse, let K be a compact subset of V. Then, for the projection π of $\langle A|V\rangle$ to the quotient space $\langle A|V\rangle/\ker(A;V)$, the set πK is compact and $\pi V = V + \ker(A;V)$. Let $T,\varepsilon > 0$ be given and denote by $E \subset \pi K$ a minimal $(T,\varepsilon,\pi K,\pi V;\pi\Phi)$-spanning set with respect to the flow $\pi\Phi$ on $\langle A|V\rangle/\ker(A;V)$, say $E = \{\pi x_1,\ldots,\pi x_r\}$ with $x_j \in K$ and $r = r_{\text{sub}}(T,\varepsilon,\pi K,\pi V)$. Note that $V + \ker(A;V) = V$. Hence it follows that for all $x \in K$ there is $j \in \{1,\ldots,r\}$ such that for all $t \in [0,T]$

$$\begin{aligned}\inf_{z\in V}\left\|e^{tA}x - e^{tA}x_j - z\right\| &= \operatorname{dist}(e^{tA}x - e^{tA}x_j, V + \ker(A;V)) \\ &= \operatorname{dist}(e^{\bar{A}t}\pi x - e^{\bar{A}t}\pi x_j, \pi V) < \varepsilon.\end{aligned}$$

We have shown that the set $\{x_1,\ldots,x_r\} \subset K$ is (T,ε)-spanning for Φ and hence the minimal cardinality $r_{\text{sub}}(T,\varepsilon,K,V)$ of such a set is equal to or less than $r_{\text{sub}}(T,\varepsilon,$ $\pi K,\pi V)$. Thus the assertion follows. □

This result shows that we have to project things to πV for every time t. Observe that $\dim e^{\bar{A}t}(\pi V) = \dim(\pi V)$. However, the projection of $e^{\bar{A}t}(\pi V)$ to $\langle A|V\rangle/\pi V$ need not have constant dimension. Slightly more generally, we have the following situation: Consider a linear map A on $\mathcal{X}$ and a subspace V of $\mathcal{X}$ which is not invariant under A. Due to Proposition 13 we know that the topological entropy is an upper bound. The following examples show that the subspace entropy may be equal to the topological entropy or less than the topological entropy.

Example 16. Consider a complex conjugate pair of eigenvalues and a one-dimensional subspace V of the real eigenspace. Let K be a compact neighborhood of the origin in V. This can be a controlled invariant subspace: Consider $V = \mathbb{R} \times \{0\}$ and with $\lambda > 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

i.e.

$$\dot{x}_1 = \lambda x_1 - x_2$$
$$\dot{x}_2 = x_1 + \lambda x_2 + u(t)$$

If we choose $u = -x_1 - \lambda x_2$, then every initial point with $x_2 = 0$ remains in this subspace.
For $u = 0$, the solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_1^0 \cos t - x_2^0 \sin t \\ x_1^0 \sin t + x_2^0 \cos t \end{bmatrix}.$$

Initial values $(x_1^0, 0) \in V$ have as second component

$$x_2(t) = e^{\lambda t}[x_1^0 \sin t + x_2^0 \cos t] = e^{\lambda t} x_1^0 \sin t.$$

Hence the projection of the solutions to $\mathbb{R}^2/V$, identified with the second component, gives for $K \subset V$

$$x_2(t) = e^{\lambda t} \sin t \cdot x_1^0,\ x_1^0 \in K.$$

The solutions $x_2(t)$ move apart with $e^{\lambda t}$, if we consider the limit superior: choose $t = (2n+1)\frac{\pi}{2}$. Hence the subspace entropy is $h_{\text{sub}}(V) = \lambda$. Observe that the image of the projection depends, naturally, on t. In $\mathbb{R}^2/V$ it is one-dimensional, except for $t = n\pi, n \geq 0$, where it drops to 0. In this example, the Lyapunov exponent in L_j determines the subspace entropy.

Example 17. Consider with $\lambda > 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

The eigenspace is $\mathbb{R} \times \{0\}$. The subspace $V = \{0\} \times \mathbb{R}$ is controlled invariant, since we may choose $u = -\lambda x_2$. One has

$$e^{tA} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Thus $e^{tA}V \to \mathbb{R} \times \{0\}$ in projective space for $t \to \infty$. The solution in $\mathbb{R}^2/V$ identified with $\mathbb{R} \times \{0\}$ is given by

$$x_1(t) = te^{\lambda t} x_2^0.$$

The solutions $x_1(t)$ move apart with $e^{\lambda t}$, hence the subspace entropy is given by $h_{\text{sub}}(V) = \lambda$. Again, the Lyapunov exponent in L_j determines the subspace entropy. Note that here $e^{tA}V$ converges to the orthogonal complement of V.

Example 18. Consider with $\lambda > 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

The eigenspace of A is $\mathbb{R}\times\{0\}\times\{0\}\times\{0\}$. The subspace $V=\{0\}\times\mathbb{R}^2\times\{0\}$ only contains the trivial A-invariant subspace and V is controlled invariant, since we may choose $u_1=-\lambda x_2-x_3, u_2=-\lambda x_3-x_4$. One has

$$e^{tA}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=e^{\lambda t}\begin{bmatrix}1&t&\frac{t^2}{2}&\frac{t^3}{3!}\\0&1&t&\frac{t^2}{2}\\0&0&1&t\\0&0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=e^{\lambda t}\begin{bmatrix}t\\1\\0\\0\end{bmatrix}, e^{tA}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=e^{\lambda t}\begin{bmatrix}\frac{t^2}{2}\\t\\1\\0\end{bmatrix}.$$

Thus $e^{tA}V\to\mathbb{R}^2\times\{0\}\times\{0\}$ in the Grassmannian $\mathbb{G}_2$ for $t\to\infty$. The solution in $\mathbb{R}^4/V$ identified with $\mathbb{R}\times\{0\}\times\{0\}\times\mathbb{R}$ is given by

$$\begin{bmatrix}x_1(t)\\x_4(t)\end{bmatrix}=\begin{bmatrix}e^{\lambda t}\frac{t^2}{2}\\0\end{bmatrix}.$$

The solutions in $\mathbb{R}^4/V$ move apart with $e^{\lambda t}\frac{t^2}{2}$, hence the subspace entropy is given by $h_{\text{sub}}(V)=\lambda$. One the other hand, the topological entropy of V in $\mathbb{R}^4$ is 2λ. Note that here $\dim V=2=\dim\mathbb{R}^4/V$.

We impose the following assumption: Let $v_1,\dots,v_k$ be an orthonormal basis of V. Then there is $\gamma>0$ such that for a sequence $t_i\to\infty$ the absolute value of the volume of the parallelepiped spanned by $\pi(e^{t_iA}v_1),\dots,\pi(e^{t_iA}v_k)$ is bounded below by a positive constant times the absolute value of the volume of the parallelepiped spanned by $e^{t_iA}v_1,\dots,e^{t_iA}v_k$. More formally, we require:

There are an orthonormal basis $v_1,\dots,v_k$ of V and $\gamma>0$ such that for a sequence $t_i\to\infty$

$$\left\|\pi(e^{t_iA}v_1)\wedge\cdots\wedge\pi\left(e^{t_iA}v_k\right)\right\|\geq\gamma\left\|e^{t_iA}v_1\wedge\cdots\wedge e^{t_iA}v_k\right\|. \tag{7}$$

Note that this assumption can only hold, if $n-k=\dim\mathcal{X}/V\geq k=\dim V$.

Proposition 19. *Let V be a subspace of $\mathcal{X}$ and suppose that condition (7) holds . Then for $A:\mathcal{X}\to\mathcal{X}$ the subspace entropy is given by*

$$h_{\text{sub}}(V)=h_{\text{top}}(V).$$

Proof. In view of Proposition 13 it only remains to show that $h_{\text{sub}}(V)\geq h_{\text{top}}(V)$. A consequence of (7) is that for all i

$$\log\left\|\pi(e^{t_iA}v_1)\wedge\cdots\wedge\pi\left(e^{t_iA}v_k\right)\right\|\geq\log\gamma+\log\left\|e^{t_iA}v_1\wedge\cdots\wedge e^{t_iA}v_k\right\|,$$

and hence

$$\limsup_{t\to\infty}\frac{1}{t}\log\left\|\pi(e^{tA}v_1)\wedge\cdots\wedge\pi\left(e^{tA}v_k\right)\right\|\geq\limsup_{t\to\infty}\frac{1}{t}\log\left\|e^{tA}v_1\wedge\cdots\wedge e^{tA}v_k\right\|. \tag{8}$$

Let K be a neighborhood of the origin in V. Then K contains a parallelepiped and we may assume that K contains the parallelepiped P spanned by $v_1,\dots,v_k$. Then the set $e^{tA}K$ is a neighborhood of the origin in the k-dimensional subspace $e^{tA}V$ and it contains the parallelepiped spanned by

$$e^{tA}v_1, \cdots, e^{tA}v_k.$$

The projected set $\pi(e^{tA}K)$ is a neighborhood of the origin in $\pi(e^{tA}V)$ and, for $t = t_i$, it contains the parallelepiped $\pi(e^{t_iA}P)$ spanned by $\pi(e^{tA_k}v_1), \cdots, \pi\left(e^{tA_k}v_\ell\right)$. By Colonius and Kliemann [5, Theorem 5.2.5] one finds

$$\lim_{t\to\infty}\frac{1}{t}\log\left\|e^{tA}v_1\wedge\cdots\wedge e^{tA}v_\ell\right\| = \sum_{i=1}^{l}k_i\lambda_i,$$

where $(k_1,\ldots,k_l)$ is an element of the index set $I(k)$ given by (2).

It remains to relate the volume growth to the subspace entropy. We argue as in Colonius, San Martin, da Silva [6, Proposition 4.1]:

For $t > 0$ the k-dimensional volume of $\pi(e^{tA}P)$ satisfies

$$\mathrm{vol}^k(\pi(e^{tA}K)) \geq \mathrm{vol}^k(\pi(e^{tA}P)) = \left\|\pi(e^{tA}v_1)\wedge\cdots\wedge\pi\left(e^{tA}v_k\right)\right\|.$$

Let $\varepsilon > 0, T > 0$, and consider a (T,ε)-spanning set $R = \{x_1,\ldots,x_r\} \subset P$ of minimal cardinality $r = r_{\mathrm{sub}}(T,\varepsilon,P,V)$ for the subspace entropy. Then (by the definition of spanning sets) the set $\pi(e^{TA}P)$ is contained in the union of r balls $B(\pi(e^{TA}x_j);\varepsilon)$ of radius ε in $\mathcal{X}/V$,

$$B(\pi(e^{TA}x_j);\varepsilon) = \{z\in\mathcal{X}/V \mid \left\|z-\pi(e^{TA}x_j)\right\| < \varepsilon\}.$$

Each such ball has volume bounded by $c(2\varepsilon)^{n-k}$, where $c > 0$ is a constant. Thus

$$\mathrm{vol}^k(\pi(e^{TA}P)) \leq r\cdot c(2\varepsilon)^{n-k}.$$

This yields

$$\begin{aligned}\log r_{\mathrm{sub}}(T,\varepsilon,P,V) &\geq \log\mathrm{vol}^k(\pi(e^{TA}P)) - \log\left[c(2\varepsilon)^d\right]\\ &= \log\left\|\pi(e^{TA}v_1)\wedge\cdots\wedge\pi\left(e^{TA}v_k\right)\right\| - \log\left[c(2\varepsilon)^{n-k}\right],\end{aligned}$$

and hence

$$\begin{aligned}&\limsup_{T\to\infty}\frac{1}{T}\log r_{\mathrm{sub}}(T,\varepsilon,P,V)\\ &\geq \limsup_{T\to\infty}\frac{1}{T}\log\left\|\pi(e^{TA}v_1)\wedge\cdots\wedge\pi\left(e^{TA}v_k\right)\right\|.\end{aligned}$$

Together with (8) one obtains the assertion for $\varepsilon \to 0$. □

As a consequence of the discussion above, we obtain the following characterization of the subspace entropy. It presents a stepwise reduction of the problem.

Theorem 20. *Let $A : \mathcal{X} \to \mathcal{X}$ be a linear map on a finite dimensional normed vector space X and consider a subspace V. Decompose the associated flow $\Phi_t := e^{tA}$ into the center-stable and the unstable parts $\Phi^{-,0}$ and Φ^+, respectively.*
(i) Then the subspace entropy satisfies

$$h_{\text{sub}}(V,\Phi) = h_{\text{sub}}(V,\Phi^+).$$

(ii) Let $\langle A|V\rangle$ and $\ker(A;V)$ denote the smallest A-invariant subspace containing V and the largest A-invariant subspace contained in V, respectively. Then the reduced flow $\Phi_t^{\text{red}} = e^{tA^{\text{red}}}$ which is induced on $\langle A|V\rangle/\ker(A;V)$ satisfies

$$h_{\text{sub}}(V,\Phi^+) = h_{\text{sub}}(V/\ker(A;V),\Phi^{\text{red}}).$$

(iii) The topological entropy of the subspace $V/\ker(A;V)$ for the reduced flow Φ^{red} is an upper bound of the subspace entropy $h_{\text{sub}}(V/\ker(A;V),\Phi^{\text{red}})$,

$$h_{\text{sub}}(V/\ker(A;V),\Phi^{\text{red}}) \le h_{\text{top}}(V/\ker(A;V),\Phi^{\text{red}}). \tag{9}$$

(iv) If the reduced flow Φ^{red} on $\langle A|V\rangle/\ker(A;V)$ and the subspace $V/\ker(A;V)$ satisfy assumption (7), then equality holds in (9).
(v) The topological entropy of the subspace $V/\ker(A;V)$ for the reduced flow Φ^{red} is determined by certain eigenvalues of A: Let $k := \dim V/\ker(A;V)$. Then

$$h_{\text{top}}(V/\ker(A;V),\Phi^{\text{red}}) = \sum_i k_i \max(0,\lambda_i), \tag{10}$$

where λ_i are the real parts of the eigenvalues of A^{red}, and the k_i are given by the chain recurrent component $\mathcal{M}^k_{k_1,\dots,k_l}$ of Φ^{red} in the k-Grassmannian $\mathbb{G}_k(\langle A|V\rangle/\ker(A;V))$ containing the ω-limit set $\omega(V/\ker(A;V))$.

Proof. Assertion (i) follows from Proposition 12, (ii) is a consequence of Proposition 15 and (iii) follows from Proposition 13. Assertion (iv) holds by Proposition 19 and (v) follows by Proposition 19 applied to the reduced flow Φ^{red}. Finally, (v) is a consequence of the characterization of topological entropy in Theorem 3 applied to the reduced flow. □

In particular, Theorem 20 characterizes the invariance entropy $h_{\text{inv}}(V)$ of a controlled invariant subspace V of a linear control system of the form (5). By Theorem 10 it coincides with the subspace entropy of $\Phi_t = e^{tA}$. One obtains the following corollary to Theorem 20.

Corollary 21. *The invariance entropy of a controlled invariant subspace V of a linear control system of the form (5) is bounded above by the topological entropy of the flow Φ^{red} induced by A on $\langle A|V\rangle/\ker(A;V)$, where $\langle A|V\rangle$ and $\ker(A;V)$ denote the smallest A-invariant subspace containing V and the largest A-invariant subspace contained in V, respectively. Hence*

$$h_{\text{inv}}(V) \le h_{\text{top}}(V/\ker(A;V),\Phi^{\text{red}}) = \sum_i k_i \max(0,\lambda_i),$$

where the sum is over the eigenvalues λ_i of A as in (10). Equality holds, if the subspace $V/\ker(A;V)$ satisfies assumption (7) for Φ^{red}.

Acknowledgments

The author was supported by DFG grant Co 124/17-2 within DFG Priority Program 1305 "Control of Digitally Connected Dynamical Systems".

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